ENCOURAGING MEANINGFUL ENGAGEMENT with pictorial patterning tasks

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Introduction

Pattern generalisation has become an important feature of mathematics classrooms around the globe. Sometimes these activities focus purely on given numerical terms, but the use of pictorial or figural patterns is now becoming part of the standard repertoire for such generalisation exercises. From a pedagogic point of view, the investigation of pictorial patterns potentially allows for a meaningful way of arriving at and exploring algebraically equivalent expressions of generality.

A typical approach to presenting such a patterning task is shown in Figure 1. Students are generally required to determine (a) the number of dots in the next few terms, (b) the number of dots in one or two terms further along in the sequence, for example the 10th or 50th terms, and (c) an expression for the number of dots in the *n*th term, i.e., an algebraic expression of generality.

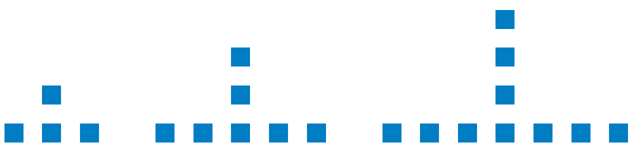

Figure 1. A typical pictorial pattern generalisation activity.

However, the presentation of consecutive terms often results in such potentially powerful activities becoming reduced to nothing more than rote exercises in which the numerical values of the terms are divorced from the figural structures that gave rise to them. Once the pictorial context has been reduced to a sequence of numerical terms, the general rule can then readily be determined by using any number of standard algorithmic approaches.

The danger with such an approach is that the focus becomes "the development of an algebraic relationship, rather than the development of a sense of generality" (Thornton, 2001, p. 252). Indeed, as Hewitt (1992, p. 7) succinctly remarks, the problem with divorcing patterns of numbers from their original context is that any generalised statements become "statements about the results rather than the mathematical situation from which they came". Such disconnected algebraic formulation neither illuminates the problem nor provides a means for validating the generated functional relationship (Noss, Healy & Hoyles, 1997). This becomes particularly problematic in situations where the justification of the general rule assumes significance (Byatt, 1994). The ability to justify a general formula is by no means commensurate with a student's proficiency in deriving such a generalisation.

One way of attempting to encourage visual as opposed to numeric approaches to pattern generalisation activities is to present the pictorial scenario by means of two non-consecutive terms (see Figure 2). This is supported by the research literature (e.g., Healy & Hoyles, 1996; Hershkowitz et al., 2002; Samson, 2007) which suggests that non-consecutive terms would be more appropriate in terms of encouraging attention to be focused on the visual stimulus.

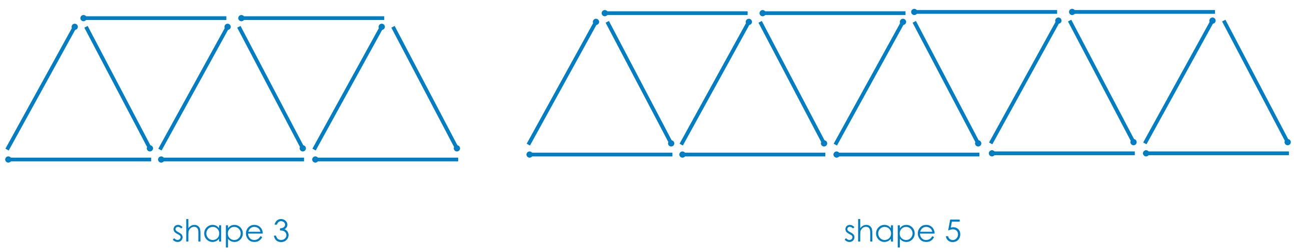

Figure 2. Two non-consecutive terms.

Although the presentation of generalisation tasks is an important aspect in terms of how students are likely to engage with such exercises, this is not the only consideration to be taken into account. Teachers still need a toolbox of pedagogical strategies which they can draw on to encourage visual engagement with the pictorial context. What this article contains is such a toolbox of strategies.

Strategy 1:
Encourage engagement with the pictorial context

Firstly, encourage students consciously to engage with the pictorial terms by using the following strategies:

- Search for structural features that contain as many elements as the term number (n), or that occur as many times as the term number. By way of example, the pictorial terms shown in Figure 2 could be seen to contain, amongst other things, n upward pointing triangles.
- Apply the above strategy more generally by searching for features or structural units that contain *nearly* as many elements as the term number itself (e.g., $n \pm 1$ or $n \pm 2$) or that occur nearly as many times as the term number. By way of example, and referring to Figure 2 once again, the top row of horizontal matches contains $(n - 1)$ matches. Alternatively, one could focus on the $(n - 1)$ *downward* pointing triangles.
- Identify elements of symmetry such as left–right equivalence or symmetrical structures radiating out from a central point, for example the three arms radiating out from a central dot in Figure 1.
- Identify visually striking geometrical features that could be used as structural keystones for particular apprehensions. These features could

either be recurring elements in the diagrams or solitary items. By way of example, consider the two diagrams shown in Figure 2. If we think of the leftmost and rightmost pairs of matches as supporting 'brackets' or 'braces', then we could potentially achieve a structural reorganisation which foregrounds the downward pointing triangles. This has the potential to lead to the general expression $T_n = 3(n-1) + 4 + (n-2)$, where the $3(n-1)$ represents the three matches required for each of the $n-1$ downward pointing triangles, the 4 represents the constant two pairs of matches at each end acting as 'brackets' or some similar structural support, while the $(n-2)$ represents the remaining $n-2$ horizontal matches positioned between the points of the downward pointing triangles. This scenario is demonstrated for Term 4 in Figure 3.

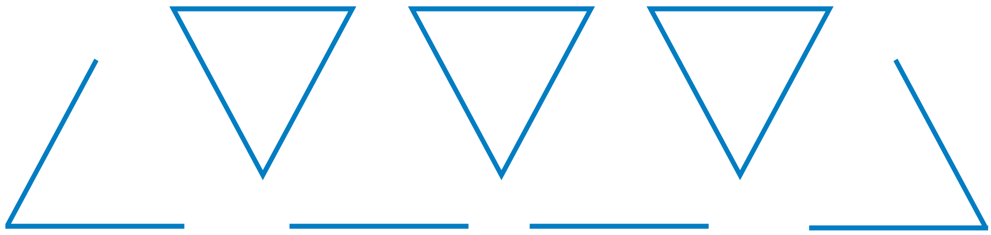

Figure 3. Term 4 visualised as $T_n = 3(n-1) + 4 + (n-2)$.

In addition, students should guard against the pitfalls of single-case concreteness. Within the context of pattern generalisation the crux of the enterprise lies in an evolving sense of generality. Prolonged focus on a single pictorial term may well act against this central endeavour. Students should thus be encouraged to look for commonalities between different terms, preferably non-consecutive terms since this is more likely to occasion a more holistic structural perception where attention is not necessarily focused on the additive unit.

Strategy 2: Look for regularities within even-numbered or odd-numbered terms

Encourage students to look for comparative regularities between only even-numbered or odd-numbered terms. Unexpected visual commonalities may be perceived in this manner that could serve as crucial triggers to occasion the evolution of new general formulae. Although the general formula thus determined may not necessarily make visual sense with respect to all the terms in the pictorial sequence, they would still be algebraically correct. This observation in itself could open up interesting classroom discussion. By way of example, consider the two terms shown in Figure 4.

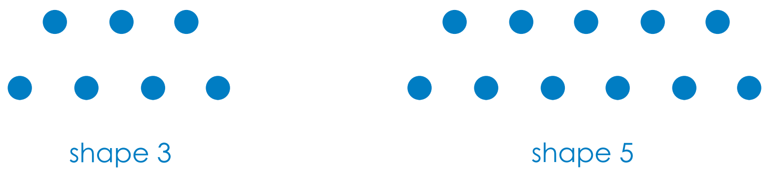

Figure 4. Two non-consecutive terms.

There are any number of equivalent algebraic expressions for the general term of the sequence. One could, for instance, argue that the top and bottom horizontal rows of dots contain n and $(n+1)$ dots respectively, thus arriving at the general expression $T_n = n + (n + 1)$. But one could also identify other structural elements such as upward or downward pointing triangles, zigzag shapes, oblique pairs of dots, or even parallelograms. One particular visual subdivision would be in terms of non-overlapping upward pointing triangles, as shown in Figure 5.

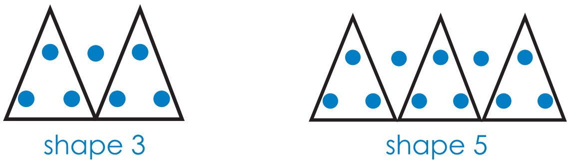

Figure 5. Subdivision of T_3 and T_5 into non-overlapping triangles.

By investigating this scenario with other terms one could potentially arrive at the realisation that, in terms of this particular figural visualisation, there is a structural difference between odd-numbered and even-numbered terms. One may thus begin to notice commonalities *within* these two sub-groups (Figure 6).

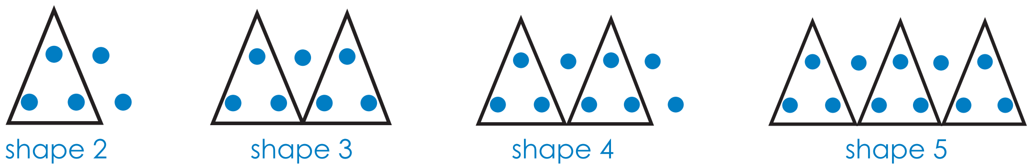

Figure 6. Subdivision of T_2, T_3, T_4 and T_5 into non-overlapping triangles.

One could then potentially make the observation, *focusing only on the odd-numbered terms*, that the number of triangles plus the number of the remaining dots gives the Shape number, n the independent variable, in each respective case. One could then possibly be able to express the number of triangles in each shape as

$$\frac{n+1}{2}$$

the number of remaining dots as ,

$$n-\left(\frac{n+1}{2}\right)$$

and hence the total number of dots in the nth term as .

$$\left(\frac{n+1}{2}\right)\times 3+n-\left(\frac{n+1}{2}\right)$$

In this particular case it would be very difficult to transfer the visual reasoning that inspired this formula onto even-numbered pictorial terms since the number of triangles in each shape

$$\left(\frac{n+1}{2}\right)$$

would be a fraction. However, since one can show that the expression simplifies to $2n + 1$, it will nonetheless provide the correct numerical answer for even-numbered terms. Furthermore, by focusing on only even-numbered terms with a similar sort of reasoning, one could potentially arrive at the following general formula:

$$T_n = \left(\frac{n}{2}\right) \times 3 + n - \left(\frac{n}{2}\right) + 1$$

Thus, for certain pictorial patterns, a pedagogical strategy of focusing on only even- or odd-numbered terms may not only be useful with respect to the generalisation process itself, but also in terms of its potential educational value.

Strategy 3: Towards an expression of generality

What we have focused on thus far relates to a visual engagement with the pictorial context. However, there is often a big leap from seeing a structural regularity, understanding the generality of that regularity, and finally arriving at a representative algebraic expression. Where students are able to describe perceived visual regularities but are unable to express this regularity in an algebraically useful manner, the following strategies may be useful:

- Tabulate a summary of structural features along with the total number of occurrences of each structural feature for specific terms.
- Make use of (i.e., draw or construct) pictorial terms further along in the sequence (e.g., $n \geq 6$) to search for structural regularities. Larger terms often act as more efficient triggers than smaller terms.
- Investigate Term 1. There are often structural anomalies or subtle differences with smaller terms that may well trigger structural understanding.

When teachers present pictorial patterns to the class they should take care not to make use of diagrams in which the term number also represents the number of elements in structural features that are likely to be brought forth by students. So, for instance, if there is a likelihood of students focusing on squares in a particular sequence, avoid Term 4. Similarly avoid Term 3 if it contains triangular structures (or any other potential three-unit features) that could act as triggers. This should help avoid confusion arising from situations where the same numerical value represents different conceptual aspects of the given pictorial term. By way of example, consider the two terms shown in Figure 7. If we focus on Shape 3 then we could calculate the number of dots in Term 3 as follows: $T_3 = 3 \times 3 - 3$ based on the reasoning that the triangle of dots has 3 sides, each of which contains three dots, but since the three corner dots overlap (and as such have been counted twice) we need to subtract 3 from the final tally. The choice of Term 3 as a generic reference point could potentially lead to confusion in terms of understanding the generality of the visualisation/calculation. Not only does 3 represent the Term number as well as the constant number of dots that need to be subtracted, but crucially it is also the number of elements in the

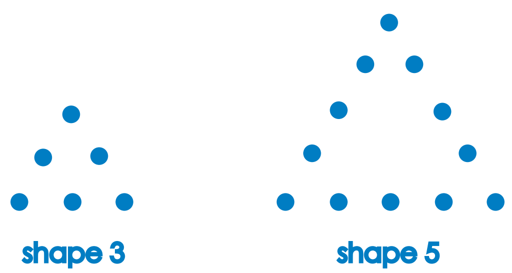

Figure 7. Two non-consecutive terms.

identified structural feature as well as the number of times this structural feature occurs in the pictorial term itself. Choice of any other Term number would have avoided the semantic ambiguity (Samson, 2011a) of the portion of her general term, since it was only in Term 3 that this could be ambiguously interpreted as either "n groups of three dots" or as "three groups of dots". Thus, a useful cautionary strategy to keep in mind when consciously searching for structural elements in a pictorial term is to make use of bigger terms (e.g., $n \geq 5$) where there is less chance of such ambiguity obfuscating the generalisation process.

In terms of expressing different visualisations in the form of algebraic expressions, some students may find it useful to make use of a stepwise process of semiotic contraction. By way of example, verbal expressions such as, "I multiplied three by one less than the shape number," could first be expressed in the form, "Three times the shape number minus one," as an interim step en route to the algebraic symbolism $3(n - 1)$. The advantage of this approach is that the interim verbal syntax is far more closely aligned with the desubjectified algebraic symbolism.

Students should be encouraged to look out for serendipitous numerical observations that could lead to the development of general algebraic expressions. For example, if $T_6 = 21$ one could make the numerical observation that $21 = 3 \times 7$. Since 7 in this instance is 1 more than the term number this could lead to an investigation to assess whether this situation is always true, in which case $T_n = 3(n + 1)$ could be an appropriate algebraic formula for the general term. Having determined this general rule numerically, one could then search for an associated visual justification.

Strategy 4: Choice of specific pictorial contexts

Certain features of pictorial patterns tend to encourage particular generalisation strategies (Samson, 2007; 2011b). Since one would want students to be able to experience a range of strategies, a range of pictorial patterns should be included in patterning tasks. These should include:

- questions where the growth pattern occurs in a single direction and where progression from one term to the next can be accomplished by the direct attachment of the additive unit—i.e., the extra matches or dots that need to be added to a given term to create the next term in the sequence (e.g., Figure 2 and Figure 4);
- questions in which the growth pattern occurs in more than one direction (e.g., Figure 1);
- questions in which progression from one term to the next can only be accomplished by the *insertion* of the additive unit *into* the previous term as opposed to the *direct attachment* of the additive unit *onto* the previous term (e.g., Figure 8).

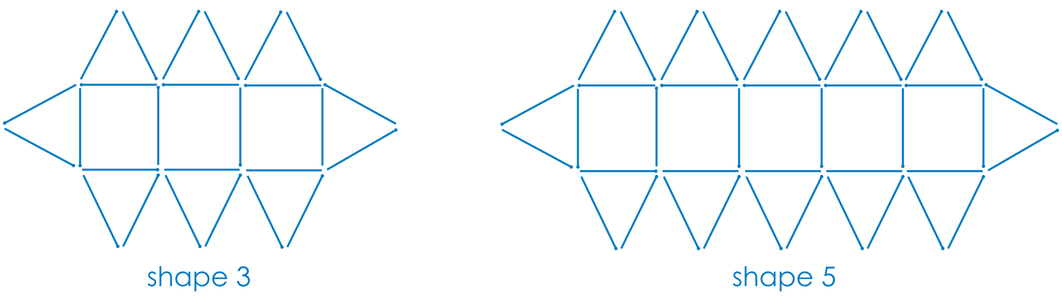

Figure 8. A pictorial sequence requiring an insertion of the additive unit.

Strategy 5: The use of manipulatives

Finally, while a conscious search for structure is a useful generalisation strategy, so too is unstructured exploration and interaction with the pictorial context—a process which could lead to the serendipitous awareness of structural regularity. Students should be encouraged to make use of physical manipulatives (such as matchsticks and plastic counters) to encourage such unstructured exploration. Even students who are sceptical about the use of manipulatives, and who profess to preferring more visual or abstract engagement, should be encouraged to make use of them. It is often the tactile, physical and whole-body engagement of such activity that leads to unconscious moments of mathematical play that could serve as crucial pivots for the evolution of new 'ways of seeing.'

Concluding comments

The insights presented in this article gradually emerged and evolved during the course of micro-analysing data stemming from a broader study (Samson, 2011b). These insights are synthesised here in relation to possible pedagogical strategies that could be used to support pictorial pattern generalisation activities. As such, I hope that they represent a useful toolbox of strategies that teachers can draw on to encourage visual engagement with patterning activities presented in a pictorial context.

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