Diversions

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Hilbert and Sierpinski space-filling curves, and beyond

More than a decade ago I was an active part of the AAMT mathematics education interest-group listserv. One of the messages posted to the listserv members said:

> I am a Year 9 student, and I was looking through my school's library and found a book called Algorithms & Data Structures. Since I am fond of computers I had a look. I turned to pages 141–147 and found the Hilbert and Sierpinski curves. I looked at the source code in Pascal and with my limited knowledge of it I translated them to BASIC. The results were spectacular. I was wondering, are there any links, source code, information or diagrams with any other forms of curves available. I am also interested in fractals. I saw my first fractal tree bloom on the Mac SE when I was about 5. I have seen many since then (in colour!) and I want to know how to replicate this effect.

Let me assume that what I said, as a reply to this Year 9 query, was helpful at the time. But let me also assume that this kind of question is perennial. Here is a version of what I said, years ago, offered in the hope it may still be of some wider, general interest.

Hilbert and Sierpinski curves are space-filling curves. They are related to fractals, in that they have self-similar patterns. That is, when we magnify one section of the fractal object, we find we are looking at a very similar version of the object, such as 1 kilometre length of coast line, measured in units of metres, for example, resembling the kind of wiggly coastal-boundariness of 100 kilometres, measured in units of 100 metres.

Such space-filling curves were originally developed as conceptual mathematical 'monsters', counter-examples to Weierstrassian and Reimannian treatments of calculus and continuity. (The mathematical philosopher Imre Lakatos explored the role of examples, and counter-examples, including 'monsters' to illustrate and test mathematical conjectures and proofs—like a kind of experimental scientific method; see Worrall & Zahar, 1976).

These were curves that were everywhere-connected but nowhere-differentiable (or some similar paradoxical combination of conditions): that is, there were no breaks in the curves, but they were so extremely and discontinuously wiggly that ordinary differentiation did not apply to them. Paradoxical little beasts, indeed! Moreover, they showed that a 'line'—specifically a 'curve', rather than a 'straight line'—could fill two-dimensional space! Remarkable!

As early as 1940, the great mathematics popularisers Kasner and Newman discussed the Koch snowflake, the anti-snowflake, and bizarre space-filling 'curves' as examples of what Kasner and Newman called "pathological" shapes. Pathological, because the two-dimensional snowflake curve, for example, is contained within a finite area but is itself infinitely long, while the three-dimensional counterpart is a space-filling curve that is infinitely long and completely fills a finite volume. Similarly, the Peano and Hilbert curves pass through every point of a finite area. Very pathological!

You might enjoy reading some of the great 'popularisers' of mathematics. Here are a few suggestions. (These days, Google, Wikipedia and other online possibilities are also good alternatives to standard paper-based book-resources!)

- Gleick, J. (1987). *Chaos: Making a new science*. New York: Viking.
- Julia, G. (1918). Mémoire sur l'iteration des fonctions rationelles. *Journal de Mathematiques Pure et Applied, 8*, 47–245.
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Both Ian Stewart (1992, chapter 17, pp. 240–242; and 1996, chapter 16, pp. 238–241) and James Gleick (1987, pp. 98–103) discuss the way chaos theory and fractals evolved from 'monster' examples of bizarre mathematical objects:

> curves that fill an entire square, curves that cross themselves at every point, curves of infinite length enclosing a finite area, curves with no length at all … in 1872 Weierstrass showed … a class of functions which are continuous everywhere but differentiable nowhere. (Stewart, 1992, p. 240)

Stewart also mentions the amazing, monstrous Cantor set, which is made by taking an interval and deleting the middle third, then deleting the middle third of the two remaining intervals, and so on, for ever. "The total length removed is equal to the length of the original interval; yet an uncountable infinity of points remain" (Stewart, 1992, p. 241). Gleick calls this remaining set "Cantor dust" (1987, p. 93).

Of course the Sierpinski triangular gasket and rectangular carpet, and the Menger sponge, which has infinite surface area and zero-volume, are two- and three-dimensional analogues of Cantor's remarkable 1-D set of points along the real number-line.

Let me emphasise that I am referring to 'dimension' in a strictly traditional Euclidean sense, not as a fractional (fractal) dimension of, say, 1.35 which describes the way a curve (which would otherwise have traditional Euclidean dimension of 1) wiggles so much that its dimension approaches 2, the traditional Euclidean dimension of an area.

The basic idea of a fractional dimension is this: a one-dimensional object, such as a line, or the arc of a circle, completely fills the 'surface' of a line or a circumference. Similarly, a two-dimensional object, such as a finite-area square or an infinite-area parabolic region, completely fills the two-dimensional surface of that object. Compare these two kinds, one-dimensional, and two-dimensional, with a 'tangled' or 'bumpy' kind of line, such as a coastline, or the contour of a cloud, or the cross-section of a cauliflower, or the threads of a river delta. In each of these cases, we have something which is not entirely two-dimensional, or does not entirely fill a two-dimensional plane, but which does occupy a *considerable portion* of a two-dimensional plane, and certainly occupies or fills far more of the two-dimensional surface than does a simple one-dimensional line or curve. In these cases, it makes sense to use a fractional dimension, a number such as 1.7 or 1.34, between the Euclidean integer dimension values of 1 and 2, to represent to space-filling dimensionality of these objects. Felix Hausdorff and A. S. Besicovich were the first two mathematicians to suggest such a fractional dimension (Stewart 1989, (rev. ed. 1997), pp. 205, 207).

Also, to explore more about fractals, try the 'logistic equation' $x_{next} = rx(1 - x)$, in which an initial input value of x is combined with some fixed constant r, and each subsequent output value x_{next} is used as the next input value of x, with the same r.

For example, if $x = 0.5$ and $r = 2.1$, then $x_{next} = 0.525$; then using this as the new value of x, the next value of x_{next} is 0.5236875.

James Gleick (1987, p. 80) quotes Robert May as saying: "The world would be a better place ... if every young student were given a pocket calculator and encouraged to play with the logistic difference equation", testing the different effects for different starting values of x and different coefficients r — "chaos should be taught", and it is "time to recognise that the standard education of a scientist [gives] the wrong impression" (see also Paulos, 1991, p. 36).

Using a graphic calculator or computer simulation of the logistic equation is better, and practicable at secondary level. The original Julia set, for example, used by Mandelbrot, is based on iteration with complex numbers, possibly a Year 11 or 12 topic.

A complex number z (or $a + ib$) is a member of a Julia set, for a given real number c, if z is the first member of a sequence, defined by an iterative function, which remains bounded.

If the sequence created by the iteration diverges indefinitely ("approaching $\pm \infty$"), then z is not in the set.

The sequence is defined by the function $z_{output} = z_{input}^2 + c$, substituting z for the first input, and iteratively using each z_{output} as the new value for z_{input}.

Plotting each z in a Julia set on an Argand diagram—plotting (a, b) on ordinary Cartesian axes—constructs a fractal object.

This is ideal computer simulation and plotting material!

Recursive and iterative processes exist as curriculum topics on the borderline between mathematics, logic and computer science at upper secondary and undergraduate tertiary levels—or for any interested and able secondary student!

When I first drafted my reply to the listserv query, I had not found any secondary-level curriculum unit for fractals which did anything more substantial than look at a few intriguing fractals. Nor, at the time, did any book or curriculum document connect fractals and chaos with the existing curriculum core in any coherent way, although Fritjof Capra (1996, chapter 6) attempts to discuss the connections between calculus and the investigation of linear systems and the non-linear focus of chaos theory.

Peitgen, Jürgens, and Saupe (mentioned above) is an excellent almost-school-level book about chaos and fractals, with much programming; also two of the pioneering originals are Julia, and Mandelbrot.

Finally, Logo (or a dialect) is an excellent programming language for investigating chaos and fractals, because Logo is purpose-built to handle recursion.

References

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From Helen Prochazka's

Scrapbook

Hunting the Hidden Dimension

In the television program "Hunting the Hidden Dimension", Benoit Mandelbrot talks about the beginning of his fascination with the visual side of mathematics.

"It is only in January, '44, that, suddenly, I fell in love with mathematics. And not mathematics in general. With geometry in its most concrete, sensual form - that part of geometry in which mathematics and the eye meet. The professor was talking about algebra. But I began to see, in my mind, geometric pictures which fitted this algebra. And once you see these pictures, the answers become obvious. So I discovered something which I had no clue before. That I knew how to transform, in my mind, instantly, the formulas into pictures."