Four Seventh Grade Students Who Qualify for Academic Intervention Services in Mathematics Learning Multi-Digit Multiplication with the Montessori Checkerboard

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Abstract

This article describes the positive impact of Montessori manipulative materials on four seventh grade students who qualified for academic intervention services because of previous low state test scores in mathematics. This mathematics technique for teaching multi-digit multiplication uses a placemat-sized quilt with different color-coded squares for place value, color-coded bead bars for representing digits, and small numeral tiles in a procedure related to lattice multiplication. The article presents a brief introduction to the Montessori approach to learning, an overview of Montessori mathematics, and an explanation of the Checkerboard for Multiplication with related multiplication manipulatives. Pretest/posttest results of the four students indicated that all increased their understandings of multiplication. The results of an attitude survey showed students improved in enjoyment, perceived knowledge, and confidence in solving multiplication problems.

Keywords
multiplication, mathematics, manipulatives, Montessori education,
Checkerboard for multiplication

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Introduction

All students should have the opportunity and the support necessary to learn mathematics with understanding, as well as develop efficient, accurate, and generalizable methods for computation necessary to solve complex and interesting problems (National Council of Teachers of Mathematics [NCTM], 2000). Despite the concerted efforts of many who are involved in mathematics education, students in the vast majority of classrooms are not learning the mathematics they need or are expected to learn (Beaton, Mullis, Martin, Gonzalez, Kelly, & Smith, 1996; Kenney and Silver 1997; Mullis, Martin, Beaton, Gonzales, Kelly, & Smith, 1997). The demands on classroom teachers increase as school populations become more diverse linguistically and culturally, as employment pressures require families to be more mobile with parents spending more time away from the home, and as students with learning disabilities and other special needs are included in the regular classroom (U.S. Department of Education, 1996, 1998). Meeting students' varied needs and learning preferences demands that classroom teachers have a large repertoire of approaches and strategies for mathematical problem solving. Montessori mathematics manipulatives can provide an alternative approach for implementing the *Standards* (NCTM, 1998) while helping students develop a deep understanding of computational algorithms.

Montessori was the first to advocate careful observation of the child.

This article provides: a brief introduction to Montessori and the Montessori approach to learning, an overview of Montessori mathematics, an explanation of the Checkerboard for Multiplication and related multiplication manipulatives, and the results of our use of this manipulative in helping four seventh grade students who qualified for academic intervention services to review and learn multi-digit multiplication.

Background of the Montessori Approach

In 1896, after defeating many obstacles in an all-male field, Maria Montessori became the first female to graduate from the University of Rome School of Medicine, and thus the first woman physician in Italy. Montessori studied pediatrics, and translated the writings of Jean Itard and Edouard Seguin into Italian. She incorporated their ideas of using sensory teaching materials into her work as director of a practice demonstration school for children who were identified at that time as having mental retardation, but who may have had other social, emotional, or cognitive difficulties. Her success in teaching these children to care for themselves, and pass exams on par with typical children led her to be regarded as an educator rather than a physician. She continued her professional development by taking education courses at the university, and in 1907 opened the Casa dei Bambini (Children's House) in the San Lorenzo slums of Rome. Here she created a prepared environment that provided the children with experiences their homes lacked.

Montessori, observing and learning from her pupils, prepared a variety of educational materials. Many of these have become familiar sights in toy stores and nursery schools, including geometric-shaped puzzles, movable alphabets for spelling, and child-sized furniture.

Over the next decade, Montessori's ideas spread over Europe and to America. Montessori was the first to advocate careful observation of the child to understand human development. This she termed her most important contribution to education, the true "discovery of the child." In 1939, after Hitler had closed all Montessori schools in Germany, Italy, and Spain, Montessori traveled to India, where she established a training center and began a passionate quest for pursuing world peace through education. She was nominated for the Nobel Peace Prize three times. Interested readers will find more details of her life, work, and method in Montessori (1964), Standing (1957), Hainstock (1978), Loeffler (1992), and Lillard (2005). Today, private Montessori preschool and elementary programs abound in the United States, while public schools are increasingly finding Montessori education an attractive alternative for elementary magnet programs (Mathews, 2007).

Montessori Mathematics

Montessori methods and materials are especially powerful for teaching mathematics because they form a coherent curriculum, progressing from concrete representations of concepts to levels of increasing abstraction, culminating in paper and pencil algorithms (American Montessori Society [AMS], 2002). This solid foundation of hands-on work with simple, appealing materials affords students a deep understanding of place value, number, and operation concepts. Individualized instruction insures that the needs of each student are met.

Montessori Mathematics Materials

Montessori mathematics materials are characterized by two color-coding systems for place value and number concepts. Montessori mathematics materials have the following properties (Lubienski-Wentworth, 1999):

- *Beauty*. Materials are of high quality and carefully designed to give a clear, uncluttered image. Bright colors, smooth, polished wood, and cool, shiny glass beads appeal to a child's senses, aesthetic enjoyment, and thus the materials support task commitment (Rule, in review; Rule, Sobierajski, & Schell, 2005).
- *Dynamic*. Mathematical materials are designed to be manipulated rather than observed.
- *Simple*. The same basic materials can be used to illustrate many different concepts. For example, bead chains can be used for counting, skip counting, learning multiples, and illustrating squares and cubes of numbers.
- *Order*. Order is revealed in the hierarchy of sets of materials (the system of ranking place value and number through color-coding of manipulatives) and in the steps and layout of materials during a lesson. Many materials are color-coded for place value: green for the ones place of all families, blue for the tens place, and red for the hundreds place. Numeral cards, colored beads on the rungs of bead frames, patches on the Checkerboard for Multiplication, and skittles (colored bowling pin-shaped pawns used to represent the divisor) used in division are all similarly color-coded in green, red, and blue for place value. The color system of the units bead stair (the step-like set of bead bars representing the digits one through nine) is also consistent throughout the materials: individual bead bars, chains, squares, and cubes are all systematically color coded to facilitate recognition and understanding of multiples and powers of numbers.
- *Developmental*. Materials are designed to lead the child from the beginning simple, concrete representations of numbers and mathematical concepts to the more com-

plex, abstract mathematical ideas (AMS, 2002).

Montessori Mathematics Lessons

In addition to the unique materials generally not found in public school classrooms, Montessori mathematics lessons incorporate the following qualities (Lillard, 2005).

- *Daily practice*. Consistent repetition and practice build strong mental connections that allow the child to access information more easily (Driskell, Willis, & Copper, 1992). This ease, in turn, builds self-confidence and facilitates further learning.
- *Impressionism*. Many lessons are designed to pique interest and appeal to the child's dramatic or impressionistic side. An example of such a lesson would be the gathering of many bead bars, hundred squares, or thousand cubes for the addition or multiplication of large numbers (AMS, 2002).
- *Varied learning preferences addressed*. Many different approaches are used during lessons. Many lessons encourage kinesthetic movement in reaching for, transporting, stacking, aligning, and sorting materials. Children may move around, sit at a quiet desk, work in a group, or spread materials out on the floor.
- *Connections*. Mathematical themes and concepts are revisited again and again through different lessons using different materials. The "Timeline of Mathematical Ideas", emphasizing contributions of different cultures, is one of the "Great Lessons" presented annually in all classrooms in different ways. Connections are regularly made between mathematics, art, other academic subjects, and everyday life (Chattin-McNichols, 2002).
- *Individualization*. Montessori classrooms are usually multi-age (serving students of a two or three year age range and therefore a range in grade levels), allowing the opportunity for children to act as followers, peers, and leaders with different classmates. Lessons are given to individuals or small groups of children who are ready for the concept. Each child may therefore progress at the child's own pace, not having to "catch up" with or "wait" for others. Research indicates that multi-age classrooms are psychologically healthy environments (Miller, 1995; Pratt, 1986; Veenman, 1995).
- *Assessment*. A Montessori lesson consists of three periods (Kroenke, 2006). First, the teacher instructs ("This is…"). Second, the teacher checks for knowledge ("Show me…"), Finally the teacher assesses comprehension ("What is…"). This structure allows the teacher to track student understanding during the lesson. The teacher monitors each child, keeps careful records, and sets appropriate short and long-term goals.
- *Control of error*. Design of lessons and materials guarantee student success. Lessons are taught in small chunks so that a child's difficulty can be isolated and retaught in a subsequent lesson. Materials are designed so the child can detect when an error has been made. This allows the child to work independently and self-correct, supporting self-esteem (Herz & Gullone, 1999).
- *Problem solving*. Children "come to the rule;" that is, they discover a pattern by

The curriculum progresses from concrete representations of concepts to levels of increasing abstraction.

using the materials and then they state the rule, rather than being told what the rule or generalization is and how it works. Structured learning games that involve problem solving reinforce learning and provide the practice necessary for "memorization" or automaticity of facts.

- *Respect*. The teacher observes the child and prepares the environment, lesson presentation, content, and materials to meet the needs of the child. The teacher takes the child at his/her current level and moves him/her forward. The teacher does not impart knowledge to the child, but helps him/her find out the rules himself/herself through a careful succession of lessons. The child is intrinsically motivated to complete mathematics assignments because lessons are presented to challenge the child at the appropriate level (Malone and Lepper, 1987).

The Checkerboard for Multiplication

An especially effective application of these qualities of Montessori Mathematics is the Checkerboard for Multiplication. The Checkerboard for Multiplication has three components: a place mat-sized board with green, blue, and red alternating squares, numeral tiles, and a set of bead bars (See Figure 1).

Figure 1. The Checkerboard for Multiplication with box of bead bars and numeral tiles.

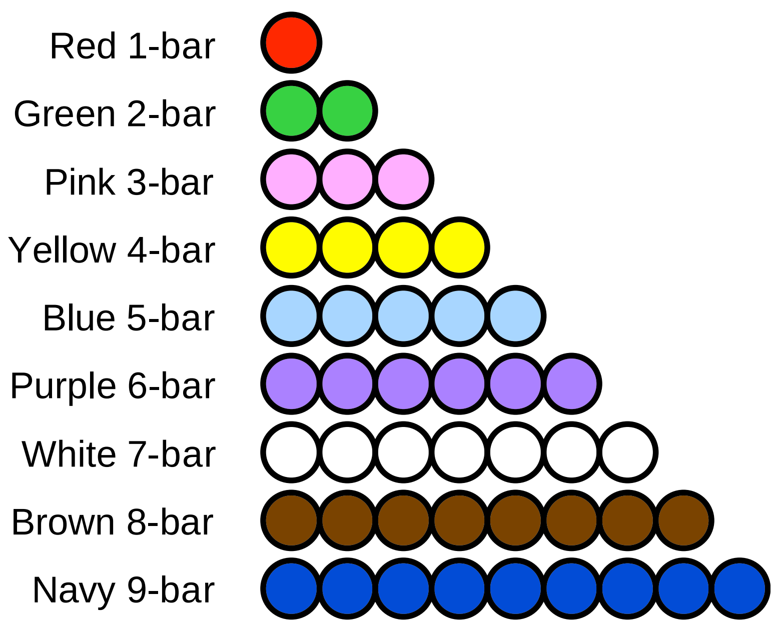
The colored checkerboard squares represent places for digits of the partial and final products of a multiplication problem. Numeral tiles placed along the bottom and right side of the board designate the digits of the multiplicand and multiplier respectively. Bead bars – beads strung on a wire or stiff, knotted cord – range from one to nine beads and are used to represent quantities and digits in the place value squares of the board. The color-coding of bead bars enables students who are accustomed to the materials to choose and recognize quantities quickly, while still allowing others who are less familiar to count beads. An ordered set of bead bars representing each number from one to nine is called a "bead stair" because of its step-like configuration (see Figure 2)

The original Montessori checkerboards are beautiful wooden boards with stenciled squares of green, blue, and red. Specially made patchwork quilts are often used as checkerboards because the quilting helps keep the bead bars in place. Similarly, a light-colored woven place mat can be stenciled with acrylic paint squares. Easily stored folding checkerboards can be quickly made by gluing and laminating fadeless colored paper squares inside file folders. All materials should be constructed with care so that they are neat, colorful, and attractive.

Durable cardboard versions of the small, wooden, one-digit numeral tiles can be made by photocopying numbers onto card-stock and gluing them to larger mat board rectangles. These numerals are used to represent the multiplicand and multiplier of a multiplication problem.

The original Montessori bead bars are beautiful glass beads on wires secured with looped ends. Glass and plastic bead bars are available commercially from Montessori materials distributors (Nienhuis Montessori U. S. A., 2000; Albanesi Montessori Education Center, 2000). However, bead bars can easily be made by stringing and knotting plastic pony beads on nylon cord, all available at most craft stores. A partitioned plastic box like those sold for storing embroidery floss or fishing tackle works well for bead bar storage. Ten to twenty bead bars of each color will allow students to work most problems.

Figure 2. The Montessori bead stair.



Place Value on the Checkerboard for Multiplication

The Checkerboard for Multiplication is made of alternating squares of green, blue, and red (See Figure 3). The green squares represent the "ones" place of a place value family – the *ones* of the units family, the *one*-thousands, and the *one*-millions. The blue squares represent the "tens" place of a family – the *tens* of the units family, the *ten*-thousands, and the *ten*-millions. Finally, the red squares represent the "hundreds" place of a family – the *hundreds* of the units family, the *hundred*-thousands, and the *hundred*-

millions. Place value concepts are introduced long before multiplication in a Montessori classroom. Color-coded numeral cards are used to help students identify the place value of digits. Similarly, non-Montessori teachers will need to review place value concepts with students before using the checkerboard.

The squares along the bottom horizontal row, then, represent the following place values from right to left: green “ones,” blue “tens,” red “hundreds,” green “one-thousands,” blue “ten-thousands,” red “hundred-thousands,” and green “one-millions.” These place values are noted in black ink across the bottom white strip and will be the positions of the multiplicand. The checkerboard, however, is two-dimensional with another axis that starts in the bottom right hand corner and extends vertically along the side of the board. Place value increases in this direction with the first green square as noted before being the “ones” place, the blue square above it being the “tens” place, and the red square above that representing the “hundreds” place. These place values are noted in black along the right-hand vertical white strip and are the positions of the multiplier. Each row above the bottom row continues the place value pattern and increases in place value from right to left so that squares of the same color and place value are aligned along diagonals (See Figure 3). These materials may be adapted for a student who is colorblind by using different patterns of fabric and distinctive bead shapes for the bead bars representing different digits.

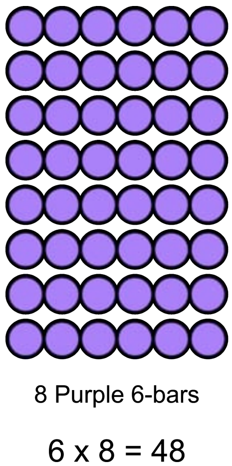
Figure 3. The place values of the squares on the checkerboard.

100,000,000 Red	10,000,000 Blue	1,000,000 Green	100,000 Red	10,0000 Blue	1,0000 Green	100 Red	100
10,000,000 Blue	1,000,000 Green	100,000 Red	10,000 Blue	1,000 Green	100 Red	10 Blue	10
1,000,000 Green	100,000 Red	10,000 Blue	1,000 Green	100 Red	10 Blue	1 Green	1
1,000,000	100,000	10,000	1,000	100	10	1	

A good way to acquaint students with the bead bar color-coding system is to have them lay out and produce a colored drawing of the “bead stair” (a set of bead bars from the red 1-bar to the navy 9-bar as shown in Figure 2). Then ask students to gather bead bars to illustrate some of the multiplication facts. For example, 6 x 8 (six taken eight times) would be illustrated by gathering eight purple 6-bars. Students who have not yet memorized this fact’s product may count the individual beads on the bead bars to determine it (see Figure 4).

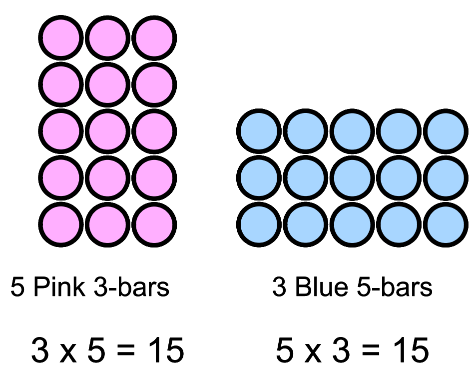
In our experience, teachers using bead bars in tutoring fifth and sixth graders who struggle with multiplication report that this activity has often been a turning point for their students. The elementary students indicated they had never before understood what they were doing in multiplication. “Times eight" did not mean anything to them; they simply set about memorizing the facts. But the act of taking a six-bar eight times suddenly made sense. They could see all of the parts of the equation: the "6" beads on each bar, the "8" bars, and the "48" beads in total.

Figure 4. Eight six-bars used to show eight taken six times.



Bead bars also can be used to illustrate the commutative property of multiplication. For example, bead bars can be used to represent 3 x 5 ("three taken five times") and 5 x 3 ("five taken three times"). Align the ends of the bead bars for each equation to form a rectangle (see Figure 5). The two rectangles, though different colors, are congruent: they can be rotated to show they have the same dimensions and counted to show they represent the same quantities (products).

Figure 5. Bead bars used to show the commutative property of multiplication.

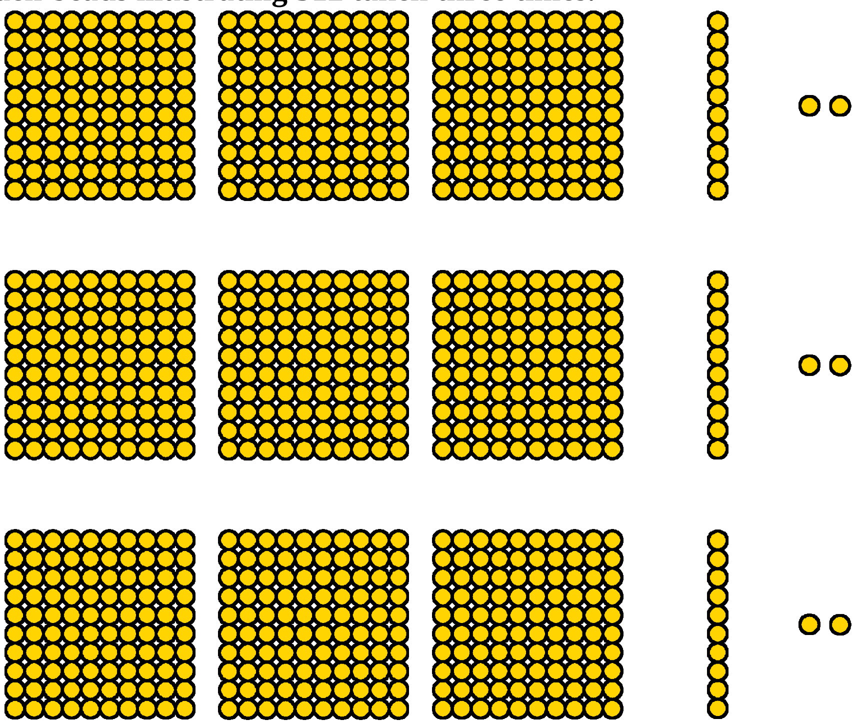


Work with Golden Beads or Base Ten Blocks

Once basic mastery of these materials has been achieved, students' understanding of place value in multiplication can be further enhanced by use of golden beads to represent problems with multi-digit multiplicands and one-digit multipliers. Montessori golden beads are analogous to the units, ten-rods, hundred-flats and thousand cubes of base ten blocks. There are golden unit beads (single gold pearly glass beads), golden ten-bars, hundred squares made of ten ten-bars wired together, and golden thousand cubes made of ten hundred squares wired together. Base ten blocks, more readily available in public school classrooms, can be used in place of golden bead bars.

Illustrate a problem such as 312 x 3 with the golden beads. Lay out the quantity three hundred twelve as two unit beads, one ten-bar, and three hundred squares, taking care to place the beads in their correct relative place value positions (See Figure 6). Do this three times. Then push the quantities together and have the students determine if any re-grouping is necessary (in this case, no re-grouping was necessary). Count the beads and determine the product, 936. The lustrous, heavy, golden beads help the student to recognize the large quantity resulting from this operation and give a very satisfying clink when gathered together.

Figure 6. Golden beads illustrating 312 taken three times.



Another problem with a larger multiplier might be attempted, perhaps 24 x 17. Ask the students, "How long would it take to lay out the quantity twenty-four, seventeen times? (They might lay out twenty-four once and then estimate). Would there be enough beads (or base ten blocks)? (Most classroom collections would not be sufficient). What if the multiplier were even larger? (There is a limit to the problems that can be represented with the available materials)." Discuss the implications of solving this problem with the golden beads: 536 x 234. Is there a practical hands-on way to solve problems with three-digit multipliers?

A Semi-Concrete Bridge to Understanding the Abstract Algorithm

In a traditional classroom, there is no hands-on way to easily illustrate problems with three-digit multipliers and multiplicands. Students move directly to abstraction - the paper and pencil algorithm. The Checkerboard for Multiplication serves as a bridge between the concrete methods described above (laying out the quantity repeatedly) and the abstract algorithm used in paper and pencil computation. The Checkerboard for Multiplication combines the abstract concept of place value squares on a board with the concrete use of bead bars to show quantities within each place. Students with the appropriate background in place value (those who can state the place value of digits in a multi-digit number and explain place value equivalencies such as ten tens equals one hundred) can use this hands-on method to multiply large quantities using manipulatives in a way that corresponds to the paper and pencil algorithm. This way, instead of simply memorizing the algorithm, they can understand how and why the algorithm works.

This is the strong point of the Montessori mathematics curriculum. Concepts are introduced concretely in many different ways with related materials. Concepts build from the simple (here, multiplication facts) to the complex (three-digit multipliers). Hands-on materials guide the student to ever-increasing levels of abstraction until the student is able to manipulate the quantities mentally or with paper and pencil using the traditional algorithms. Place value, typically difficult for students, is more completely mastered using the golden bead hierarchy and color-coded numeral cards, boards, and manipulatives (Bennett & Rule, 2005).

Hands-on materials guide the student to ever-increasing levels of abstraction.

It is beyond the scope of this paper to discuss all of the mathematics exercises that form a foundation in a Montessori program for multiplication and use of the Checkerboard for Multiplication. The work with unit bead bars and golden beads are only part of this groundwork, which includes place value activities, bead chain exercises, and work with simple multiplication facts. However, non-Montessori teachers can make use of the checkerboard activity, integrating it into their own mathematics programs to provide their students with another, and perhaps more concrete, approach to teaching multiplication.

A Simple Problem

After students have had a chance to become familiar with illustrating facts with bead bars, they will be ready for a simple problem on the Checkerboard for Multiplication. As an introduction, the problem 43 x 2 ("Forty-three taken two times") will be illustrated. Start by using the one-digit numeral tiles to represent the multiplicand and multiplier of the problem. Place a "4" and a "3" along the bottom of the board to represent the four "tens" and three "ones" of the multiplicand. Place a "2" at the "ones" place along the right side of the board to represent the multiplier (see Figure 7).

The Checkerboard for Multiplication uses a form of matrix multiplication. The green square above the "3" and to the left of the "2" is the "ones" place. This is where the operation "3 x 2" is performed (the first step in solving the 43 x 2 problem). Two pink 3-bars will represent the operation of taking three two times. Place these bead bars in the square. Now move left to the blue "tens" place square. Below this square is the "4" of the multiplicand representing four "tens". To the far right is the "2" of the multiplier indicating "four tens *taken two times.*" Since the blue "tens"-place square keeps track of the place value, gather two yellow 4-bars. Place these in the blue square.

Because more than one bead bar occupies each square, regrouping must take place. Each bead bar can be thought of as a digit; only one digit is allowed in each place. Start with the "ones" place. Count the beads, or skip-count (count by multiples of three), or recognize the multiplication fact and replace the two pink 3-bars with one purple 6-bar. Then move to the "tens" place. Two yellow 4-bars can be replaced with one brown 8-bar. Now the product is in its final form, and can be read or recorded: "six ones, eight tens," or "eighty-six" (see Figure 8).

Figure 7. Partial checkerboard showing layout of 43 x 2.

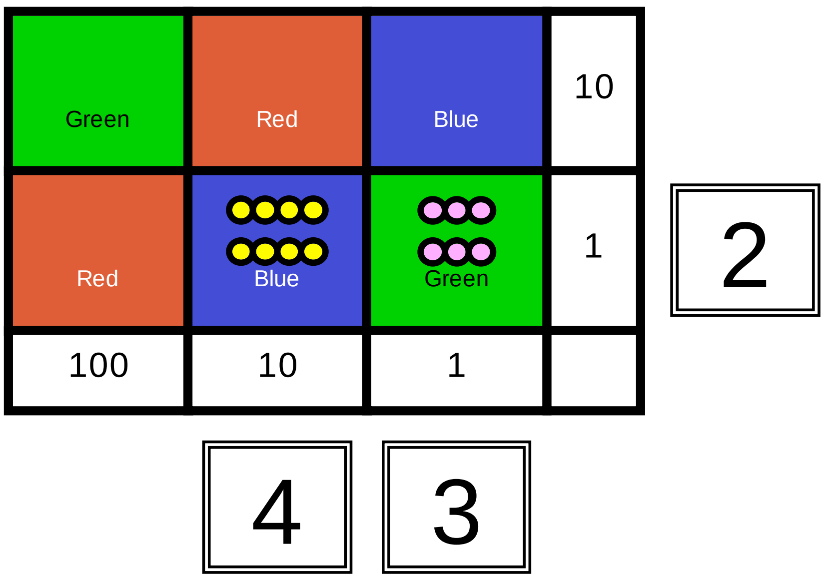


Figure 8. Final product.

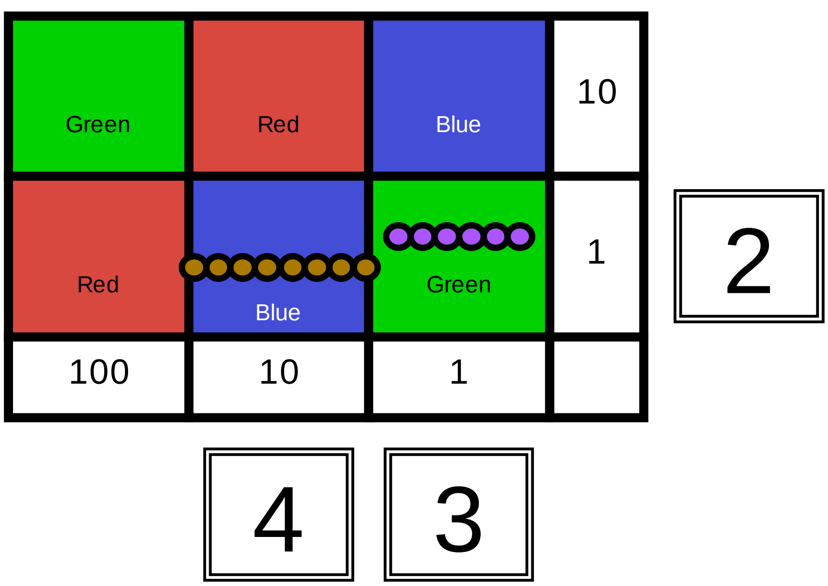


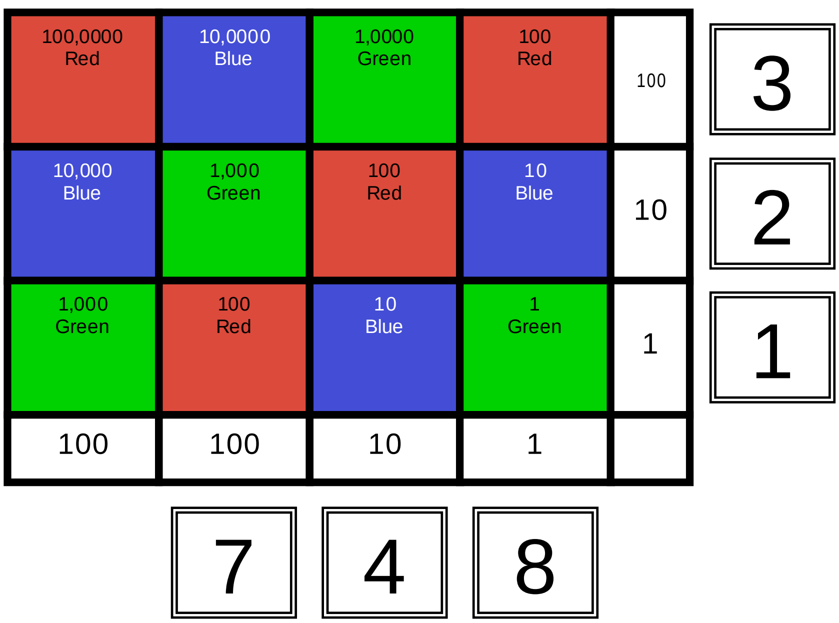
Table 1. Job descriptions for group members.

Member	Checkerboard for Multiplication Tasks
1	• Place the number tiles for the problem. • Act as "banker". Manage the bead bar box. Give bead bars for squares and exchange as requested by other team members.
2	• Place the bead bars for the problem. • Slide the bead bars to sum the partial products. • Read the digit and place value of each square when recording partial or final products.
3	• Tell the operation for each square describing needed bead bars. • Add or calculate multiplication of bead bars for regrouping. • Record partial products on paper.

A More Complex Problem

To show the power of this manipulative, a problem with three-digit multiplicand and multiplier will be illustrated next, although it would be better to present simpler problems to students when they are first learning to use the board. 748 x 321 is the more complex problem (see Figure 9). Begin by placing the numeral tiles across the bottom and along the right side of the board to represent the problem as was done in the first example. Note that care should be taken in placing the numeral tiles in their correct place value positions. Transposition of digits will result in the wrong problem being solved. Students should carefully check that they have placed the digits in the correct place value positions. Table 1 shows suggested roles for students working in small groups.

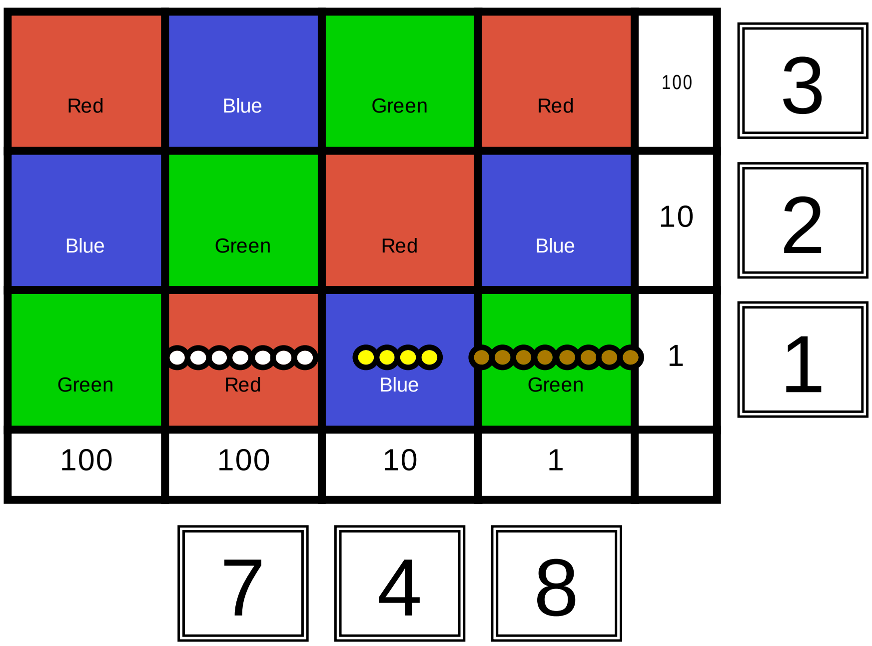
Figure 9. Partial checkerboard showing tiles for multiplicand and multiplier.



First Partial Product

Begin the multiplication problem on the bottom row of the board, starting in the green "ones" square. Below that square is the digit "8" indicating the "ones" place of the multiplicand. To the right of the square is the digit "1" which is the "ones" place digit of the multiplier. Analogous to paper and pencil methods, the first operation is to take the eight one time (8 x 1). That is accomplished by taking one brown 8-bar and placing it in the square. Then the student moves on to the blue "tens" place in the bottom row. The digit below this square is a "4" indicating the four tens of the multiplicand. Because this blue square is in the bottom row, the "1" digit of the multiplier to the right will be applied to this digit. Therefore, take one yellow 4-bar and place it in this square. Now move left to the red "hundreds" square. Directly below this square is the "7" indicating seven hundreds of the multiplicand. Again, this will be multiplied by one or taken one time because the "1" of the multiplier is at the far right end of the row. Take a white 7-bar and place it in this square. Because there is only one bead bar corresponding to one digit in each square, no regrouping is necessary. The first partial product may now be recorded in the 'copy-book" (a small booklet of multiplication problems). Start in the green "ones" place square and tell the digit (number of beads on the bead bar in that square) for each place. In this case the student would say, "Eight ones, four tens, seven hundreds." The first partial product of this problem is 748 (See Figure 10).

Figure 10. Checkerboard showing first partial product.



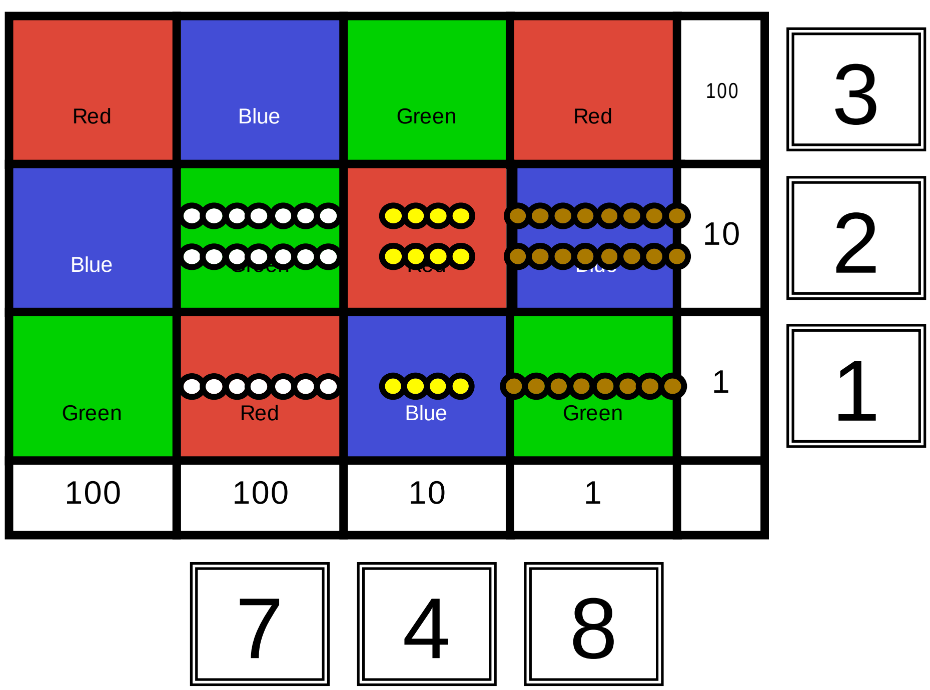
Second Partial Product

Leave the bead bars of the first partial product in place and turn attention, instead, to the next row of the checkerboard. Begin in the blue "tens" place square next to the "2" of the multiplier. There is no green "ones" place square in this row. This is because the multiplicand will now be multiplied by a "tens" place digit. The lack of a "ones" place is analogous to automatically placing a "zero" in the "ones" place of the second partial product in paper and pencil methods. (A pre-service teacher enrolled in the second author's mathematics curriculum and instruction course once spontaneously remarked that for the first time, she understood why a zero is automatically recorded in the ones place of the second partial product during the paper and pencil algorithm. She had methodically done this for years, but never had understood why!)

Place your finger in the blue tens square. Below this square, along the bottom edge of the checkerboard, is the digit “8” of the multiplicand. To the right of this square is the “2” digit of the multiplier. So the operation that will occur in this square is eight taken two times. The “2” actually represents two tens because the place value of the blue square is the “tens” place. Put two brown 8-bars in this square. Similarly, move to the next square in this row, the red “hundreds” square, and place two yellow 4-bars in the square. Then place two white 7-bars in the next square, the green “thousands” square (see Figure 11). However, the second partial product cannot yet be recorded because there is more than one bead bar per square. Regrouping must take place.

To regroup, start with the square at the right end of the row, the blue “tens” square. There are two bead bars in this square – two 8-bars. If the student knows the multiplication fact 8 x 2 = 16, then the student may continue. But if the student does not know this fact, the student may still continue by simply counting the beads. This is one of the most powerful contributions of this manipulative. Students who do not yet know all of their multiplication facts can still complete more advanced multiplication problems. Students practice and learn multiplication facts by counting the bead bars involved.

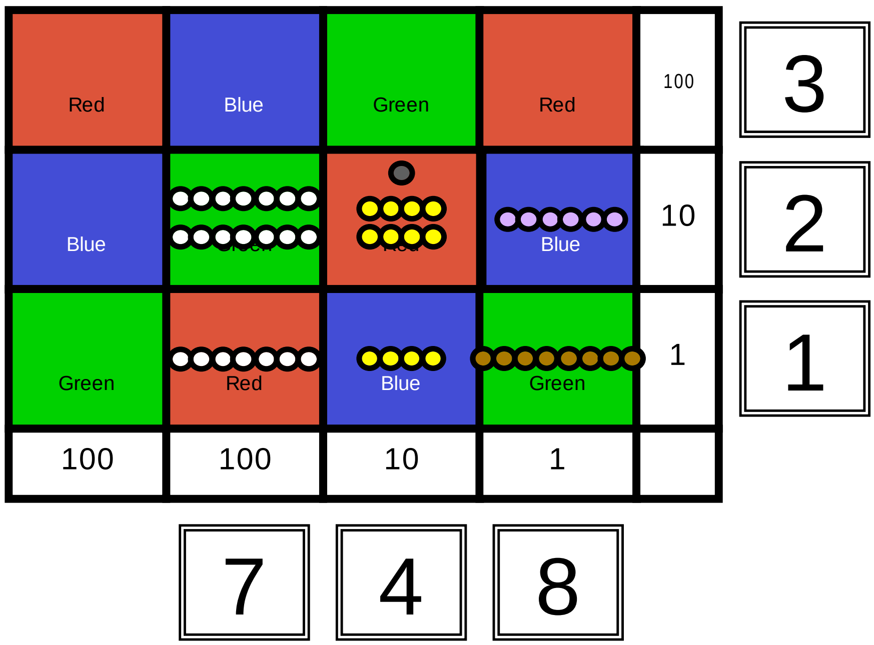
Figure 11. Checkerboard showing second partial product before regrouping.



Now comes the trickiest part of this manipulative: the renaming of ten beads as one bead of the next place value. The student has determined that there are sixteen beads in this blue “tens” square. The student knows that no more than nine beads (equivalent to a single digit) can be in a square (equivalent to a place value position). How will the student represent "16" using single digits? By placing a purple 6-bar for the "6" in the current square, and a red 1-bar for the "1" in the next square of higher place value. Because the current blue square was the "tens" square, the "16" represented sixteen tens. In regrouping, we have converted the sixteen tens to six tens and one hundred. Be sure to remove and exchange the old bead bars for their new regrouped configuration (see Figure 12).

Now move to the red hundreds square. There are two yellow 4-bars here and a red 1-bar. The student can multiply 4 x 2 = 8 and add the red 1-bead bar to make 9 or can count the individual beads. Exchange these bead bars for the equivalent navy 9-bar (see Figure 13).

Figure 12. Partly-regrouped second partial product.



Finally, move to the next green "one-thousands" square. It contains two white 7-bars from the 7 x 2 operation where seven hundreds were taken twenty times. Seven times two equals fourteen, so exchange these beads for a red 1-bar and a yellow 4-bar. Originally, because the fourteen was in the green "one-thousands" square, it represented fourteen one-thousands. Now, place the yellow 4-bar in the green "one-thousands" place to represent four thousands, and the red 1-bar into the next blue square to represent one ten-thousand. Because each place has only one bead bar (a single digit) the second partial product can now be recorded in the copybook. Start with the far right end of the row. The blue square is the "tens" place. Be sure to write a zero in the "ones" place before recording the digit for the "tens" place, since this second partial product results from a multiplier of two tens. Then continue along to the left, recording each digit. A student would say, "zero "ones", six "tens", nine "hundreds", four "one-thousands", one "ten-thousand." The second partial product is 14,960 (see Figure 14).

Figure 13. Regrouping of the hundreds place in the second partial product.

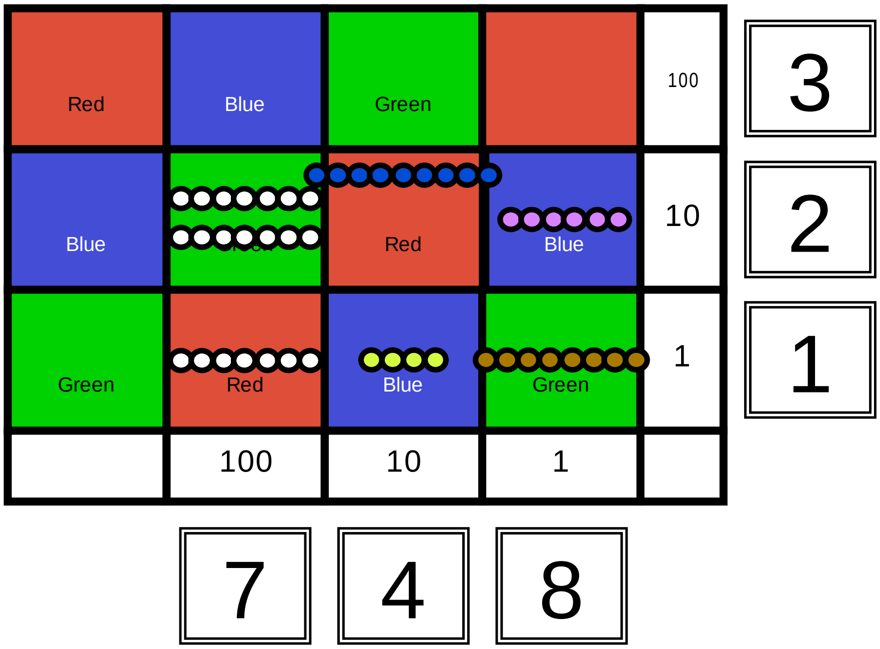
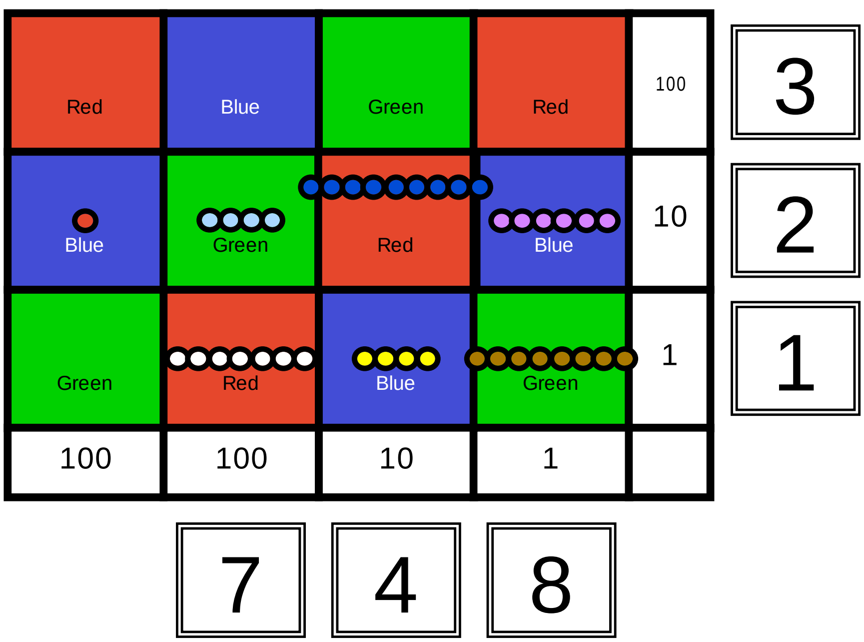


Figure 14. Completed regrouping of second partial product.



Third Partial Product

In a manner similar to that described above, the student may place the bead bars for the third partial product on the third row of the checkerboard. Note that the first place value in this row is the "hundreds" place, denoted by a red square. This is because the multiplicand will now be multiplied by *three hundred*. Figure 15 shows the bead bars before regrouping and Figure 16 shows them after regrouping. The third partial product is 224,400. Remind students to determine the place value of the first square in order to record the appropriate number of zeroes in the partial product.

Summing the Partial Products

Summing of the partial products to generate the final complete answer can also take place on the Checkerboard for Multiplication by an operation known as “sliding along the diagonal”. Recall that squares of the same place value are aligned on diagonals. Therefore, to sum the partial products, carefully slide the bead bars down and left on the diagonal to the equivalent place value squares in the next row until all bead bars are in the bottom row (See Figure 17 for direction of sliding). Figure 18 shows the bead bars for the sample problem after sliding.

Figure 15. Third partial product before regrouping.

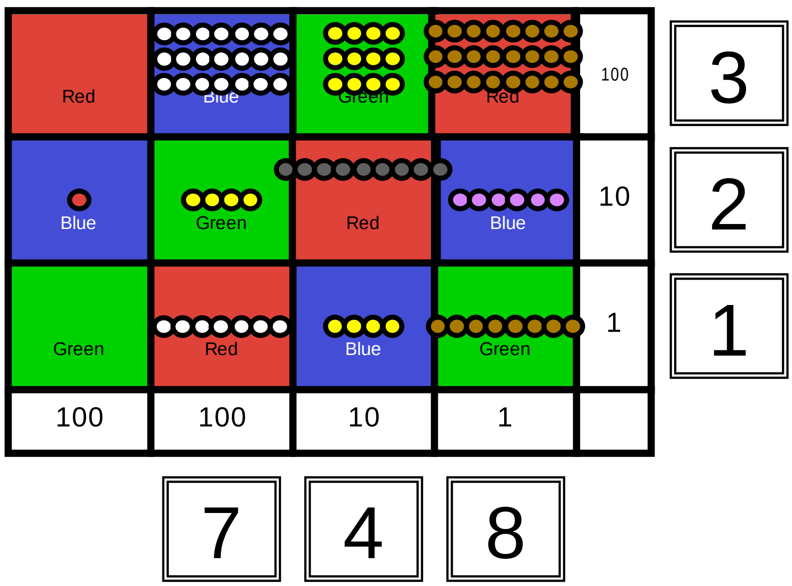
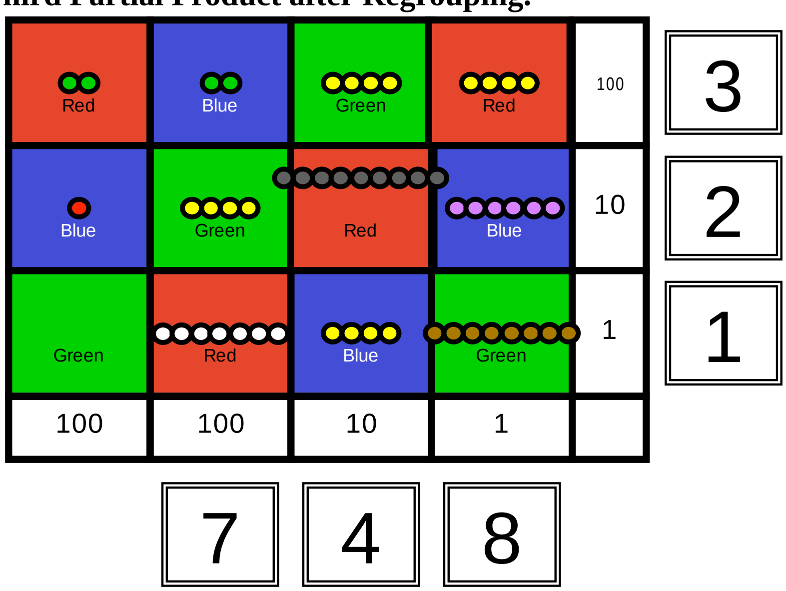


Figure 16. Third Partial Product after Regrouping.



Notice that now there is more than a single bead bar in most squares. The bead bars will have to be regrouped before the final product can be determined. Regroup in the same manner as described earlier. Students who are familiar with bead bar colors can perform simple addition problems, while other students can count the beads for regrouping. Take care in recording the final product: "eight ones, zero tens, one hundred, zero one-thousands, four ten-thousands, two hundred-thousands." Figure 19 shows the final product after regrouping, 240,108.

Figure 17. Arrows showing direction of sliding for summation of partial products.

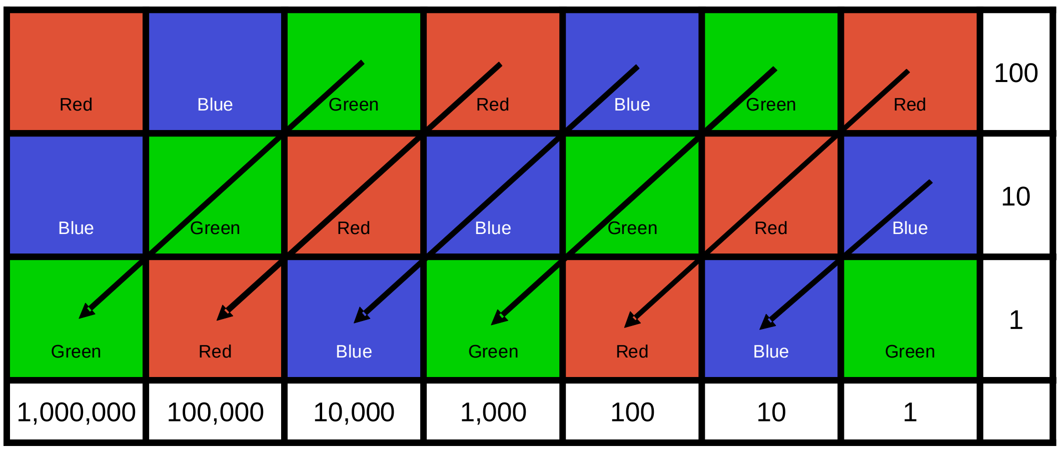
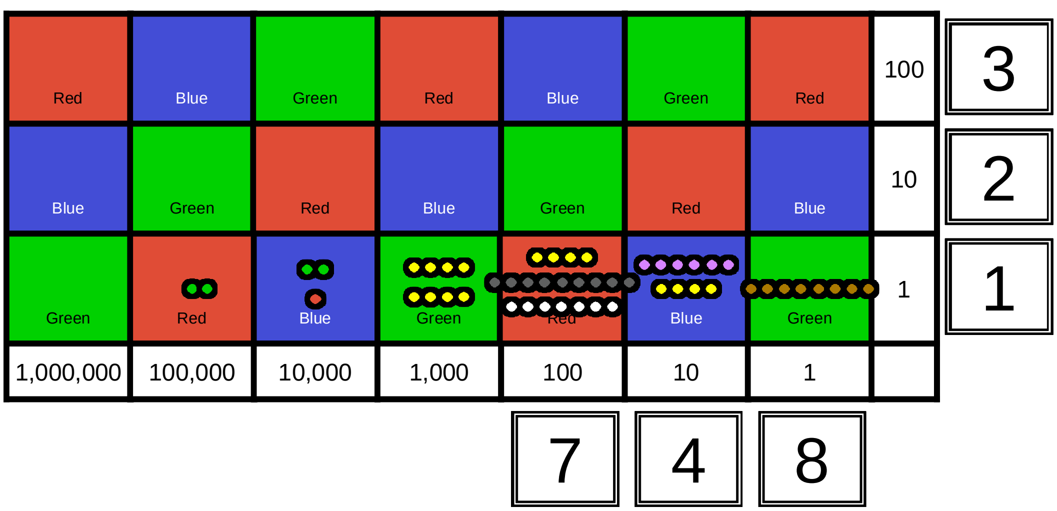


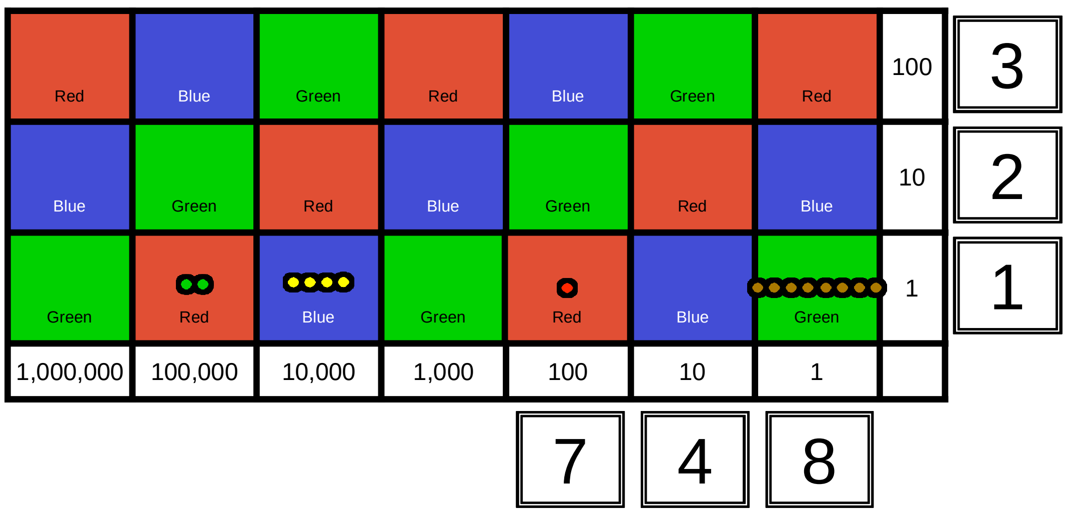
Figure 18. Bead bar positions after sliding has occurred.



Shortcut of Immediate Regrouping on the Checkerboard for Multiplication

After students have had practice with the above process on the Checkerboard for Multiplication, they will be ready for a procedure that more closely approximates the paper and pencil algorithm. For example, in the second partial product, when multiplying 8 x 2, instead of placing two 8-bars in the square, the student can regroup immediately. That is, if the student knows that 8 x 2 = 16, the student can place a purple six-bar in the blue tens square and a red 1-bar in the square to the left of it, the red hundreds square. This is analogous to the regrouping (movement of the "1", representing the ten of the "16", to the tens column) that would take place immediately in the paper and pencil method. Additionally, if there are bead bars already in the square from regrouping that took place earlier in the square to the right, the student can immediately add these to the current computation and complete the regrouping in one large step. Be sure to allow students ample practice in the longer, more concrete process before showing them these shortcuts.

Figure 19. Final product after regrouping.



Discussion of Checkerboard Technique

Applying the Standards to the Checkerboard for Multiplication

Although the procedure for the Checkerboard for Multiplication seems complex when described in words, in actual practice, students quickly grasp the principles as the teacher demonstrates. (In my own former inner-city classroom, second grade children readily learned to multiply large numbers using the checkerboard.) This tool's power to teach children very abstract algorithms through manipulatives on a place value board is invaluable to the contemporary teacher of mathematics.

The principles of Standard 1: Number and Operation (National Council of Teachers of Mathematics, 1998) can easily be applied to the Checkerboard for Multiplication for multiplication.

- When students represent a multiplicand of 543 as five hundreds, four tens, and three ones, they are showing their understanding of place value in our base ten number system. Similarly, representing 3 x 7 by seven 3-bars or two tens and one unit on the Checkerboard for Multiplication reinforce students' concepts of number.
- Students who use the Checkerboard for Multiplication find out what multiplication means concretely as they gather bead bars to represent parts of the problem. Determining partial products and summing them for a final product teaches students how the steps relate to each other.
- The Checkerboard for Multiplication is a computational tool for students to learn

multiplication in a hands-on way. It makes use of place value to allow the student to concentrate on the multiplication operation occurring between two digits of the problem.

A Tool for Teaching the Pencil and Paper Algorithm Teaching students the multiplication algorithm with the Checkerboard for Multiplication has many advantages:

- Engaging materials: The board and beads are colorful and attractive. Students are motivated to use them.
- Reinforcement of standard paper and pencil techniques: Steps of multiplication on the board are analogous to the steps used in paper and pencil work. The partial products, summing operation, and final product are shown in a very obvious way on the board. The board arrangement guides the student through the problem.
- Focus on place value: The board controls and keeps track of the place values of all of the digits. Preservice and inservice teachers introduced to this technique often remark that this is the first time they understand the reason for the zero in the ones place of the second partial product.
- Concrete, dynamic, hands-on materials: The beads allow the student to see what the multiplication <u>means</u>. Counting beads allows students who have not memorized the multiplication facts to work more advanced problems while practicing the facts. Seeing the problem concretely promotes deep understanding. The board builds the background for more abstract work. It does not become a "crutch" because once students truly understand the concepts; they are ready to abandon the board in favor of faster paper and pencil computations.

Case Studies of Four Seventh Graders

Participants

The study was conducted with four Euro-American seventh grade students (2 male, 2 female) enrolled in an Academic Intervention Service (AIS) math program at a public middle school (not a Montessori school) in a small town in rural central New York State. Students who have low scores on the yearly state mathematics tests qualify for AIS services. The AIS classes are designed so students will have individualized time with their teachers to help students learn skills that they have trouble retaining. More one-on-one time allows teachers to answer questions and also use more concrete, hands-on materials to make instruction more effective.

Research Design

We examined student performance on identical assessments before and after eight weeks of work with the multiplication checkerboard and kept notes on student reactions and insights during the lessons. Students took the pretest in January without the use of manipulatives. As an intervention, they participated in lessons using the multiplication manipulatives described in this article for all of February and March for a total of eight weeks of work, using the checkerboard at least once a week for 40 minutes during the last four weeks. Students worked in pairs with the checkerboard materials. The assessment questions are shown in Table 2. Additionally, we asked students to rate, at the time of the pretest and posttest, their enjoyment, knowledge, and confidence regarding multiplication on a ten-point scale with 1 indicating a low score or level and 10 representing a high score or level. Student took the posttest in April, again without the use of manipulatives. The lessons are described in

Table 3. Pseudonyms will be used to describe the four participants.

Table 2. Assessment questions.

Problem	Question	Equation
1	Make a drawing and write sentences to explain how multiplication works and what it means in this problem.	7 x4 =
2	Make a drawing and write sentences to explain how the multiplication works and what it means in this problem.	23 x41
3	Name the terms in this problem.	52 x39 2028
4	Look at this problem that has been solved correctly. Explain why the zero is there.	19 x 42 38 760 798
5	Solve these problems showing all your work.	62 x73
		128 x459

Table 3. Description of lessons.

Week	Lesson Description
1	Students worked with the colored bead bars to become familiar with the color coding for 1-9. They used bead bars to represent simple multiplication facts.
2	Students learned to identify place value positions on the checkerboard. They learned to represent different multi-digit numbers by placing bead bars in different square of the checkerboard.
3	Student learned to solve simple problems using the multiplication checkerboard. Students learned to regroup the bead bars.
4-8	Students worked complex multi-digit problems using the checkerboard while simultaneously recording the problem, partial products, and final product on paper.

Results

Table 4 shows pretest and posttest scores of the four seventh grade participants and the aspects of each problem that were scored. On the pretest, students scored poorly with scores ranging from a high of fifty percent correct to a low of twenty-five percent correct. All students made large gains on the posttest with scores ranging from ninety-five to one hundred percent correct. In the following sections we discuss individual student performances.

Table 4. Pretest and Posttest Scores of Students

Problem Number and Aspect		Shawna Pre	Shawna Post	Edward Pre	Edward Post	Cindy Pre	Cindy Post	Martin Pre	Martin Post
1	Multiplication shown by grouping or array	1	1	1	1	0	1	0	1
	Multiplication is commutative	1	1	0	1	0	1	0	1
2	Multiplication shown by grouping or array	1	1	1	1	0	1	0	1
	Multiplication is commutative	1	1	0	1	0	1	0	0
	Correct form for solution	0	1	0	1	1	1	1	1
	Error-free solution	0	1	0	1	1	1	0	1
3	Multiplicand identified	0	1	0	1	0	0	0	1
	Multiplier identified	0	1	0	1	0	1	0	1
	Product identified	0	1	0	1	0	1	0	1
4	Zero is a placeholder	1	1	1	1	1	1	1	1
5	Problem 1: Regrouping in calculating partial products correct	0	1	1	1	1	1	1	1
	Problem 1: Summing of partial products correct	1	1	1	1	1	1	0	1
	Problem 1: Use of zero as placeholder correct	1	1	1	1	1	1	1	1
	Problem 1: Multiplication facts correct	0	1	1	1	1	1	1	1
	Problem 1: Final product is correct	0	1	1	1	1	1	0	1
	Problem 2: Regrouping in calculating partial products correct	0	1	0	1	0	1	0	1
	Problem 2: Summing of partial products correct	1	1	1	1	0	1	0	1
	Problem 2: Use of zero as placeholder correct	1	1	1	1	1	1	0	1
	Problem 1: Multiplication facts correct	0	1	0	1	0	1	0	1
	Problem 2: Final product is correct	0	1	0	1	0	1	0	1
Total Score		9	20	10	20	9	19	5	19
Percent Correct		45	100	50	100	45	95	25	95

Student A. Shawna performed fairly well on the pretest, being able to explain multiplication by groupings ("You got seven groups of four or four groups of seven."), by drawing arrays, which are orderly arrangements of items in equal-sized rows, and mentioning the commutative property. However, she was able only to make drawings for the simpler multiplication fact, not the two-digit multiplication problem. Although she was not able to name any of the multiplication terms, she was able to identify the zero in the ones place of the second partial product as a placeholder. When solving the final two multiplication problems, Shawna made errors in writing the correct multiplication facts and regrouping. On the posttest, Shawna showed that she had learned the multiplication terms and was now able to solve multi-digit problems correctly. After the intervention, Shawna scored well (a "3") on the New York State Mathematics Assessment and was able to graduate from academic intervention services. To be removed from AIS classes, a student must score at least a "3" out of a range from 1-4 on the NYS Math Assessment. Shawna came full circle in math class. Shawna often stopped by the math lab after school to use the multiplication checkerboard. She wanted to be able to maintain what she had learned on the checkerboard so it was fresh in her mind, as she was preparing for final exams that take place in June. Her attitude towards math had changed a great deal and the result was higher marking period grades. She gained the confidence that she needed to succeed at a high level, maintaining a quarterly average in math class of 80 or above.

Student B. Edward also showed quite a bit of knowledge of multiplication on the pretest, though not scoring quite as well as Shawna. He also was able to show the simple multiplication fact as both groupings (4 groups of seven) and as an array of seven boxes in four rows. However, for the double-digit problem, he also was not able to make a drawing. Although not able to name the terms, he explained zero's role as a placeholder ("to fill in the ones place.") and was able to solve the first of the last two problems correctly, making fact and regrouping errors on the second problem. On the posttest, Edward showed improvement in all areas of previous deficiency. He solved the final two problems correctly.

Edward was a student that appeared to lack academic motivation. Prior to using the checkerboard, he did not work to his potential. After taking part in this experiment and being introduced to the multiplication checkerboard manipulative, he made substantial improvements in math class. Edward was promoted to the next grade level after generating quarterly average percents that ranged in the upper 70's. Some of his other core area teachers also commented on his improved attitude in their classes, stating that his work become more proficient and he contributed to class discussions showing more confidence in what he had to say. Being able to work with Edward one-on-one with the multiplication checkerboard was rewarding for both him and his teachers.

Students quickly grasp the principles as the teacher demonstrates.

Student C. On the pretest, Cindy was not able to explain how multiplication works concretely through words or drawings ("Multiplying works because it's easier to use than adding"). She did not know the multiplication terms, but was able to identify the role of zero as a placeholder: "The zero is there because it tells you that you started a new place in the problem." She solved the first of the final two problems correctly, but made fact, regrouping, and addition errors in the second problem. On the posttest, Cindy was able to draw groupings and arrays to represent multiplication problems and had learned the terms for the parts of a multiplication problem. She was also able to solve the final two problems correctly.

Cindy still receives Academic Intervention Services. A large part of her difficulties was that she was not a good test taker.

She made a tremendous amount of progress after her use of the multiplication checkerboard manipulative. She became more confident because she looked forward to taking tests since she understood the material. Using manipulatives gave her a deep understanding of the concepts of multiplication. Therefore, after using the manipulatives, she was more confident and her test anxieties began to diminish because her test scores reached the low 70's.

Student D. Martin performed most poorly of the four students in this study on both the pretest and posttest, but exhibited a large amount of growth. On the pretest, he was able only to identify zero as a placeholder and solve parts of the first of the final two equations correctly, making errors in addition of the partial products. He did not solve any part of the second equation correctly. Martin made a lot of progress as evidenced by his ability to concretely portray multiplication problems as groupings and as arrays. He was able to name all the multiplication terms and solve the first of the final two problems correctly. His errors in the second problem were in not using correct multiplication facts.

Through the use of the multiplication checkerboard Martin was able to gain a better understanding of the underlying mathematical concepts. Martin was a huge fan of the multiplication checkerboard from the start. He was immediately attracted to the colors and was interested in sharpening his math skills. Although expending much effort throughout the eight weeks of lessons, Martin still needed to receive Academic Intervention Services. However, he had shown marked improvement. His score on the state math assessment was only two points below where he needed to be in order to graduate from the AIS Program.

Table 5. Student responses to attitude survey.

Question	Student							
	Shawna		Edward		Cindy		Martin	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Circle a number on the rating scale to indicate how much you like multiplication.	3	7	4	7	4	7	8	9
Circle a number on the rating scale to indicate how well you understand how multiplication works.	10	10	10	10	10	10	7	8
Circle a number on the rating scale to indicate how confident you are in doing multiplication.	5	8	10	10	8	9	8	10

Teacher's Observations. At first some of the students balked at having to learn to use the checkerboard, commenting, "Why can't we just use pencil and paper?" They were hesitant to participate because they didn't want to admit that they needed help multiplying. But rapidly, they were won over by the attractive materials and concrete representations of mathematical ideas. As the lessons on the multiplication checkerboard progressed, all of the students were actively participating. In fact, most days, students entered my classroom and asked if we were going to use the multiplication checkerboard that day. Because the checkerboard and the bead bars are colored and appealing, the students felt that they were playing a game. Soon students were remarking, “Why didn’t we learn this in elementary school?” “This is like a game,” “I can see the regrouping,” and, “It’s easy to use the beads because of the colors.” After the first lessons, all four students were enthusiastic about using the materials.

Student Attitudes. Table 5 shows student attitudes measured at pretest and posttest. Students were asked to select the rating from 1 (low) to 10 (high) that best reflected their perceptions. A neutral score would be 5 to 6, therefore scores seven or above can be interpreted as positive and those below five as more negative. All students improved, or maintained the highest level (for those who rated an aspect as a “10” on the pretest) of their reported feelings and perceptions of multiplication from the beginning to the end of the study. In particular, the first three students (Shawna, Edward, ad Cindy) moved from not liking multiplication to liking it, while Martin simply increased his liking for the topic.

Conclusion

All of the students in the study made progress towards becoming more proficient when they multiply. The poorer-performing students in the study (Cindy and Martin) were able to be successful using the multiplication checkerboard because they were able to count beads to determine the multiplication facts. The stronger students acquired an understanding for the regrouping, which allowed them to slow down and work through a problem without making careless errors. The multiplication checkerboard, color-coded bead bars, and the numeral cards were an effective way to teach multiplication to our students. Additionally, and very importantly for our struggling students, the manipulatives described here motivated our students and increased their interest and confidence in mathematics.

These Montessori mathematics materials were an effective way to teach multiplication in our public school study of seventh graders needing extra assistance in understanding multiplication. Our study is limited in that it was a small case study of four individuals. However, we hope that readers will consider trying these materials with their students. Students of elementary school teachers, special education teachers, and middle school academic intervention services math teachers may benefit by using these materials to understand multi-digit multiplication.

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