$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

Fun With The Fourier Series

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Santa Clara Valley Chapter

IEEE EMC Society

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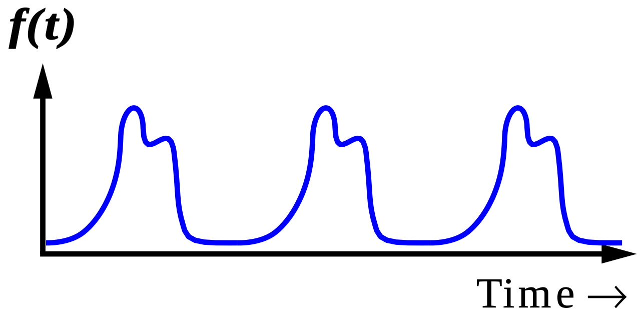
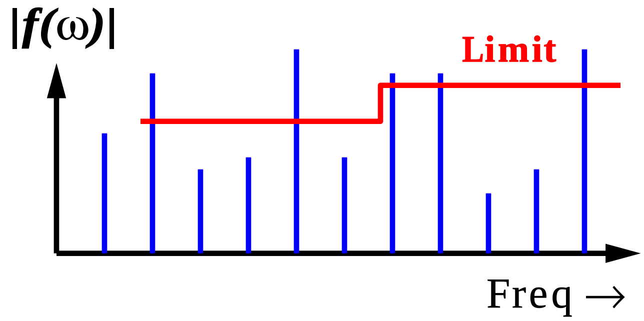
Time ⇔ Frequency Domain



Many of us are used to thinking in the time domain.

Unfortunately, many EMC specification limits such as conducted and radiated emissions are in the frequency domain.



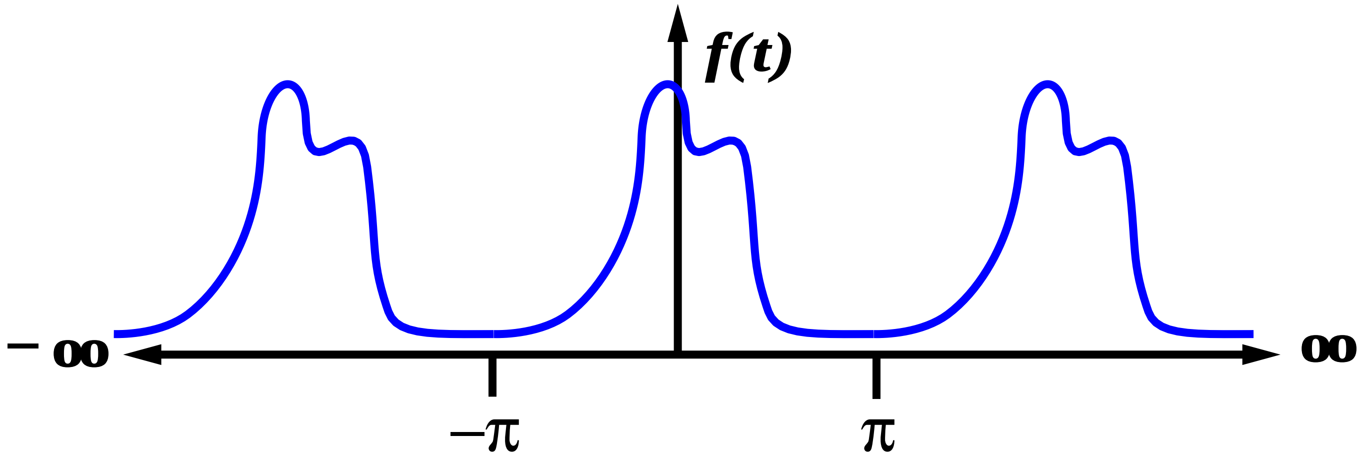
The Fourier Series is a method that allows us to travel back and forth between the time and frequency domains.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

Baron
Jean Babtiste Joseph Fourier

- Teacher
- Secret policeman
- Political prisoner
- Govenor of southern Egypt
- Prefect of Isère and Rhône
- Friend of Napoleon
- Secretary of the Académie des Sciences
- Author of "Théorie Analytique de la Chaleur"
- Inventor of the definite integral symbol

$$\int \Rightarrow \int_{-T/2}^{T/2} \Rightarrow \int_C$$

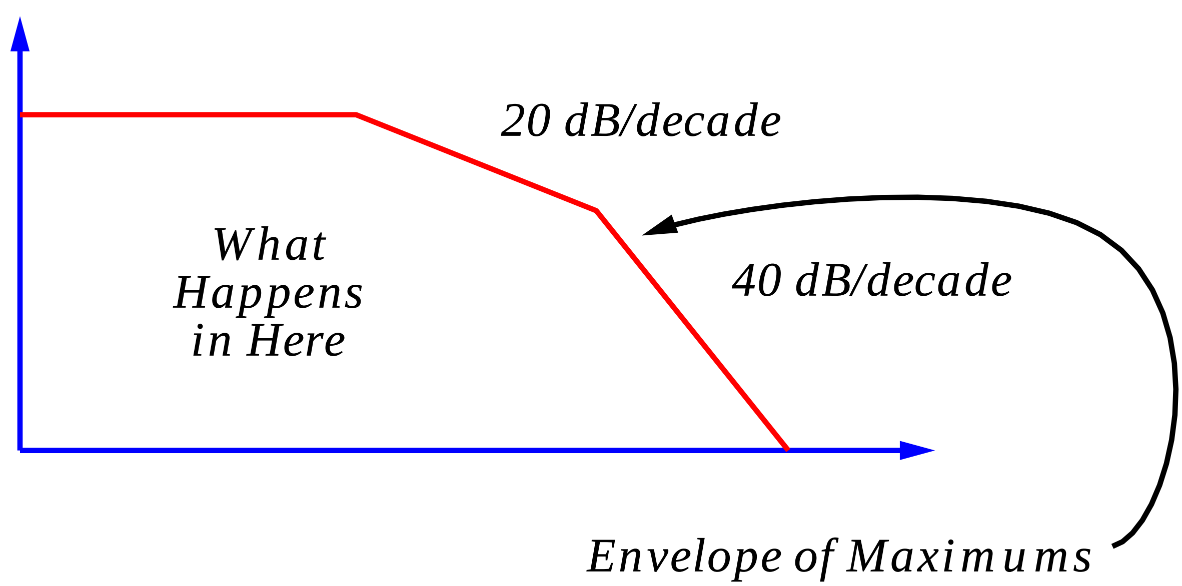


In 1807, Fourier proposed that any bounded periodic function, *f(t)*, defined in the interval [−π, π] could be expressed as the trigonometric series:

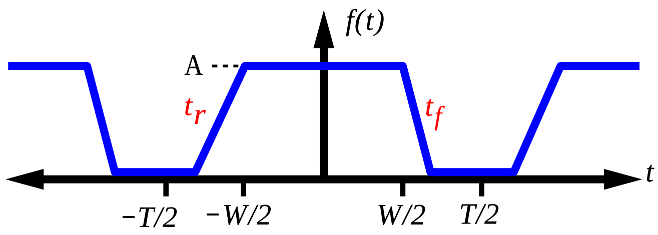
$$f(t) = \sum_{n=0}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

where a_n and b_n are real numbers that represent the magnitudes of the corresponding sine and cosine terms.

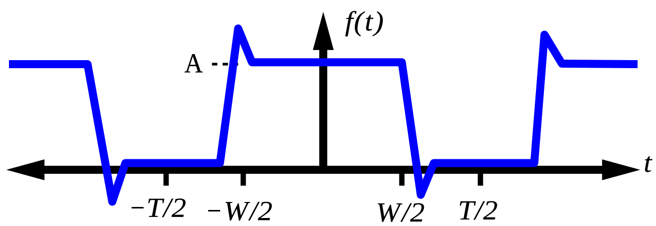
- o Square Wave
  - Only odd harmonics
  - What kind of waveform produces only even harmonics?
- o Rectangular Pulse Train
  - Sometimes envelope of harmonics looks like sinc function, sin(X)/X .
- o Trapezoidal with Equal Rise/Fall Times



- Trapezoidal with non−equal rise/fall times



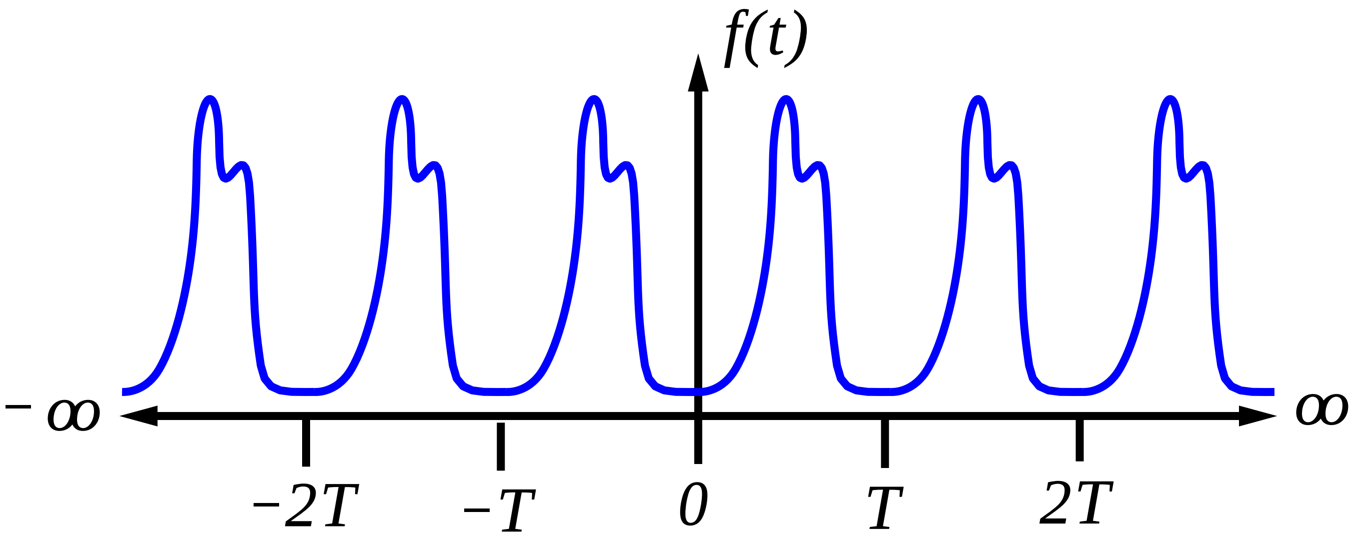
- Waveforms with over/under shoots



- Digital Differential Drivers
  - Do they really cancel each other out?

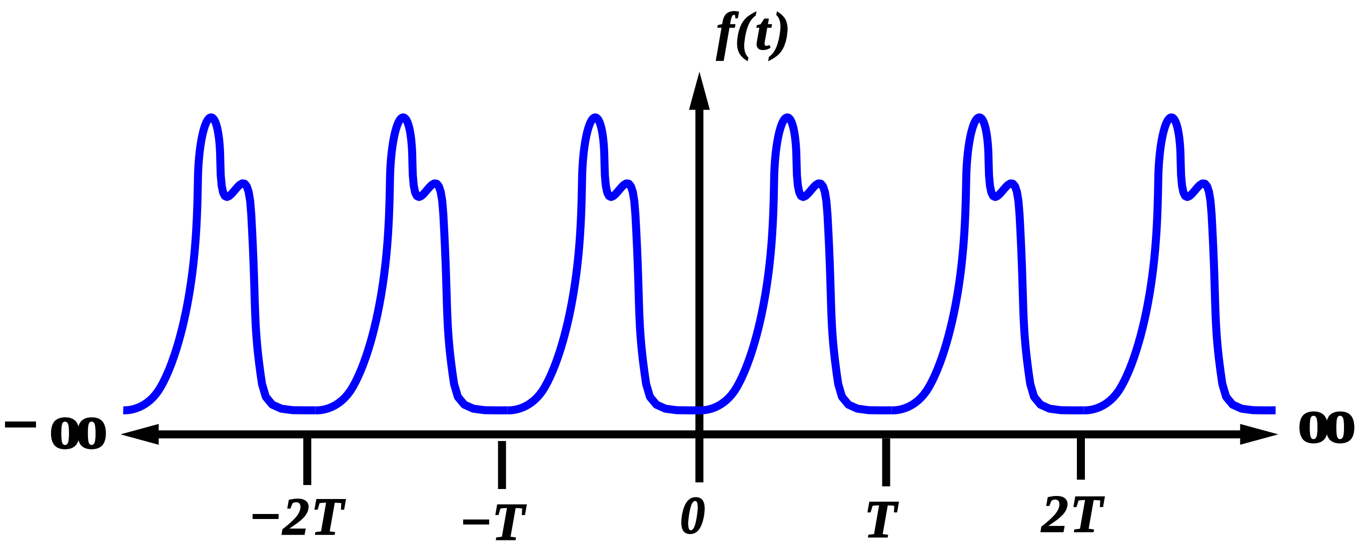
- Most spectrum analyzer displays
  - Can a connection even be made between a Fourier series and a typical spectrum analyzer display?

"Fitting" an Equation to a Periodic Waveform



- Non−periodic functions such as polynomials, log, and exp do not fit well since they all (with the exception of a polynomial constant) go to to infinity
- Periodic functions such as the tangent and cotangent functions also do not fit well because they periodically go to infinity
- Only the sine and cosine functions fit well since they are periodic and do not go to infinity

"Fitting" an Equation to a Periodic Waveform



A good start for a general (all encompassing) equation is the following:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

The only problem with this equation is it does not tell you how to go about finding the optimum values for a_0, a_n and b_n.

- Scalar (Times) Product

scalar * scalar = scalar $\qquad 2 * 2 = 4$

scalar * vector = vector $\qquad 2\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$

- Outer (Cross or Vector) Product

Only valid for vectors in 3−D space

vector x vector = vector $\qquad \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}$

- Inner (Dot) Product

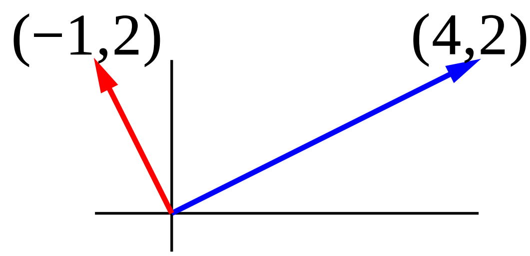
vector * vector = scalar

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2*1+3*2+4*3 = 20$$

Perpendicularity

- If the inner product of two vectors is zero then the vectors are <u>perpendicular</u> to each other

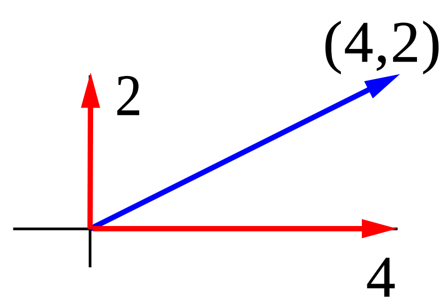
$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} * \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (-1)*4 + 2*2 = 0$$



- The inner product can be used as a "filter" to "extract" the components of a vector that align with the desired axis.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} * \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (1)*4 + (0)*2 = 4$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} * \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (0)*4 + (1)*2 = 2$$



- Perpendicular vectors exhibit the property of orthogonality

$$A = [a_0, a_1, \ldots, a_n]$$

$$B = [b_0, b_1, \ldots, b_n]$$

$$A * B = \langle A,B \rangle = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n = 0$$

$$\sum_{i=0}^{n} a_i b_i = 0$$

- The concept of orthogonality can be extended to other mathematical entities such as the definite integral of algebraic expressions

$$\int_{-T/2}^{T/2} f(t) \cos(2m\pi t/T)\, dt = 0$$

Finding a_n Using an "Orthogonal Filter"

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

$$\int_{-T/2}^{T/2} f(t)\cos(2m\pi t/T)\,dt = \int_{-T/2}^{T/2} a_0 \cos(2m\pi t/T)\,dt$$

= 0 for all m

$$+ \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} a_n \cos(2n\pi t/T)\cos(2m\pi t/T)\,dt$$

= 0 for n <> m

= $a_n T/2$ for $n=m$

$$+ \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} b_n \sin(2n\pi t/T)\cos(2m\pi t/T)\,dt$$

= 0 for all n, m

$$\int_{-T/2}^{T/2} f(t)\cos(2n\pi t/T)\,dt = a_n T/2$$

Note: $n = m$

The Fourier Series and Associated Euler Formulas

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos(2n\pi t/T)\, dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin(2n\pi t/T)\, dt$$

$$Mag_n = \sqrt{a_n^{\,2} + b_n^{\,2}} \qquad \theta_n = Tan^{-1}(b_n/a_n)$$

hint: $\cos(n\omega t) = \cos(n2\pi f t) = (2n\pi t/T)$

Other Forms of the Fourier Series

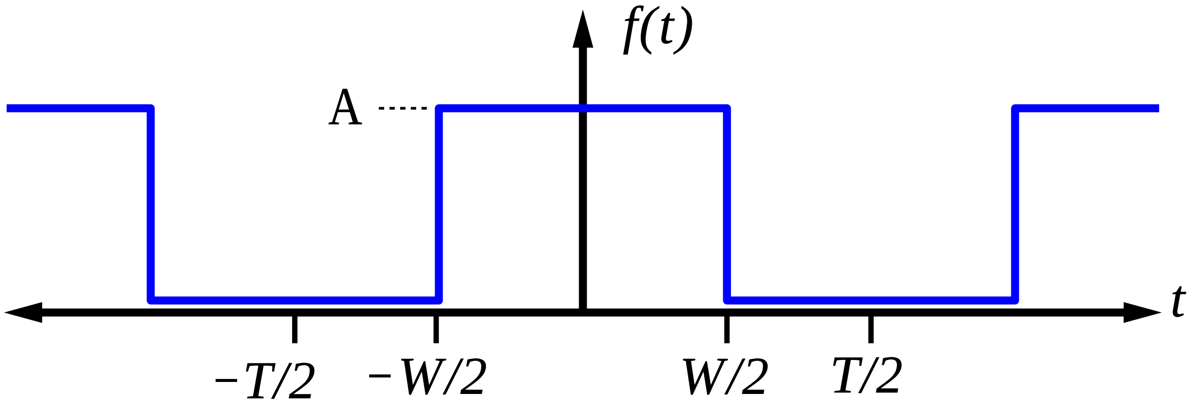
$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(2n\pi t/T + \phi_n)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n \sin(2n\pi t/T + \theta_n)$$

$$f(t) = \sum_{n=-\infty}^{n=+\infty} g_n \, e^{i2n\pi t/T}$$



$$f(t) = \underbrace{AW/T}_{\text{DC Term}} + \sum_{n=1}^{\infty} \underbrace{2A/n\pi \ \sin(n\pi W/T)}_{\text{Harmonic Amplitude } a_n} \cos\underbrace{(2n\pi t/T)}_{\text{Frequency of Harmonic}}$$

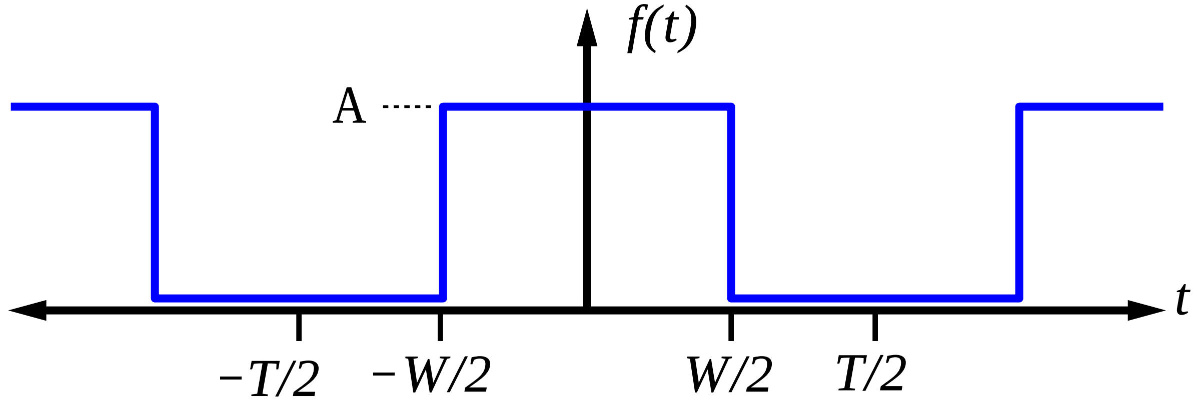
Note that there are no b_n terms, and for most EMC applications, the DC term, a_0, is of no interest and so is usually ignored.

$$Mag_n = \sqrt{a_n^2 + b_n^2} = \sqrt{a_n^2 + 0^2} = |\, a_n \,|$$

$$\theta_n = Tan^{-1}(b_n/a_n) = Tan^{-1}(0/a_n) = 0, \pi$$

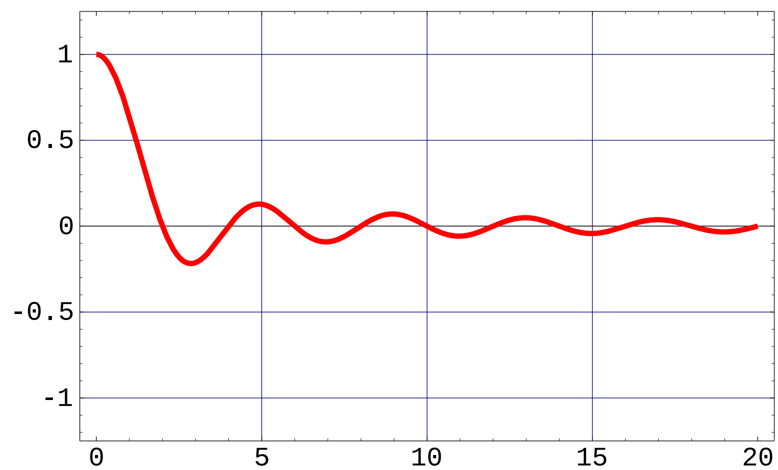
$$-\,Cos(\theta) = Cos(\theta+\pi)$$

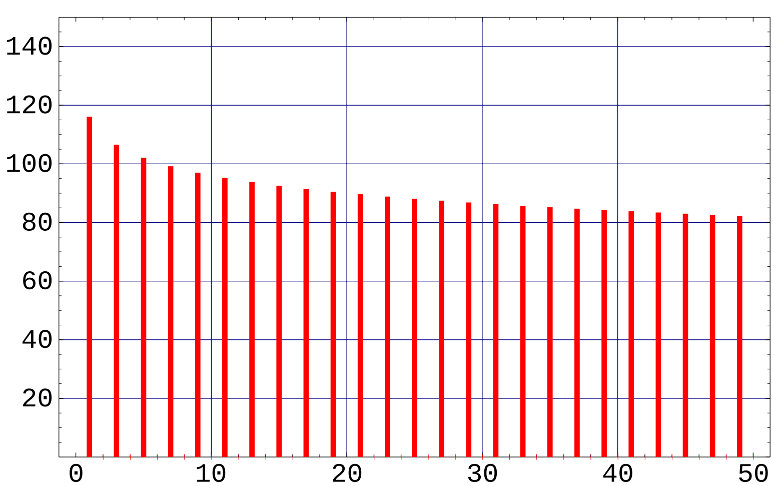
Rectangular Pulse



$$|a_n| = |2A/n\pi \sin(n\pi W/T)| = |2AT/n^2\pi^2 W \operatorname{sinc}(n\pi W/T)|$$

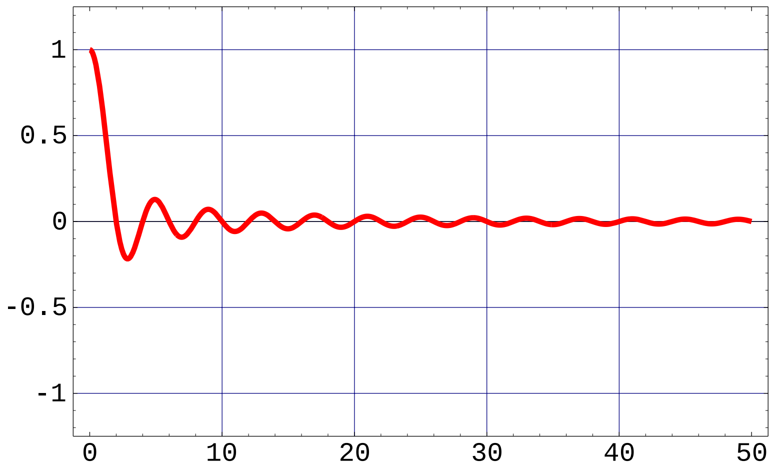
If a rectangular pulse produces a sinc *shaped spectrum....*



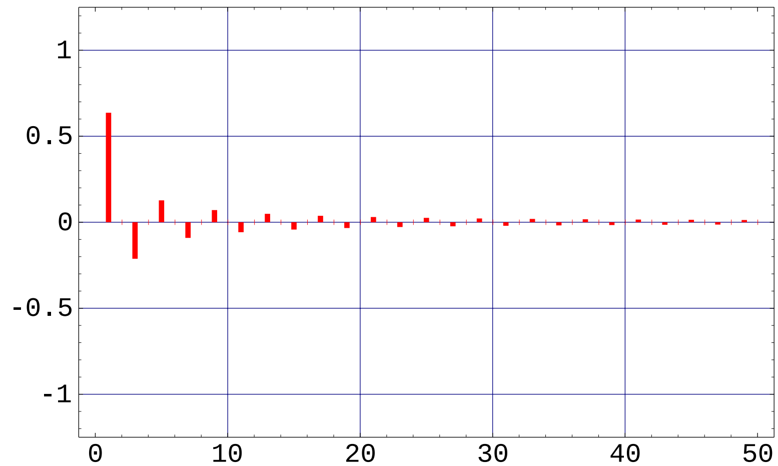


...how come the spectrum of a 50% duty cycle pulse does not look like one???

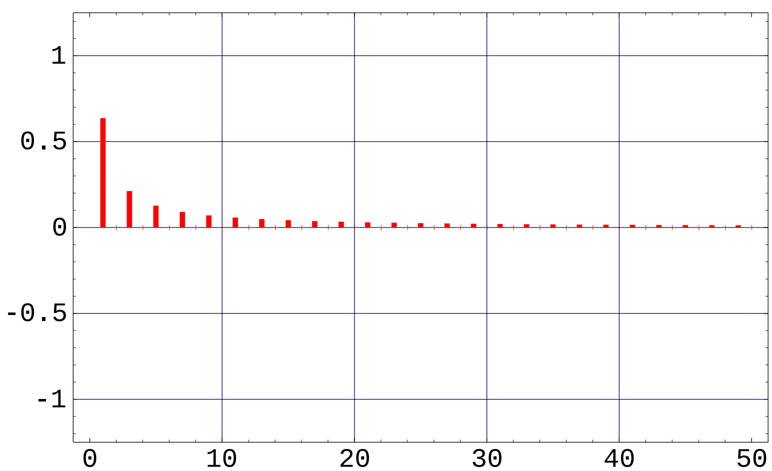
Rectangular Pulse



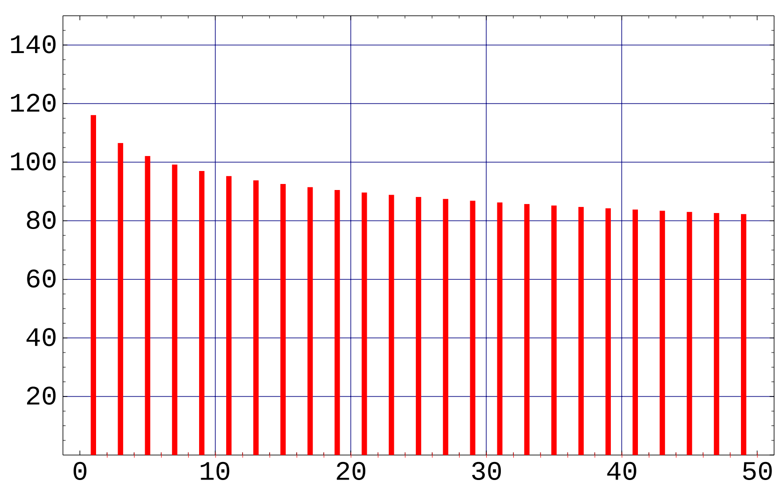
Only valid for integer n

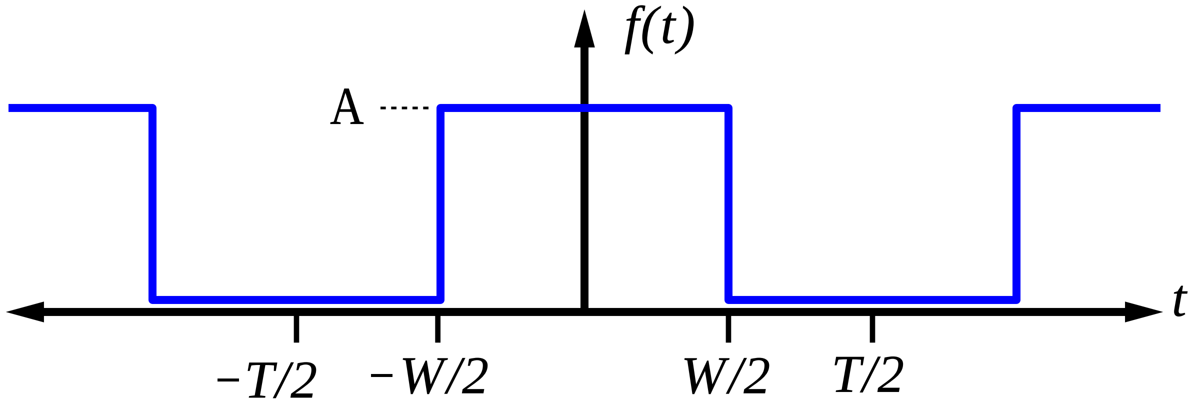


SA only displays absolute value



Display magnitude in Log mode





$$a_n = 2A/n\pi \sin(n\pi W/T)$$

a_n is at a local "maximum" when $\sin(n\pi W/T) = 1$

This occurs when $n\pi W/T = (2m-1)(\pi/2)$, $m = 1, 2, \ldots$

or $n = (m - 1/2)\,(T/W)$

For a 50% duty cycle, $W/T = 0.5 \Rightarrow n = 1, 3, 5, \ldots$

For a 25% duty cycle, $W/T = 0.25 \Rightarrow n = 2, 6, 10, \ldots$

a_n is at a local "minimum" when $\sin(n\pi W/T) = 0$

This occurs when $n\pi W/T = 2m\pi$, $m = 1, 2, \ldots$

or $n = mT/W$

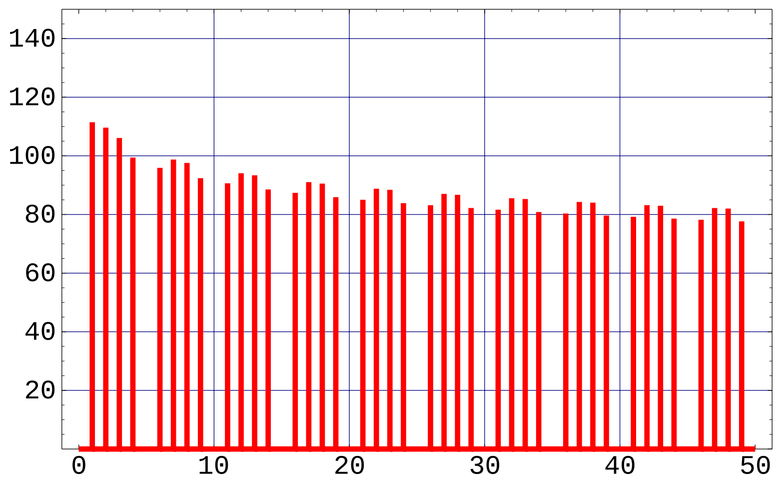
For a 50% duty cycle, $W/T = 0.5 \Rightarrow n = 2, 4, 6, \ldots$

For a 25% duty cycle, $W/T = 0.25 \Rightarrow n = 4, 8, 12, \ldots$

Rectangular Pulses

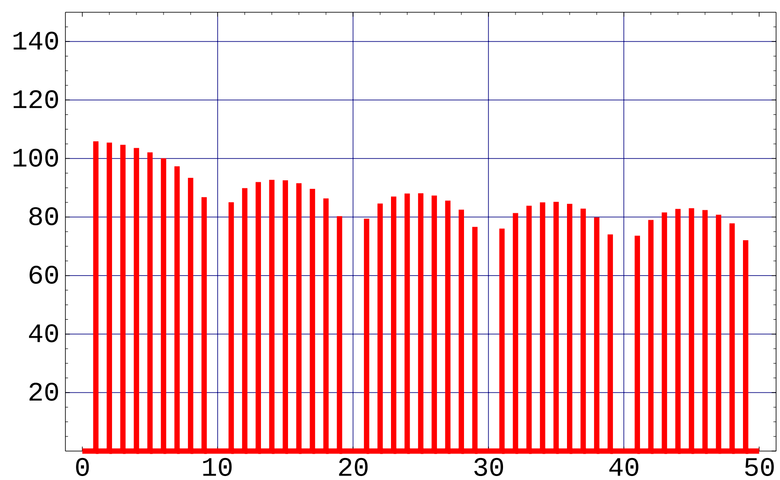
Freq (MHz)	a_n **(dBμV) 50%**	a_n **(dBμV) 25%**	a_n **(dBμV) 10%**
1	*116.1*	*113.1*	*105.9*
2	---	*110.1*	*105.4*
3	*106.5*	*103.5*	*104.7*
4	---	---	*103.6*
5	*102.1*	*99.1*	*102.1*
6	---	*100.5*	*100.1*
7	*99.2*	*96.2*	*97.3*
8	---	---	*93.4*
9	*97.0*	*94.0*	*86.8*
10	---	*96.1*	---
	↑ *Every 2nd One*	↑ *Every 4th One*	↑ *Every 10th One*

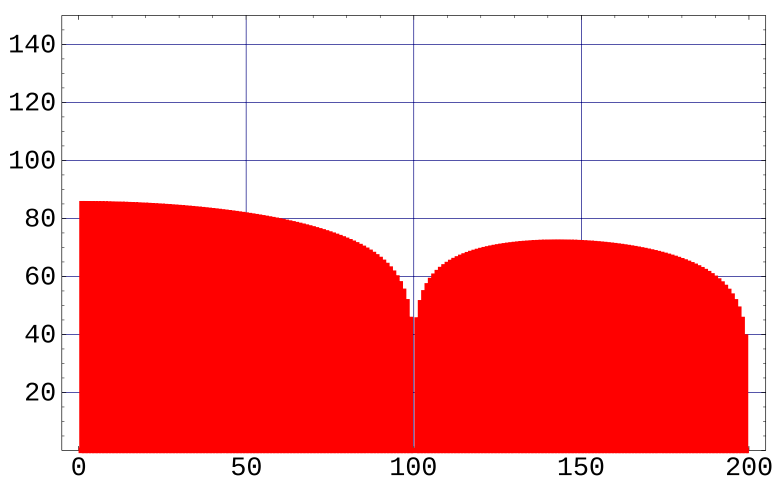
Some Examples of Rectangular Pulses



20% Duty Cycle
(every 5th harmonic =0)

10% Duty Cycle
(every 10th harmonic =0)





1% Duty Cycle
(every 100th harmonic =0)

What Goes Down
Must Come Back Up

For narrow rectangular pulses....

$$a_n = 2A/n\pi \sin(n\pi W/T)$$

$$\lim_{x \to 0} \sin(x) = x$$

$$\lim_{W \to 0} [\, a_n \,] = \lim_{W \to 0} [(2A/n\pi)\,(n\pi W/T)] = (2AW/T)$$

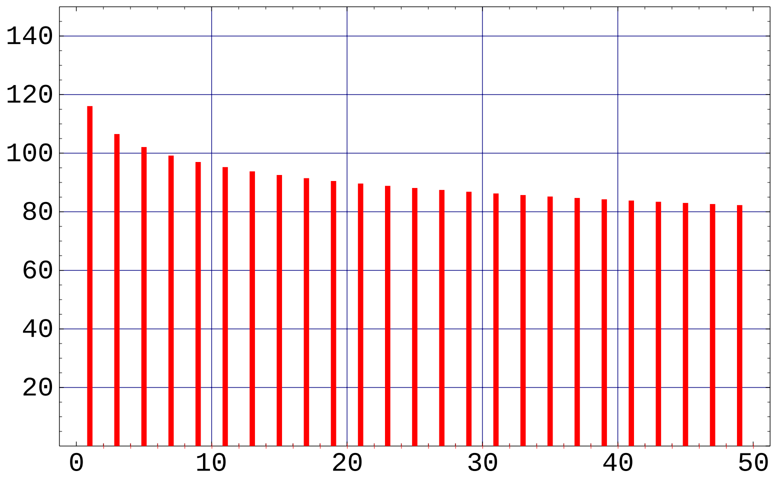
Only for very narrow pulse width do all harmonics decrease equally with duty cycle

For not−so−narrow rectangular pulses....

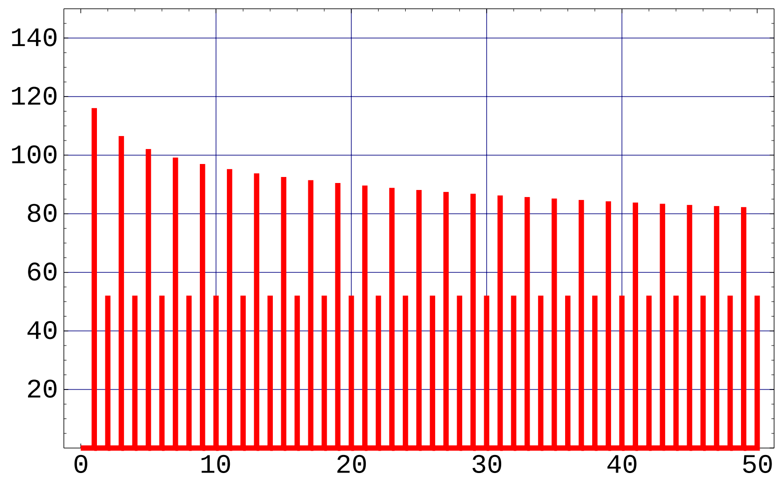
$$\frac{\partial a_n}{\partial W} = \frac{\partial [2A/n\pi \sin(n\pi W/T)]}{\partial W} = 2A/T \cos(n\pi W/T)]$$

For not−so−narrow rectangular pulses the rate of increase of low level harmonics is greater than the rate of decrease of high level harmonics

What Goes Down Must Come Back Up



50% Duty Cycle Pulse



49% Duty Cycle Pulse

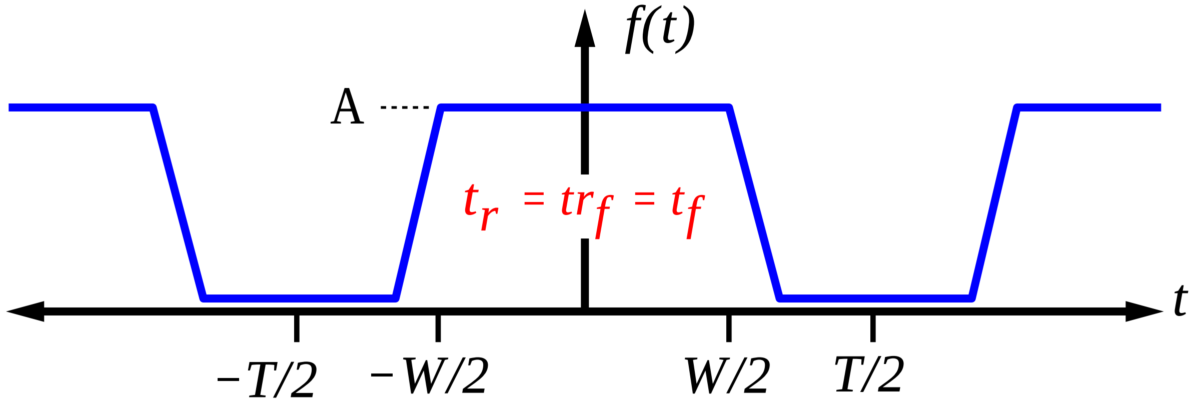
What Goes Down Must Come Back Up

Freq (MHz)	a_n (dBμV) 50%	a_n (dBμV) 25%	a_n (dBμV) 10%
1	116.1	113.1	105.9
2	---	110.1	105.4
3	106.5	103.5	104.7
4	---	---	103.6
5	102.1	99.1	102.1
6	---	100.5	100.1
7	99.2	96.2	97.3
8	---	---	93.4
9	97.0	94.0	86.8
10	---	96.1	---

Very Little Change

Very Big Change

Symmetrical 1 MHz Trapezoidal Pulses

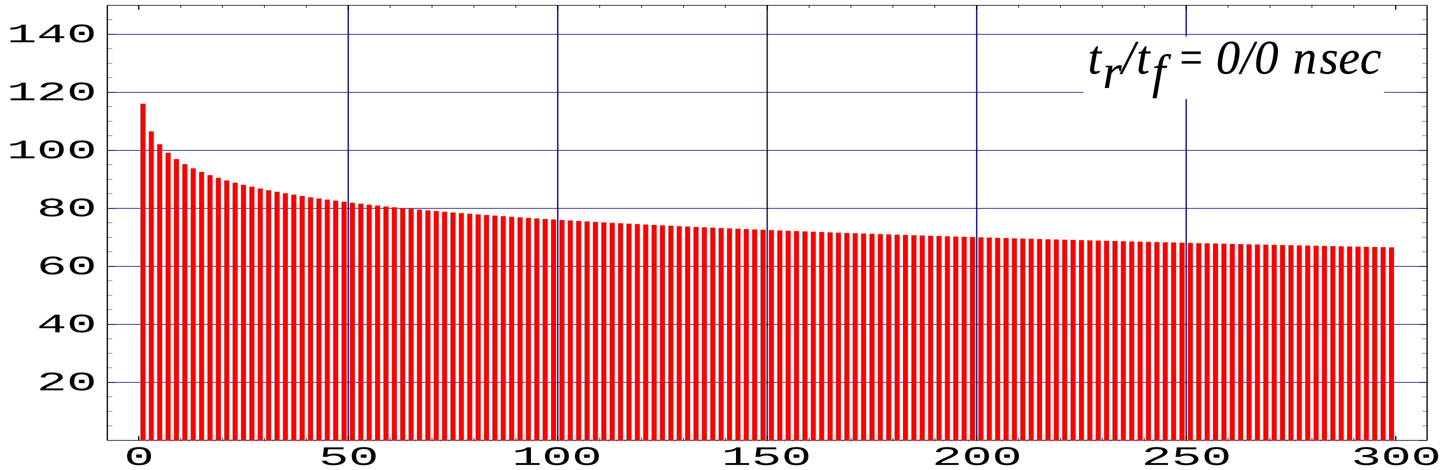


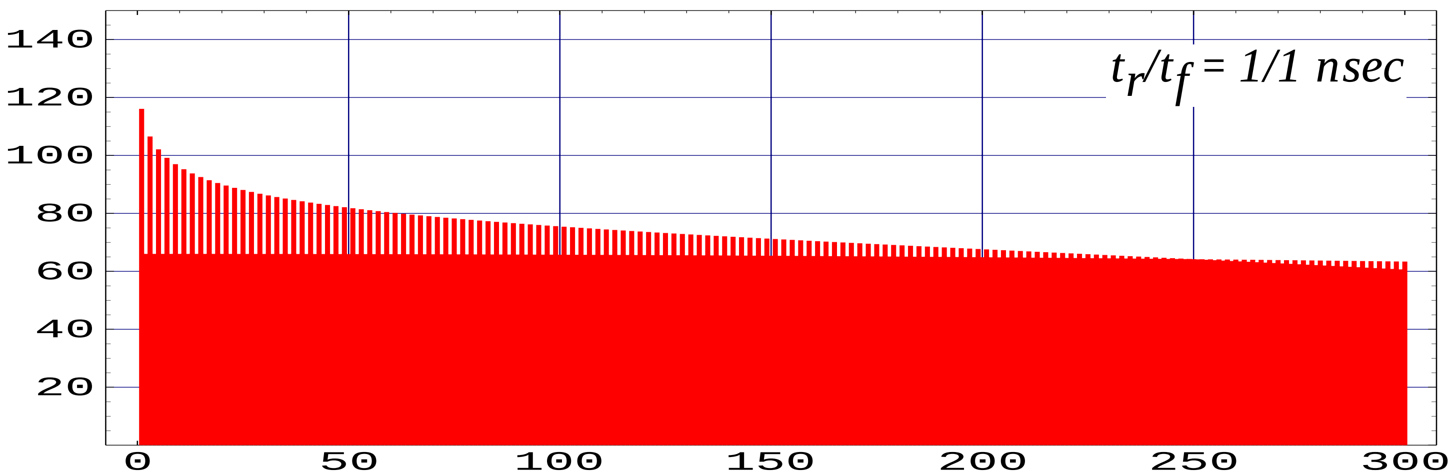
$$a_n = \frac{2AT}{n^2\pi^2 t_{rf}} \sin\left(\frac{n\pi t_{rf}}{T}\right) \sin\left(\frac{n\pi W}{T}\right)$$

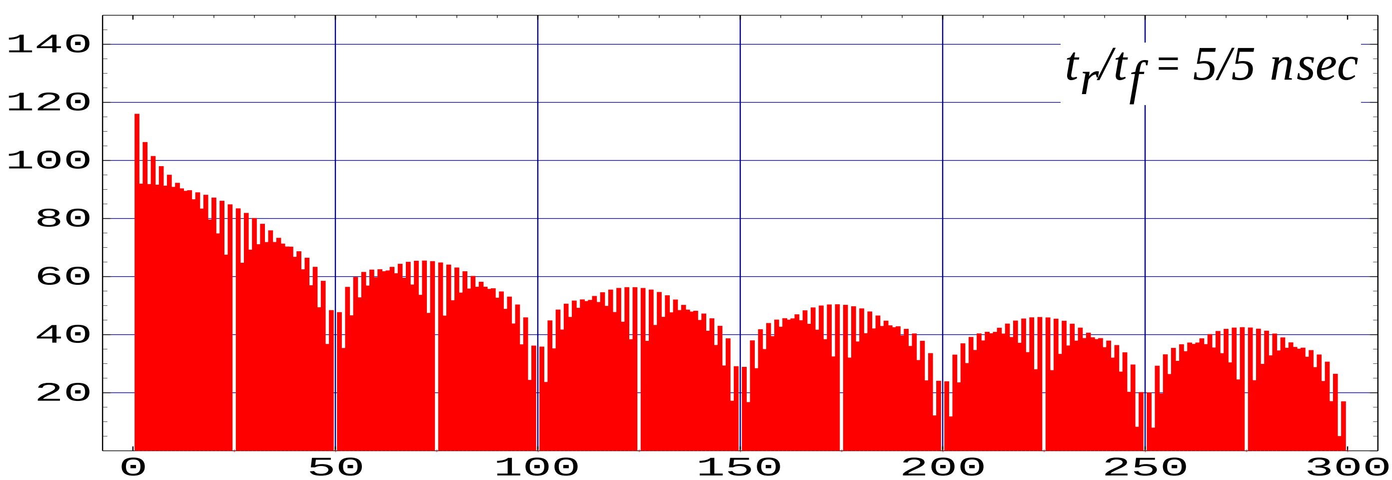
Freq (MHz)	**a_n (dBµV) t_{rf} = 0**	**a_n (dBµV) t_{rf} = 1 nsec**	**a_n (dBµV) t_{rf} = 5 nsec**
1	116	116	116
2	---	66	80
3	107	107	107
4	---	66	80
5	102	102	102
6	---	66	80
7	99	99	99
8	---	66	80
9	97	97	97
10	---	66	80
25	88	88	87
51	82	82	78
103	76	75	45
201	70	68	24
301	67	61	17

These go up while these go down

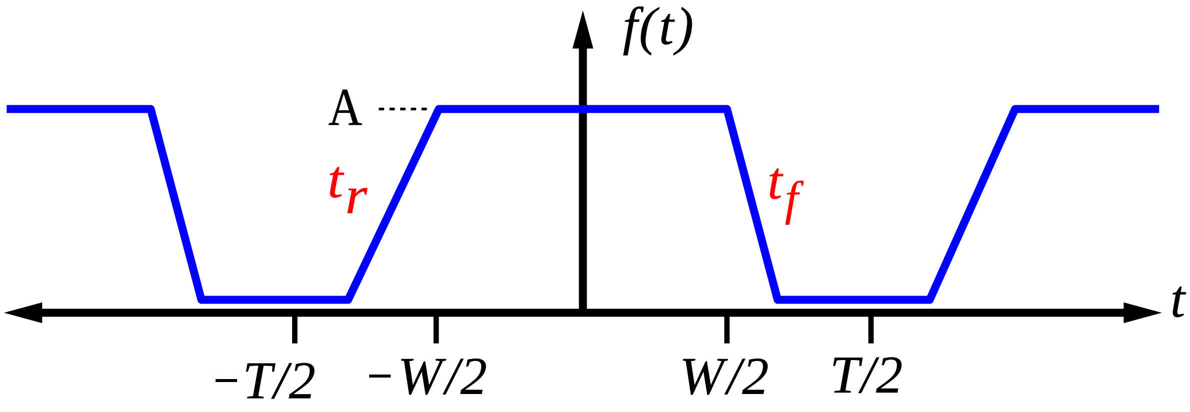
Symmetrical 1 MHz Trapezoidal Pulses







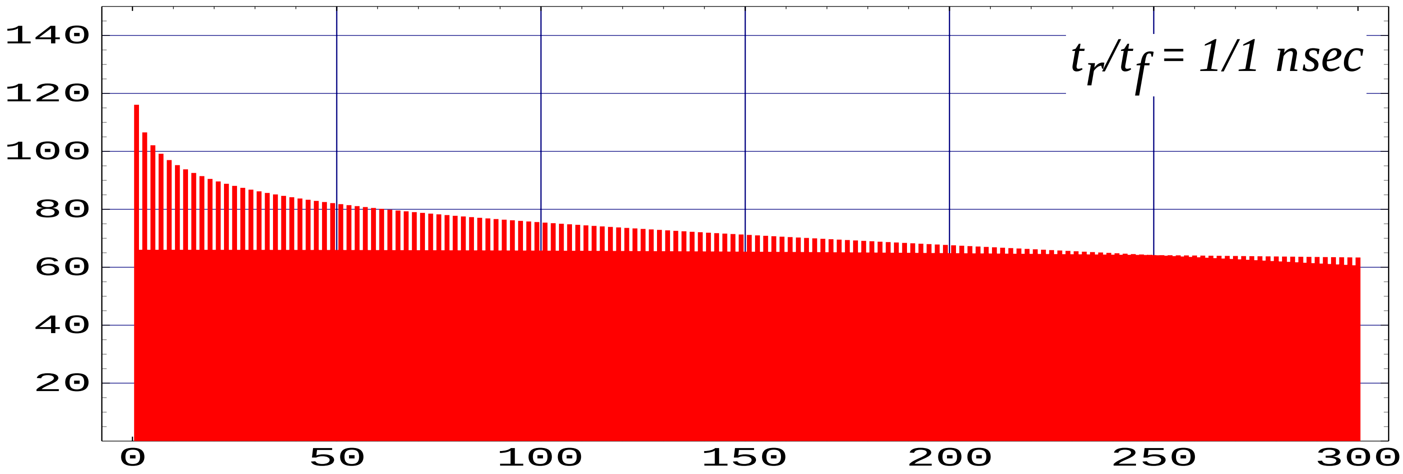
Asymmetrical 1 MHz Trapezoidal Pulses

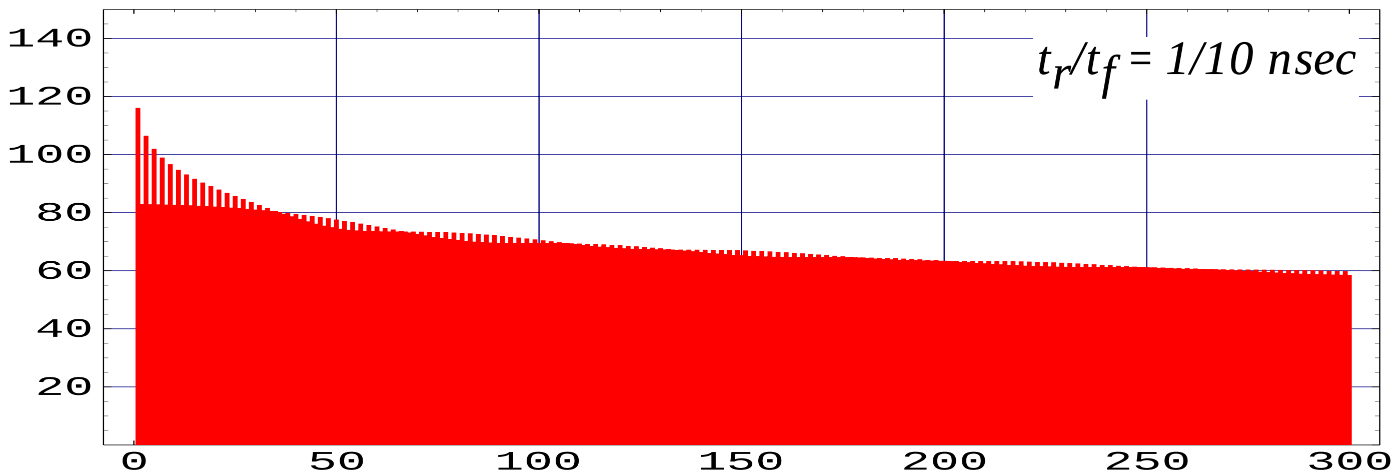


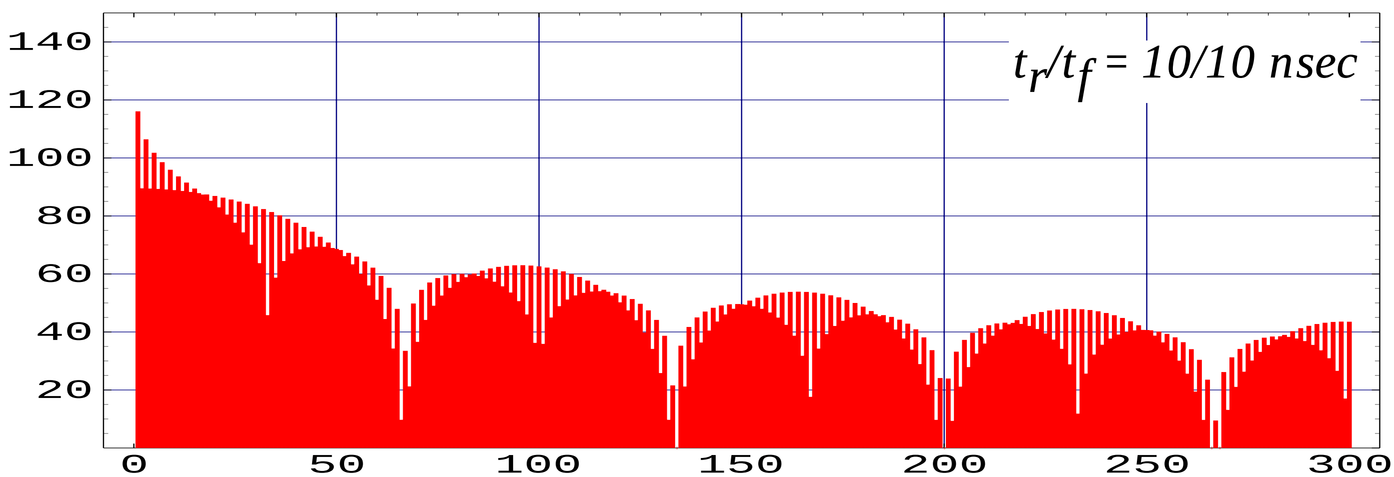
Freq (MHz)	Mag (dBμV) t_r/t_f = 1/1 ns	Mag (dBμV) t_r/t_f = 1/10 ns	Mag (dBμV) t_r/t_f = 10/10 ns
1	116	116	116
2	66	81	86
3	107	107	107
4	66	81	86
5	102	102	102
6	66	81	85.9
7	99	99	99
8	66	81	86
9	97	97	97
10	66	81	86
101	75	70	36
201	68	63	24
301	61	59	17

Only very high frequencies are substantially affected

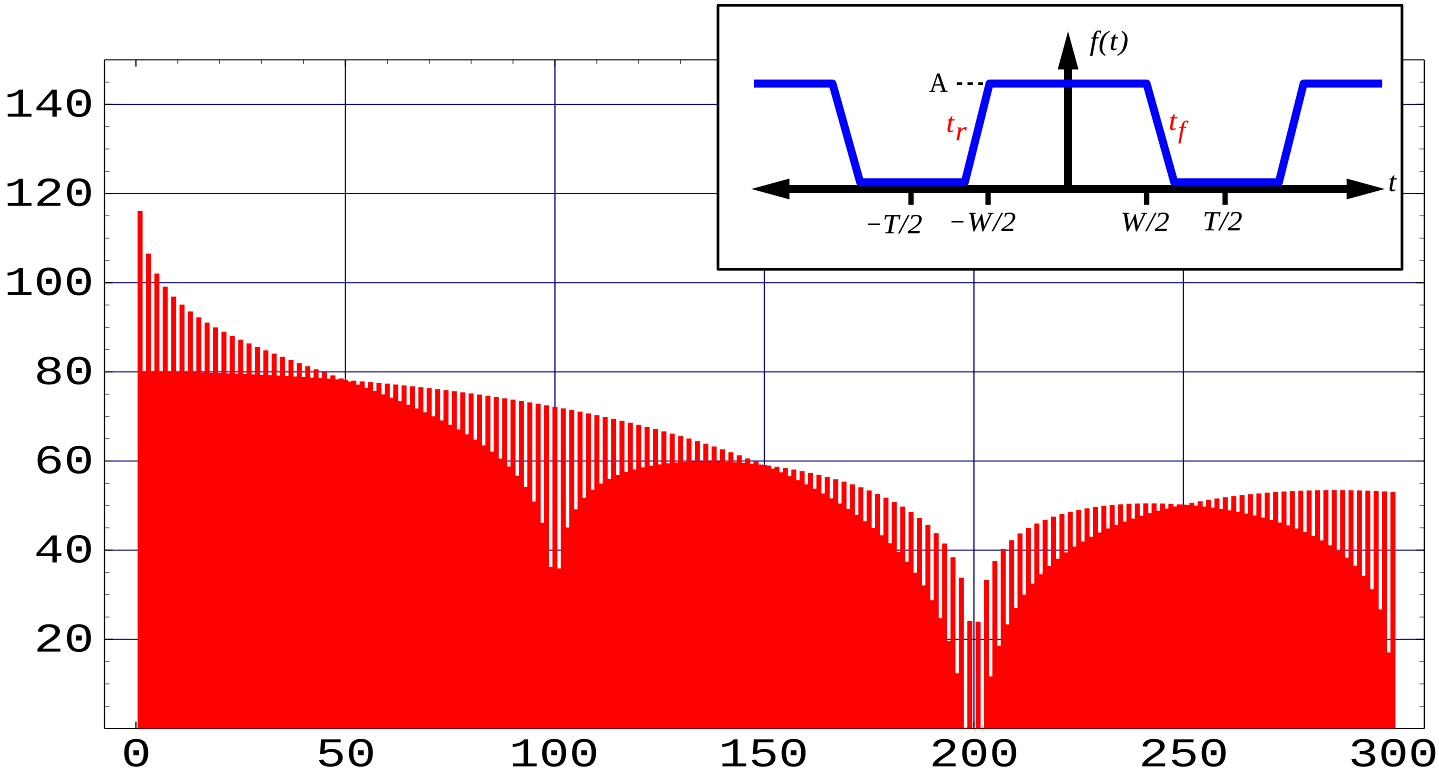
Asymmetrical 1 MHz Trapezoidal Pulses

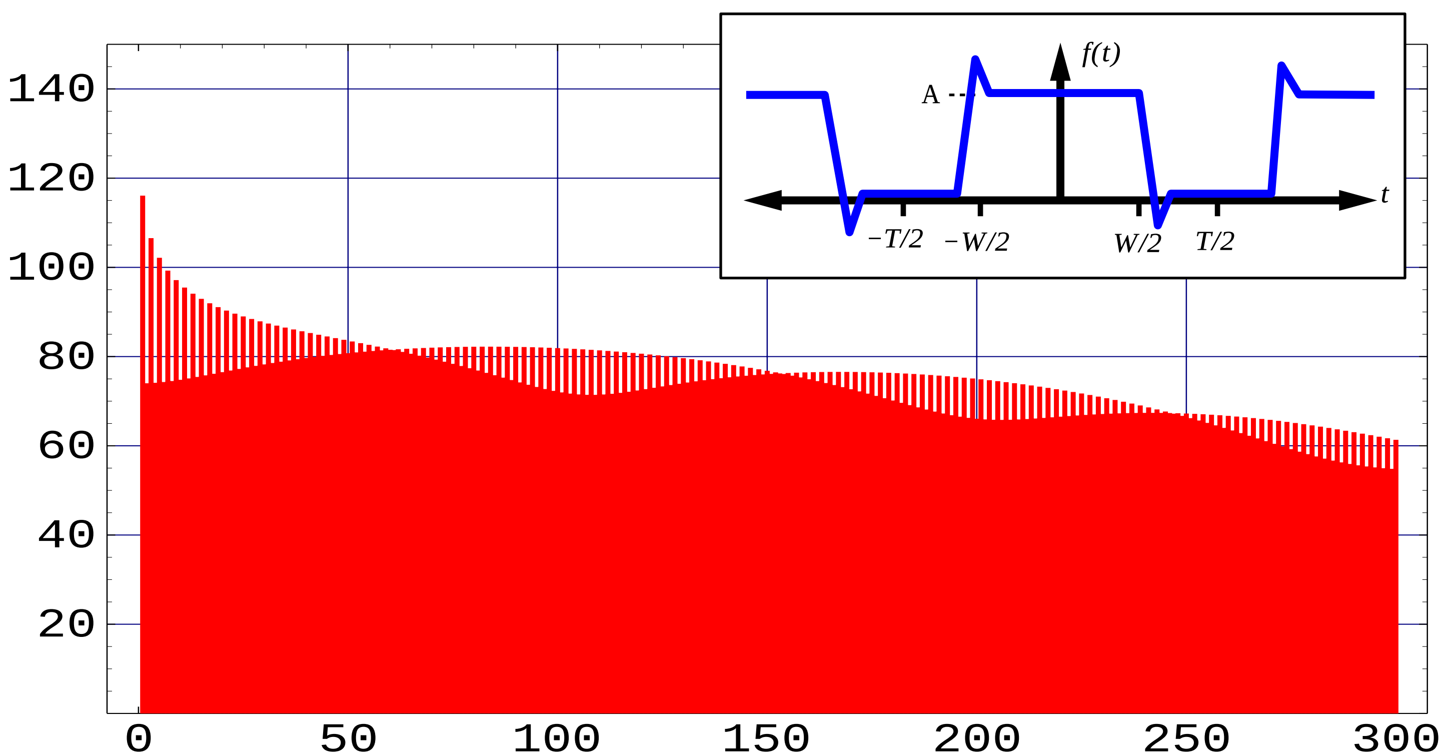




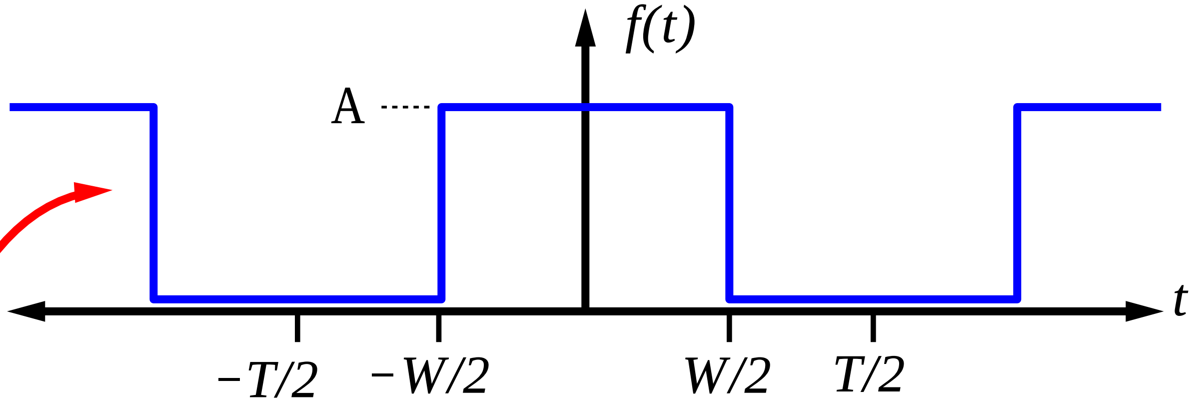


Trapezoidal Pulses With Over/Under Shoots





Rectangular Pulse



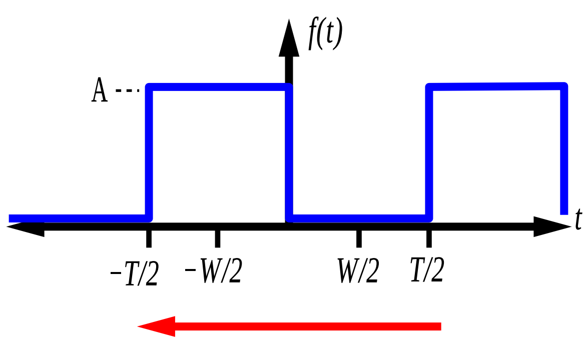
$$f(t) = \underbrace{AW/T}_{\text{DC Term}} + \sum_{n=1}^{\infty} \underbrace{2A/n\pi \ \sin(n\pi W/T)}_{\text{Harmonic Amplitude } a_n} \underbrace{\cos(2n\pi t/T)}_{\text{Frequency of Harmonic}}$$

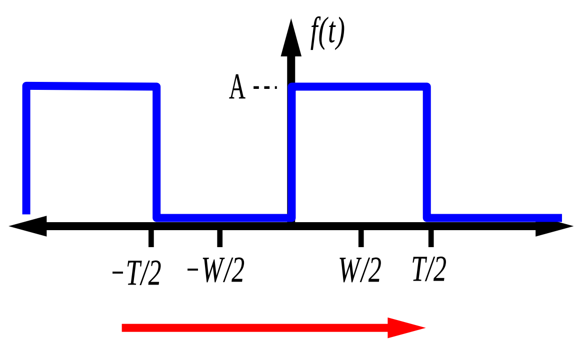
DC Term

Harmonic Amplitude $\mathbf{a_n}$

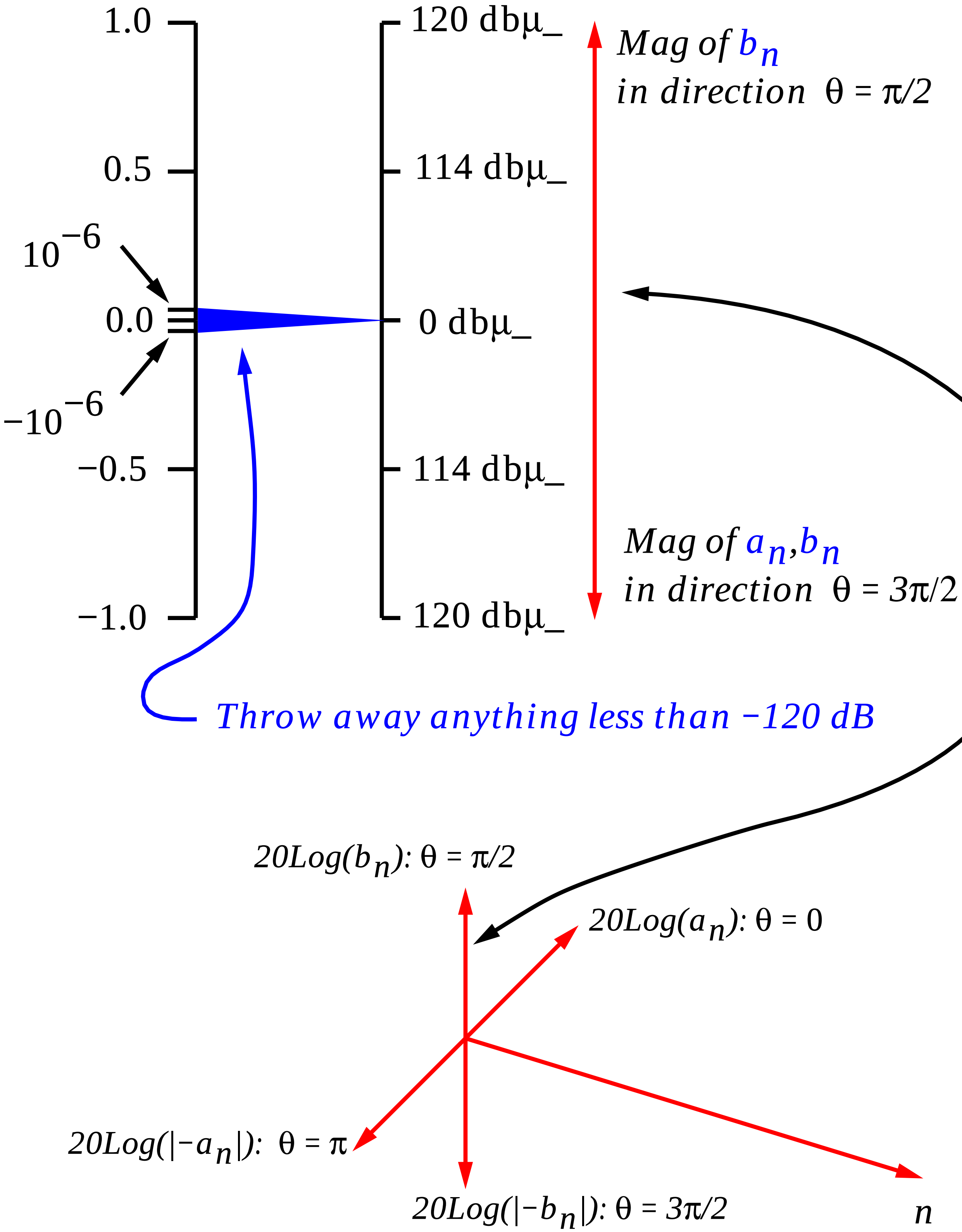
Frequency of Harmonic

What if the waveform was not centered about t = 0??





A Little Black Magic!!

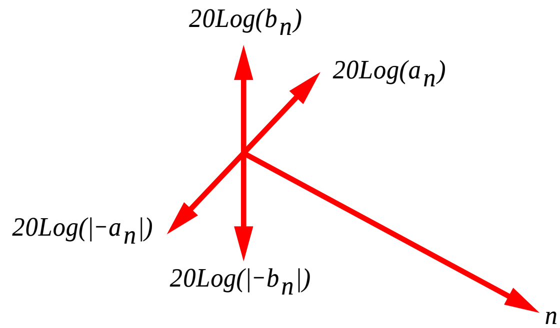


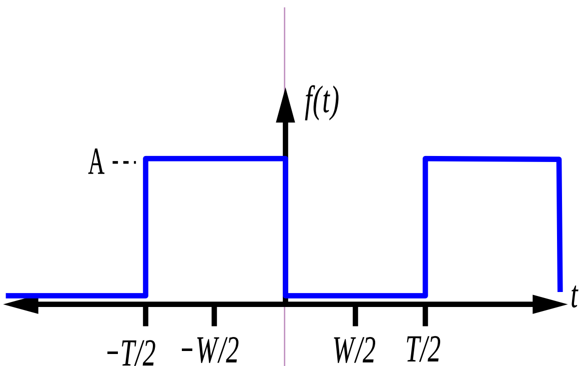
Offset Rectangular Pulse

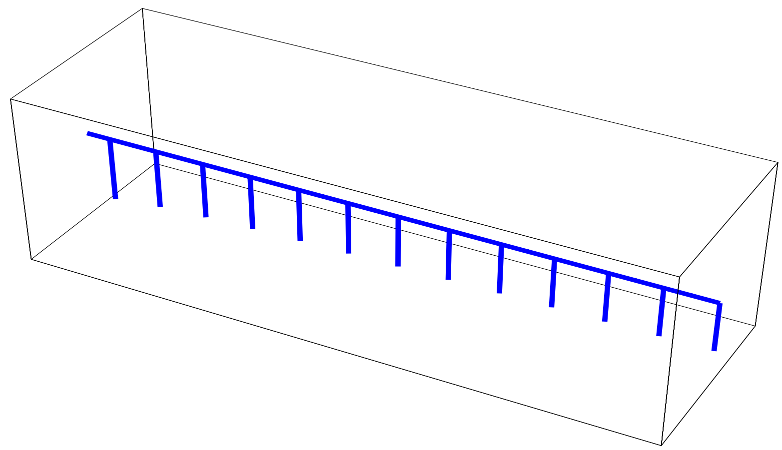
$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t/T) + b_n \sin(2n\pi t/T)$$

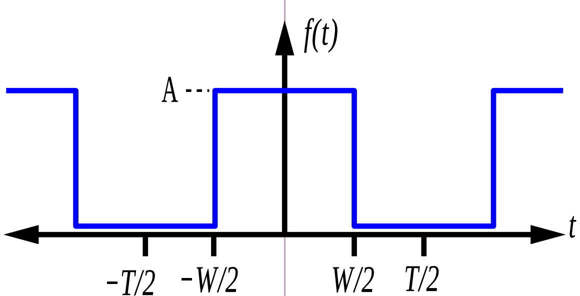
$$\sin(x) = \cos(x+\pi/2)$$

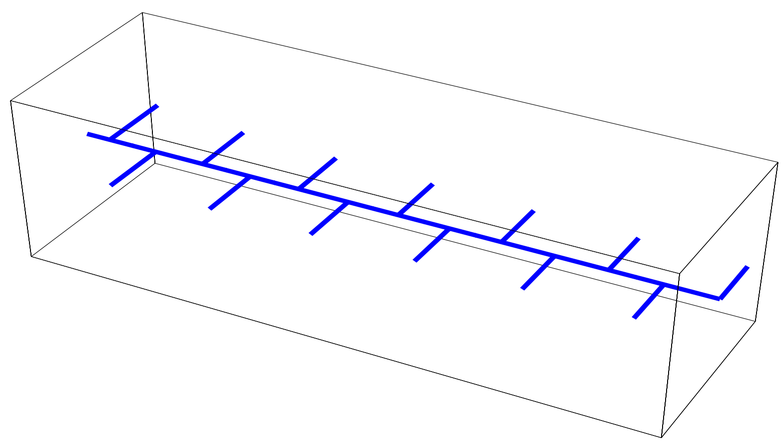
$$-\sin(x) = \sin(x+\pi)$$

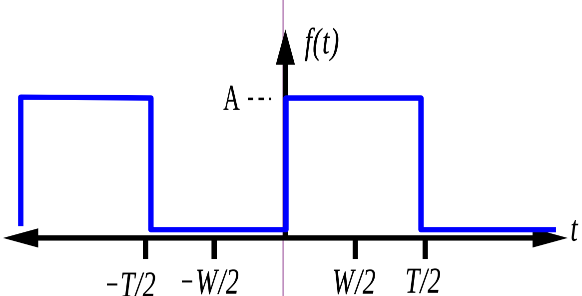


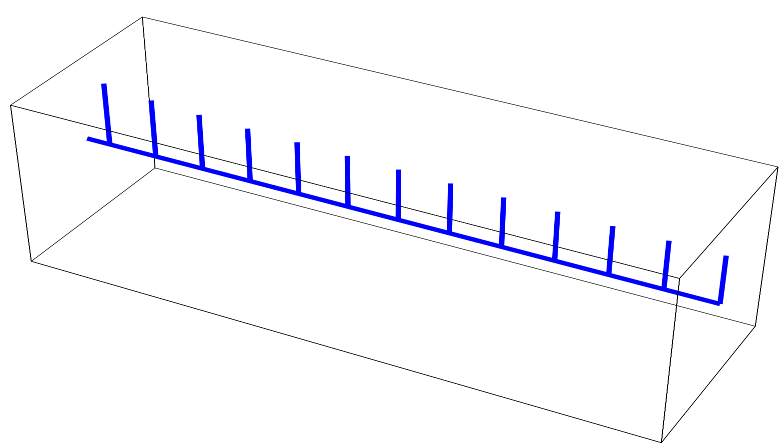






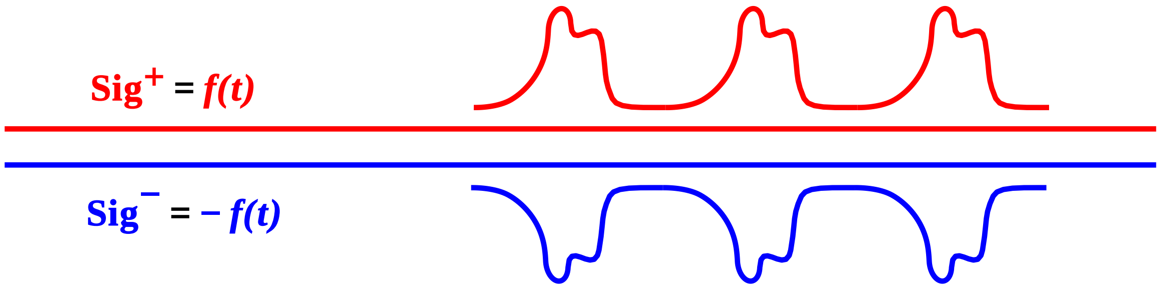






Harmonic Analysis of Differential Signals

Consider a differential signal pair [Sig$^+$, Sig$^-$] represented by the Fourier Series [$f(t)$, $-f(t)$] .



$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n \, e^{jn\omega_0 t} \qquad -f(t) = \sum_{n=-\infty}^{n=\infty} -c_n \, e^{jn\omega_0 t}$$

If we ignore for the moment the fact that the two conductors are not physically located in the same space, one can intuitively see that the two signals should "cancel each other out".

$$f(t) - f(t) = \sum_{n=-\infty}^{n=\infty} c_n \, e^{jn\omega_0 t} - c_n \, e^{jn\omega_0 t} = 0$$

Harmonic Analysis of Differential Signals

If, however, the Sig− signal is delayed in time by a small value, τ, (for example if Sig− is derived from Sig+ via an inverter, or the propagation time for Sig+ and Sig− is not the same) there will not be complete cancellation.

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} \qquad - f(t-\tau) = \sum_{n=-\infty}^{n=\infty} - c_n e^{jn\omega_0 (t-\tau)}$$

$$f(t) - f(t-\tau) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} - c_n e^{jn\omega_0 (t-\tau)}$$

$$= \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} - c_n e^{jn\omega_0 t} e^{-jn\omega_0 \tau}$$

$$= \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} (1 - e^{-jn\omega_0 \tau})$$

When $e^{-jn\omega_0 \tau} = 1$, Sig^{+}/Sig^{-} will cancel.

When $e^{-jn\omega_0 \tau} = -1$, Sig^{+}/Sig^{-} will be additive.

For other values of $e^{-jn\omega_0 \tau}$, Sig^{+}/Sig^{-} will partially cancel.

Harmonic Analysis of Differential Signals

$$1 - e^{-jn\omega_0\tau} = 1 - \cos(n\omega_0\tau) + j\sin(n\omega_0\tau)$$

$$= 1 - \cos(2n\pi f_0\tau) + j\sin(2n\pi f_0\tau) \qquad [\omega_0 = 2\pi f_0]$$

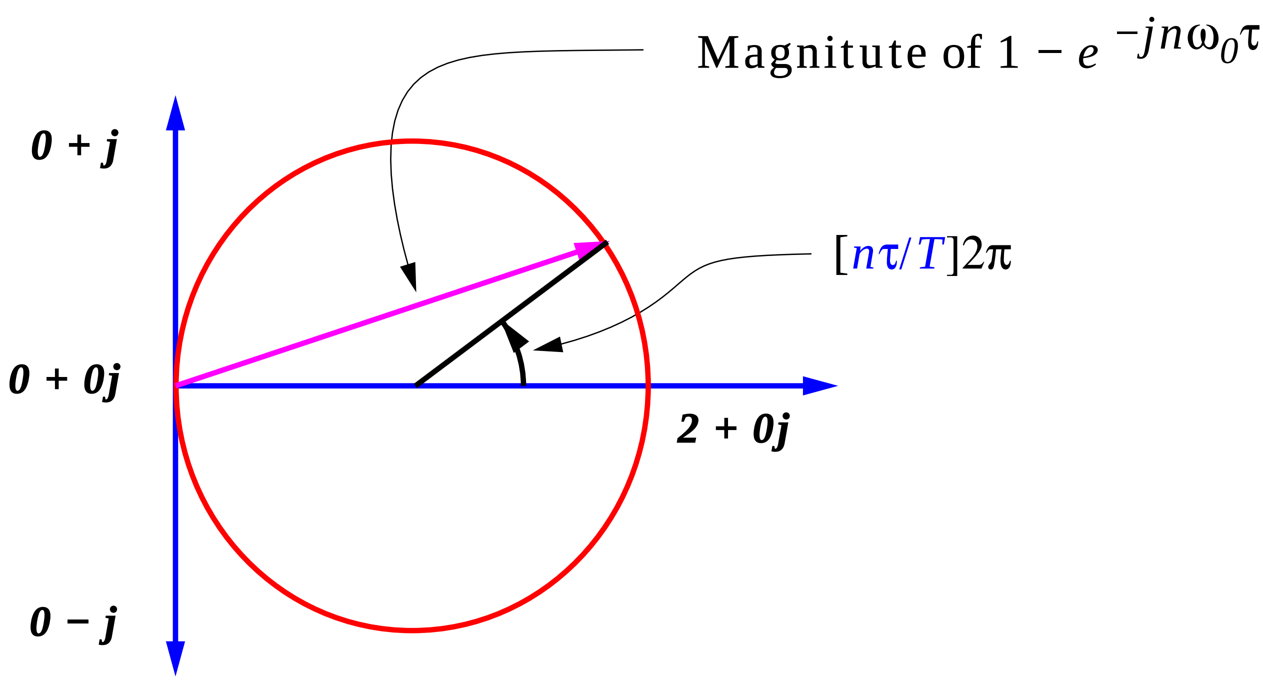
$$= 1 - \cos([n\tau/T]2\pi) + j\sin([n\tau/T]2\pi) \quad [f_0 = 1/T]$$

For $n\tau/T = 0, 1, 2, 3, ...$

$\cos([n\tau/T]2\pi) = 1$, $\sin([n\tau/T]2\pi) = 0$, $1 - e^{-jn\omega_0\tau} = 0$

For $n\tau/T = 1/2, 3/2, 5/2, ...$

$\cos([n\tau/T]2\pi) = -1$, $\sin([n\tau/T]2\pi) = 0$, $1 - e^{-jn\omega_0\tau} = 2$



Harmonic Analysis of Differential Signals

$$f(t) - f(t-\tau) = \sum_{n=-\infty}^{n=\infty} c_n\, e^{jn\omega_0 t} - c_n\, e^{jn\omega_0 (t-\tau)}$$

$$= \sum_{n=-\infty}^{n=\infty} c_n\, e^{jn\omega_0 t}\,(1 - e^{-jn\omega_0 \tau})$$

$n\omega_0\tau$	$\text{Mag}[1 - e^{-jn\omega_0\tau}]$	$20\log_{10}(\text{Mag}[1 - e^{-jn\omega_0\tau}])$
0°	0	$-\infty$
5°	0.087	−21.2 dB
10°	0.174	−15.2 dB
20°	0.347	−9.19 dB
30°	0.518	−5.71 dB
40°	0.684	−3.29 dB
60°	1.000	0.00 dB
120°	1.732	4.77 dB
180°	2.000	6.02 dB

If Sig^- shifts by 180 then Sig^- is in phase with Sig^+

Harmonic Analysis of Differential Signals

Freq (MHz)	T (nsec)	5° [−20 dB] (psec)	30° [−6 dB] (psec)	60° [0 dB] (psec)
100	10.00	138.9	833.3	1666.6
200	5.00	69.4	416.7	833.0
300	3.33	46.3	277.8	555.6
500	2.00	27.8	166.7	333.3
700	1.40	19.8	119.0	238.1
1000	1.00	13.9	83.3	166.7
1500	0.67	9.3	55.6	111.1
2000	0.50	6.9	41.7	83.3

Highest FCC measurement frequency if
108 MHz $< F_0 <$ 500 MHz

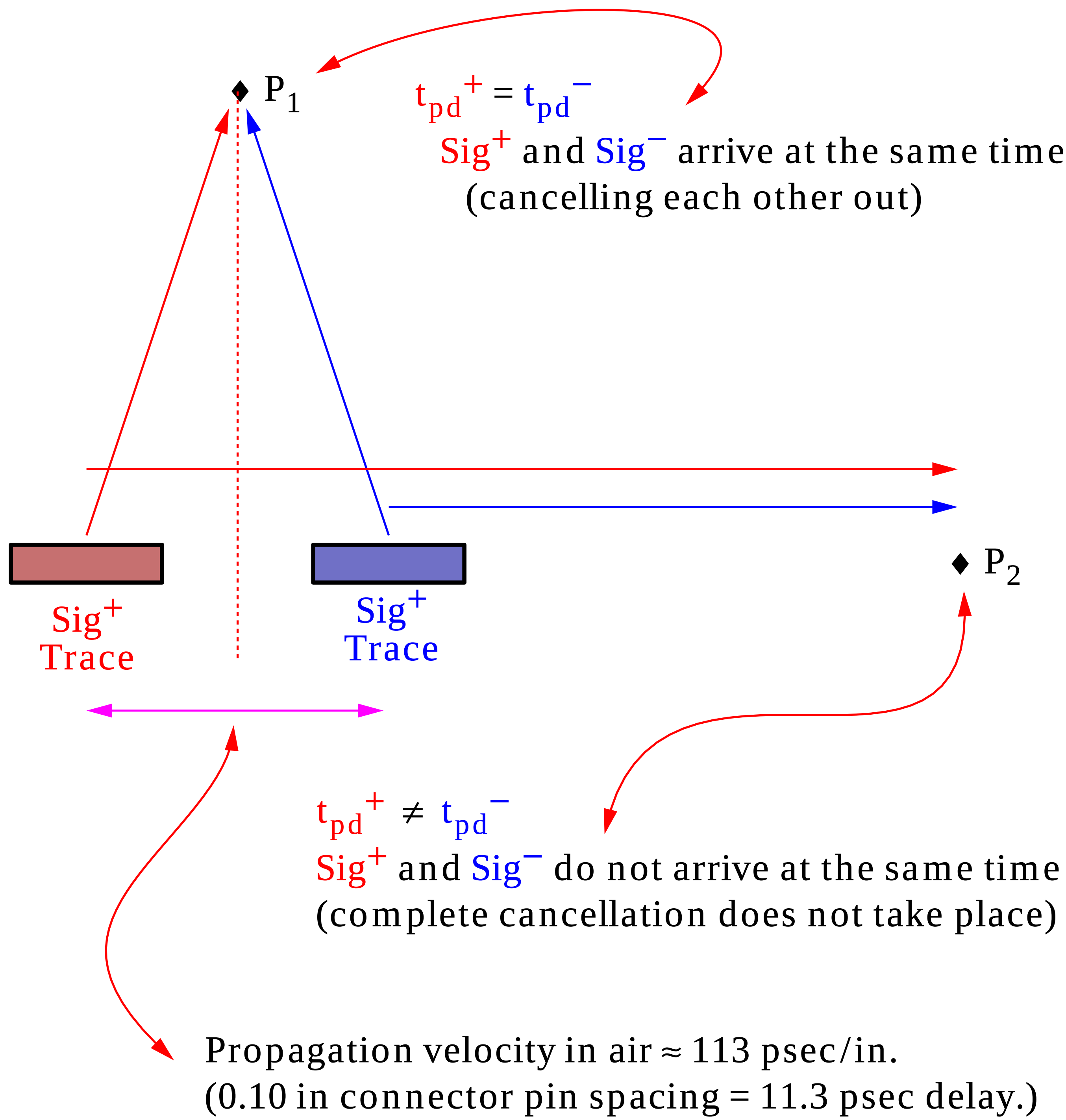
If the harmonic at 2000 MHz is delayed by more than 83.3 psec, then Sig^+ and Sig^- no longer cancel each other out. For typical PCB prop delay = 175 psec/in

6.9 psec ≈ 0.04 in (routing Δ's around connectors)

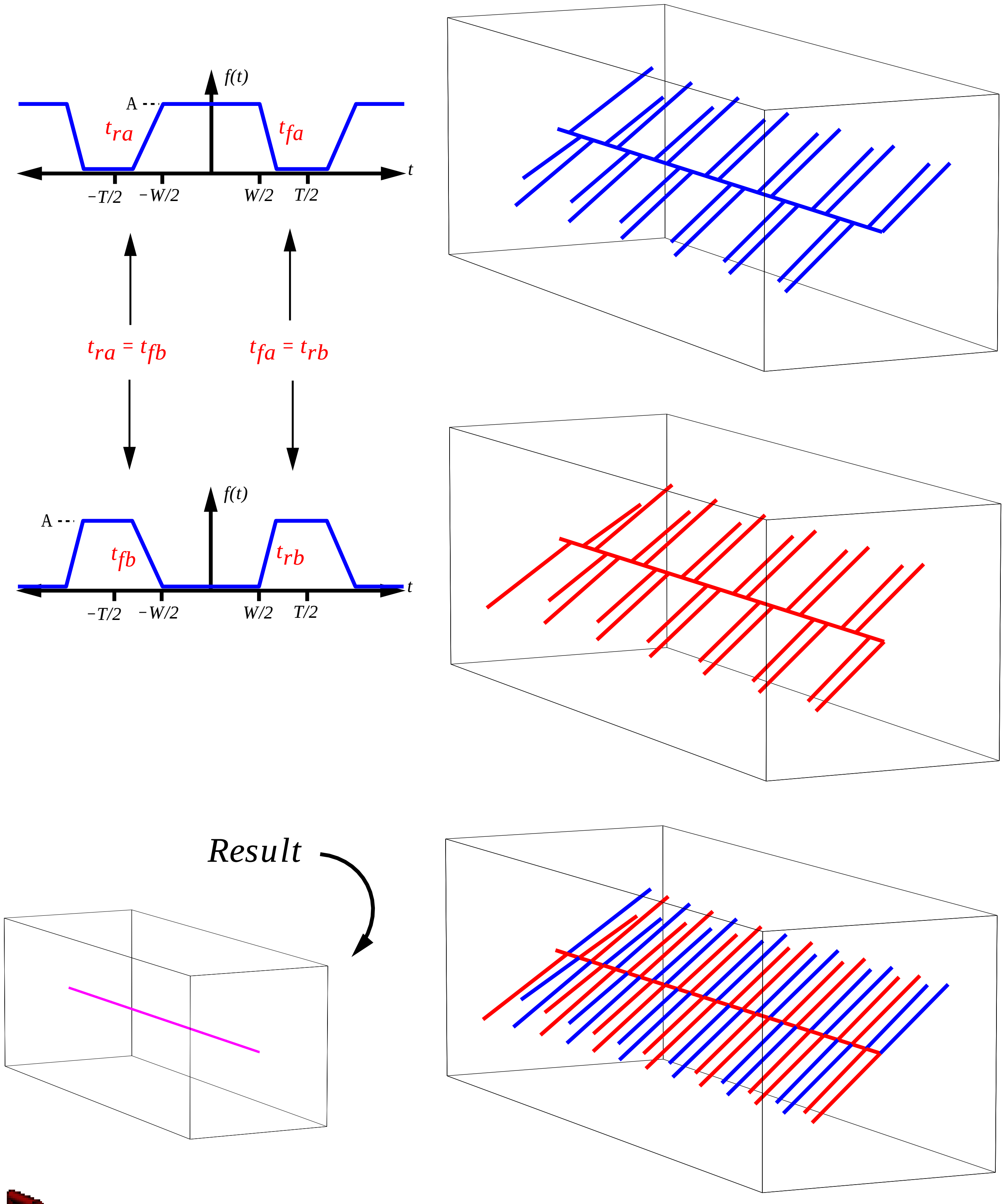
88.3 psec ≈ 0.5 in

5 nsec (inverter) ≈ 29 in

Harmonic Analysis of Differential Signals

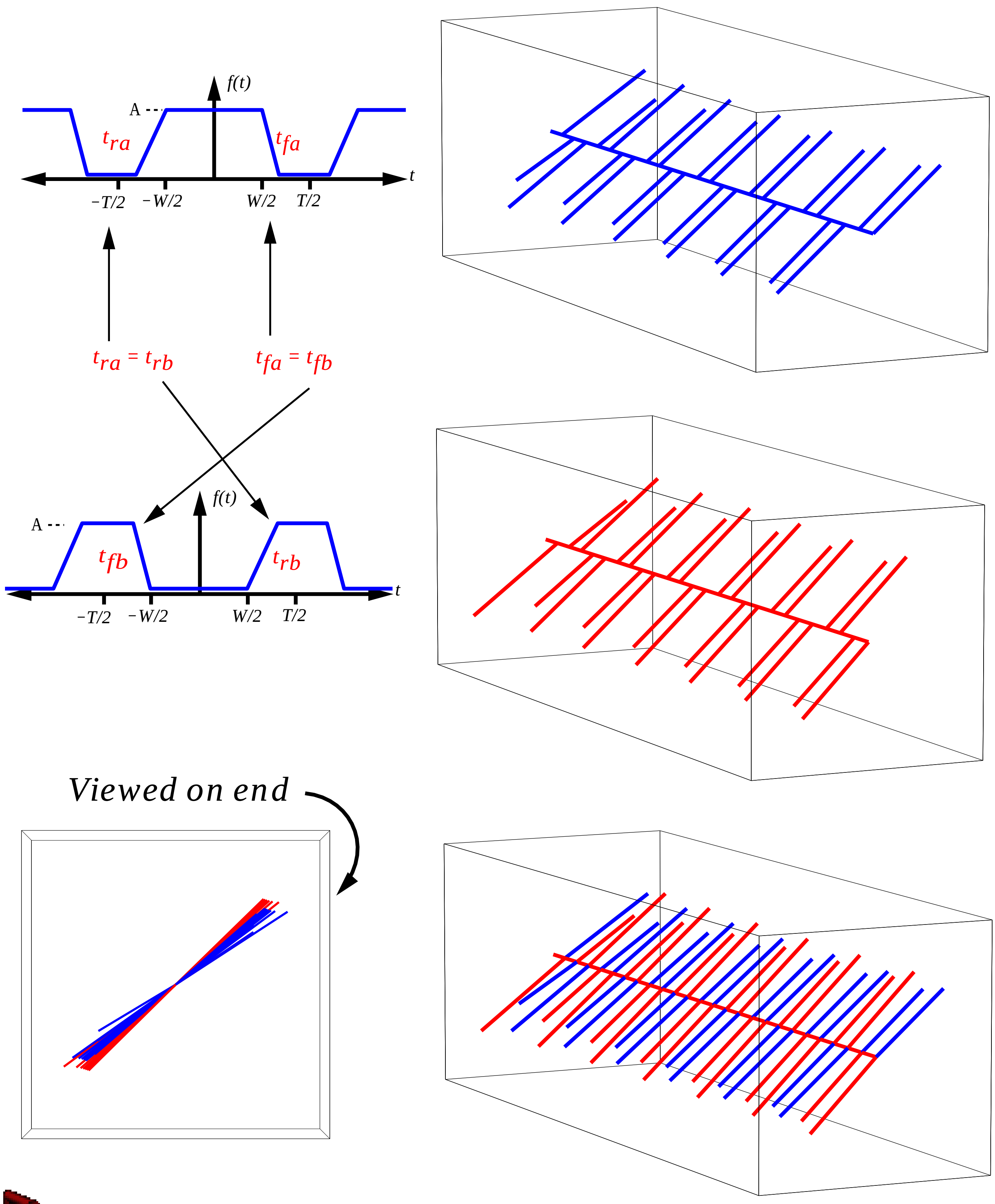


Asymmetrical 1 MHz Trapezoidal Pulses

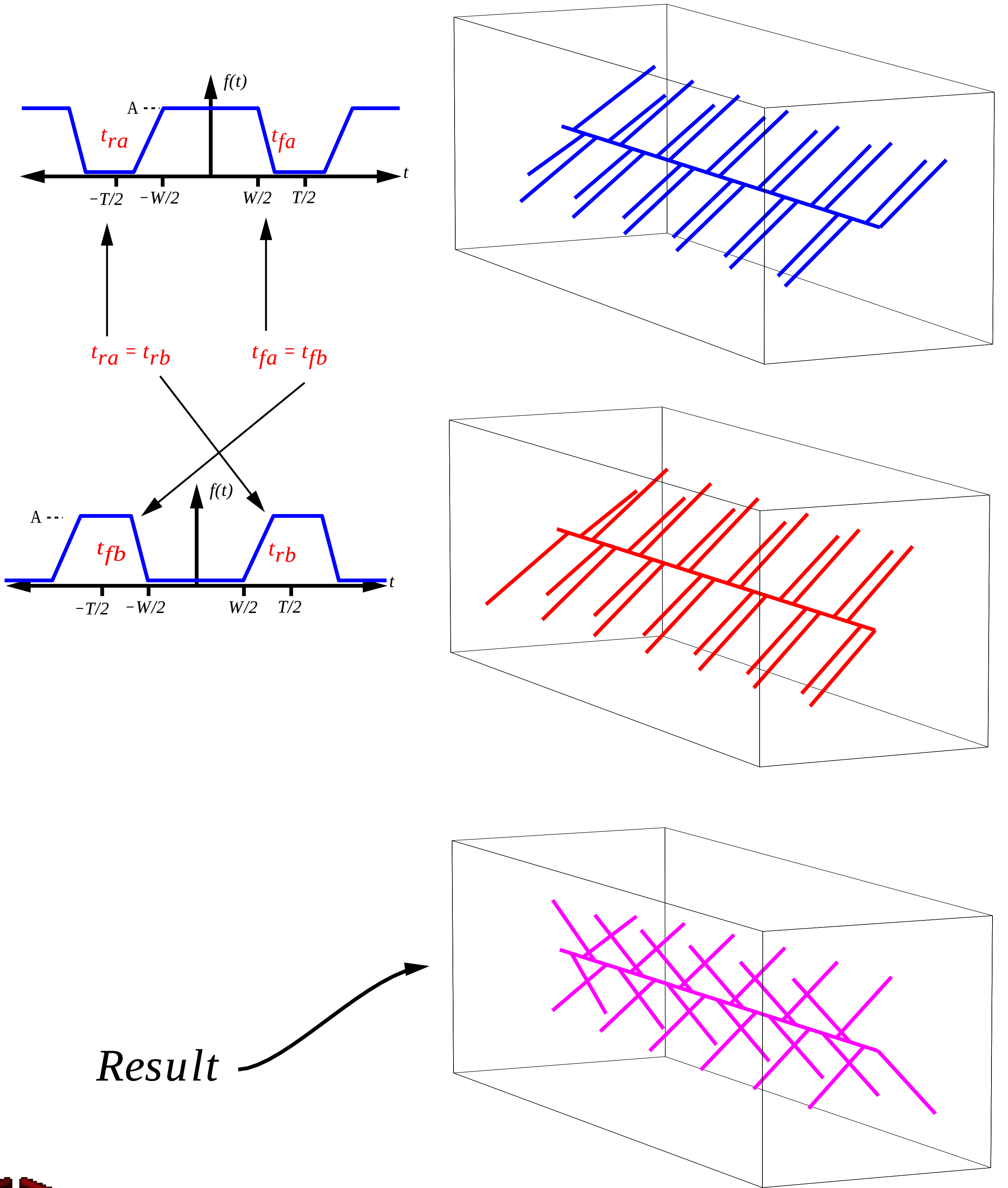


f = 1 MHz /T = 1 usec, tr = 2 nsec, tf = 8 nsec, A = 1,W = 0.5 nsec

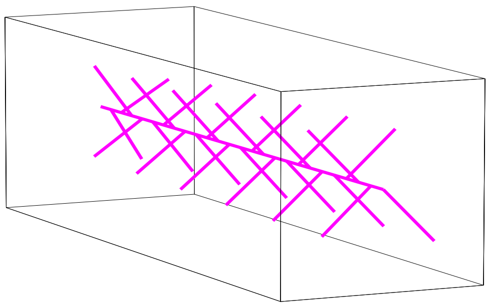
Asymmetrical 1 MHz Trapezoidal Pulses

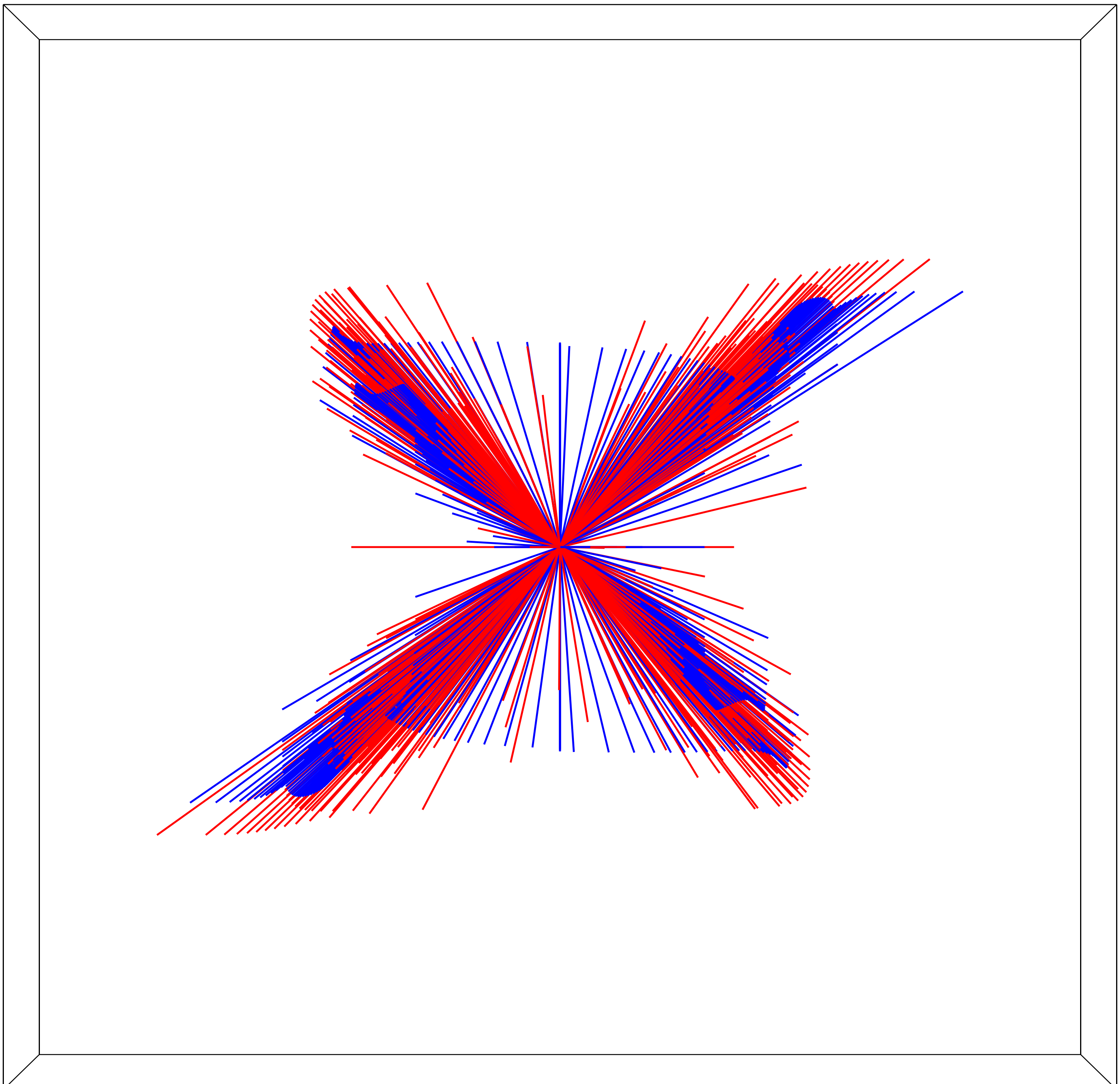


Asymmetrical 1 MHz Trapezoidal Pulses



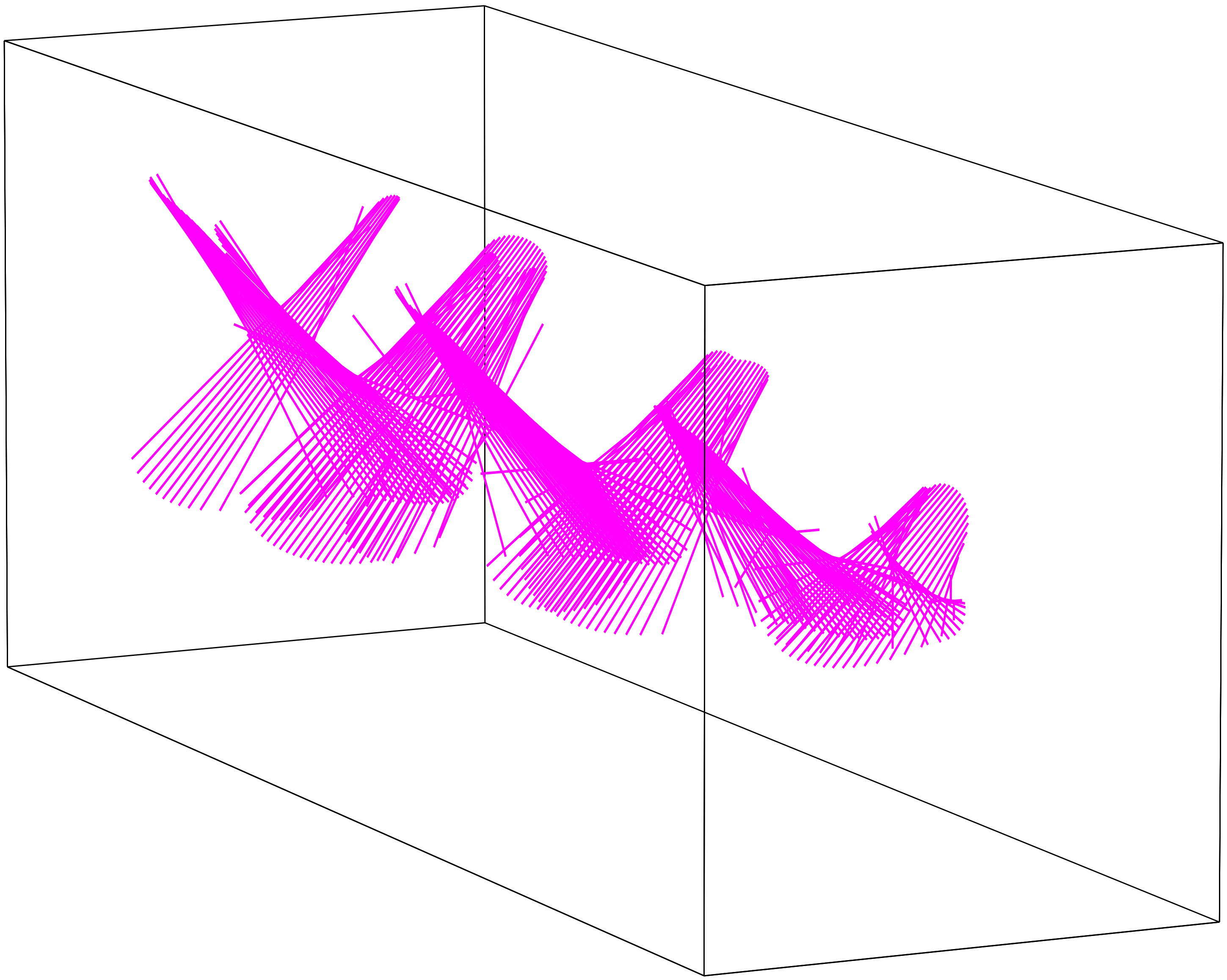
Asymmetrical 1 MHz Trapezoidal Pulses





Viewed on End (500 Harmonics) ......

Asymmetrical 1 MHz Trapezoidal Pulses



Resultant Levels (500 Harmonics) ......