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First Year Pre-service Teachers' Mathematical Content Knowledge: Methods of Solution for a Ratio Question

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In this article, pre-service teachers' mathematics content knowledge is explored through the analysis of two items about ratio from a Mathematical Competency, Skills and Knowledge Test. Pre-service teachers' thinking strategies, common errors and misconceptions in their responses are presented and discussed. Of particular interest was the range and nature of common incorrect responses for one whole-whole ratio question. Results suggested pre-service teachers had difficulty interpreting a worded multi-step, ratio (scale) question, with errors relating to ratio and/or conversion of measurement knowledge. These difficulties reveal underdeveloped knowledge of mathematical structure and mathematical connections as well as an inability to deconstruct key components of a mathematical problem. Most pre-service teachers also lacked knowledge of standard procedures and methods of solutions.

Introduction

The research reported in this article is part of a larger study that is investigating primary pre-service teachers' mathematical content knowledge. This research began by collecting primary pre-service teachers' responses to Mathematical Competency, Skills and Knowledge Tests, over a period of four years. Many first-year pre-service teachers completing a Bachelor of Education degree had difficulty with a range of questions from this test that they completed in 2008 during the first semester of their course. For this article a sample of first-year pre-service teachers' responses to two ratio questions were analysed. This study identified difficulties with worded multi-step ratio items. A sample of answers for a ratio (scale) question was then used to identify and describe these pre-service teachers' incorrect responses. The dimensions from Chick, Baker, Pham and Cheng's (2006) Pedagogical Content Knowledge [PCK] Framework were used to analyse pre-service teachers' answers to a ratio (scale) question that 89% of the cohort (n=297) answered incorrectly.

Background

Mathematical Content Knowledge for Teaching

Teacher knowledge has been described as complex and consisting of many facets (Chick, Baker, Pham, & Cheng, 2006; Fennema & Franke, 1992). Schulman (1986) was one of the first to express the complexities of the major categories of knowledge teachers need and use: content knowledge, pedagogical content

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knowledge and curricula knowledge. "Content knowledge is the amount and organisation of knowledge in the mind of the teacher" (Schulman, 1998, p. 9). Pedagogical content knowledge is a second kind of content knowledge and relates to the ways the teacher represents and formulates their content knowledge when teaching (Schulman, 1998). Since Schulman's initial research many other studies have continued to specify the different types of mathematical knowledge needed for teaching. The focus of this paper is mathematical content knowledge.

Ma (1999) described teachers' mathematical content knowledge as thorough understanding of mathematics which has breadth, depth, connectedness and thoroughness, referring to this as *Profound Understanding of Fundamental Mathematics* (PUFM). Schoenfeld and Kilpatrick (2008) also described knowing school mathematics in depth and breadth as an important dimension that proficient mathematics teachers require. They describe proficient teachers of mathematics as having broad knowledge, knowing multiple methods, as well as a deep knowledge of mathematics. Proficient teachers know the curriculum and how the ideas develop from conceptual understanding. These definitions begin to provide understanding of the complexities within mathematical content knowledge and the knowledge unique to teaching.

Teachers' mathematics content knowledge can be described as *Specialised Content Knowledge* [SCK] (Ball, Bass, & Hill, 2004). A recent model identified three sub categories of mathematics content knowledge: common content knowledge, specialised content knowledge and horizon content knowledge (Ball, Thames, & Phelps, 2008). "Common content knowledge is held by an adult who can use a method to solve a mathematical problem whereas specialised content knowledge is mathematical knowledge that is unique to teaching" (Ball, Thames, & Phelps, 2008, p. 399). Ball, Thames and Phelps (2008) compared common content knowledge and specialised content knowledge and believe an effective primary teacher needs more not less mathematical knowledge than the average adult.

Excellent teachers of mathematics must know the mathematics appropriate to the grade level and primary mathematics subjects they teach (Australian Association of Mathematics Teachers [AAMT], 2006; Schulman, 1986). A teachers' mathematical content knowledge can be demonstrated in many ways. They use their mathematical knowledge in teaching for identifying a range of solutions and mathematical connections when they are working with students, planning lessons and evaluating students' work (Aubrey, 1997; Ball, Thames, Bass, Sleep, Lewis, Phelps, 2009; Ball et al., 2008; Chick, Pham, & Baker, 2006; Leinhardt & Smith, 1985; Ma, 1999; Schoenfeld & Kilpatrick, 2008; Stylianides & Stylianides, 2006). In their work effective teachers can draw on a range of mathematical knowledge such as: procedural knowledge, procedural fluency, conceptual knowledge and mathematical connections (Ball & Bass, 2003; National Curriculum Board, 2009).

Advanced content knowledge is understanding of the mathematical horizon and is evident when the teacher demonstrates a broad understanding of how mathematical ideas connect. They know how mathematical ideas connect to the

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mathematics they are teaching and to the mathematics curriculum that their students will travel to in future years (Ball et al., 2009; Ball et al., 2008). A teacher who demonstrates knowledge of the mathematical horizon has peripheral vision, for example, they know the questions to prompt student understanding of mathematical proofs, know when to assist learning as well as when to be patient allowing the student to work through the problem independently (Ball & Bass, 2009; Ball et al., 2009; Ball et al., 2008).

A Pedagogical Content Knowledge [PCK] Framework

Two studies have investigated teachers' PCK using a framework for analysing pedagogical content knowledge (Chick, Baker et al., 2006; Chick, Pham et al., 2006). The first study introduced the PCK Framework and investigated teachers' pedagogical content knowledge related to their teaching of decimals (Chick, Baker et al., 2006). The second study used the PCK Framework for investigating teachers' strategies and use of pictorial models when they solved number items on a questionnaire (Chick, Pham et al., 2006). The PCK Framework lists the many facets that can be used and demonstrated by teachers; it can be applied to data collected from teachers about teaching and content knowledge using questionnaires, interviews or classroom observations (Chick, Baker et al., 2006, p. 298).

Chick, Baker and colleagues' (2006) PCK Framework consisted of three categories – Clearly PCK, Content Knowledge in a Pedagogical Context, and Pedagogical Knowledge in a Content Context. The second category, *Content Knowledge in a Pedagogical Context*, summarises and focuses on mathematics content knowledge in mathematics teaching and lists five descriptors of mathematical content knowledge that teachers could demonstrate (see Table 1). In this paper we apply the second category, *Content Knowledge in a Pedagogical Context*, to pre-service teachers' written responses to a ratio question from a Mathematics Competency, Skills and Knowledge Test.

Table 1.
Content Knowledge in a Pedagogical Context (Chick, Baker et al., 2006 p. 299).

PCK Category	Evident when the teacher…
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts

PCK Category	Evident when the teacher…
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a mathematical problem

Chick and colleagues (2006) identified five dimensions of *Content Knowledge within a Pedagogical Context* that may be evident during teaching. The five descriptions of the dimensions are closely linked to the literature and previous studies of mathematical content knowledge (see Table 1). The first dimension, *Profound Understanding of Fundamental Mathematics* uses Ma's (1999) definition of PUFM, that is, exhibiting breadth and depth. *Deconstructing Content to Key Components* is evident when a method is used to check an answer or estimation is used to check an answer and a teacher can identify the critical elements of the concepts (Chick, Pham et al., 2006). This dimension relates to Ball's (2000) earlier discussion of teaching, as a teacher needs to be able break up their understanding of mathematics and interpret students' correct and incorrect methods. *Deconstructing Content to Key Components* identifies the structure for understanding concepts and could be important for the teacher to use when assisting students to develop mathematical knowledge and when correcting misconceptions. Classroom mathematics needs to be unpacked for students with the use of models to represent the concepts (Ball & Bass, 2003). This is demonstrated in the dimension *Mathematical Structure and Connection* when the teacher makes connections between concepts and topics when teaching (Chick, Pham et al., 2006). Combining all of these first three dimensions may lead to a teacher demonstrating the knowledge that is special to teachers: specialised mathematical content knowledge.

The final two dimensions could be used to describe common content knowledge and are needed but not necessarily special to teaching. Teachers can use *Procedural Knowledge and Methods of Solution* (Chick, Pham et al., 2006) for solving a mathematical item in the classroom. However, when having to elaborate on the mathematics and compare different student responses the teacher would need to draw on their specialised content knowledge. To identify if a teacher was working towards demonstrating knowledge at the mathematical horizon they would be required to display all of these dimension's (Table 1) with flexibility during their daily mathematics teaching.

Complexities of Ratio and Proportion

Proportional reasoning is understanding of the multiplicative relationship between variables in proportional situations and using this knowledge to solve problems (Dole, 2008). Lamon (2007) defines proportional reasoning as the skill

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to understand the relationship in comparison problems where four values are given and/or, missing value problems where three of the four values are given. These skills are taught during the middle years of schooling and take time to develop (Department of Education and Early Childhood Development, 2009). Understanding of multiplication and division is needed for proportional reasoning which is further developed along with understanding of fractions, decimals, scale drawing and ratio (Dole, 2008). Proportional reasoning is needed for understanding many areas of the middle school curriculum and ratio is a foundation for building knowledge of situations of comparison (Dole, Clarke, Wright, Hilton, & Roche, 2008; Steinthorsdottir & Sriraman, 2009).

Ratio is the comparison between two quantities. There are three common ratio comparisons: *ratio*, part-part (for example, one part cordial and four parts water or 1:4); *proportion*, part-whole (for example, one of the five parts is cordial or 1/5); and *scaling*, whole-whole (comparing wholes to wholes, where 1cm on the map equals 1 250 000 cm on the ground) (Suggate, Davis, & Goulding, 2006). The language and range of types of ratio situations may be a reason for confusion when working with ratio situations (Suggate et al., 2006).

Recent studies indicate poor understanding of ratio and proportional thinking in the middle years of schooling. A study of year five girls in Iceland found the students developed the basics of ratio understanding but were slow to develop flexibility to solve ratio and proportion problems (Steinthorsdottir & Sriraman, 2009). An Australian study of middle years numeracy students found they had difficulty when recognising the applicability of ratio and proportion and in justifying this mathematically (Siemon, Virgona, & Corneille, 2001). Behr, Harel, Post and Lesh (1992) suggested a lack of basic understanding of ratio in the early years may cause difficulties in the middle years of schooling which results in an absence of ratio knowledge in adult life.

Within the literature, it is evident ratio and proportion are complex topics and identified as difficult for teachers to teach and students to learn (Behr et al., 1992; Lamon, 2007). Mewborn's (2001) review of the literature of elementary teachers' mathematics content knowledge, found they could perform computations successfully but many were unable to demonstrate conceptual understanding across a range of mathematical topics. Fourth grade teachers in an American study were not able to explain the difference between fractions and ratios with reference to part/whole and part/part concepts (Leinhardt & Smith, 1985). Middle years teachers also had difficulty with conceptual understanding. Responses to a survey conducted to investigate content knowledge related to proportional reasoning found middle years teachers lacked conceptual knowledge of the topic (Dole et al., 2008). Dole et al. (2008) note that teachers and students need to understand proportional reasoning as it is used regularly in many domains of the middle years mathematics curriculum. Mathematical content knowledge including knowledge of ratio concepts therefore, should be developed during pre-service teacher education.

Studies of pre-service teachers do not often include ratio and have focused on other areas of number knowledge (Southwell & Penglase, 2005). Further research of pre-service teachers' primary mathematical content knowledge of ratio, will extend understanding of their errors and common misconceptions. A misconception is a genuine misunderstanding, when a student uses an alternate concept; an error occurs when a student misreads a question or makes a slip when calculating (Drews, 2008). A review of the literature suggests further research is required to support future teaching and learning of fractions, ratio and proportion (Lamon, 2007).

Methodology

This study used simple descriptive statistics and content analysis to analyse first year pre-service teachers' responses to items in a Mathematical Competency, Skills and Knowledge Test completed in 2008. All tests (N=297) were collected at the end of first semester, after pre-service teachers had completed an education unit that introduced them to mathematical knowledge for teaching and teaching mathematics in a primary school.

The mathematical test was designed to assess pre-service teachers' mathematical knowledge of number, fractions, decimals, percentage, ratio, space, area, volume, measurement, chance and data. There were 49 test items that ranged in difficulty but generally comprised items examining knowledge and understanding of mathematics for years 5 to 8, that is, Victorian Essential Learning Standards (VELS) Levels 4 to 5, (Victorian Curriculum and Assessment Authority, 2007). A wide range of mathematical sources were used for these items, including items from past Year 5 and Year 7 AIM Tests (Victoria Curriculum and Assessment Authority, 2007). Some test items were conceptually orientated to explore pre-service teachers' responses to mathematical principles, ideas and representations of mathematical concepts used in primary mathematics teaching. All items required short answers using words or symbols (numbers) and recording of working out was encouraged.

The two most difficult items, that is, the two items with the smallest proportion of correct responses, were identified. Both these items (Item 38 and Item 49) concerned knowledge of ratio and measurement as well as multiplicative and proportional reasoning (see Table 2). For these items the pre-service teachers needed to identify the missing part of the proportion and interpret a whole-whole scale ratio problem. We were interested to analyse the nature of pre-service teacher errors and misconceptions for the most difficult items. When recording their answer for the most difficult item, most pre-service teachers wrote a response with no working out. The second most difficult item, (Item 49, ratio (scale)) was selected for in-depth analysis because the responses to this item provided samples of working out. This item, the ratio (scale) question also drew our attention as the responses illuminated a range of errors and potential misconceptions that could be analysed.

A sample of 20% of the total cohort (N=297) was then randomly selected for further analysis of responses to Item 49, the ratio (scale) question. This sample size was determined after a total of 62 (20%) responses for the ratio (scale)

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question had been tallied and the answers analysed showed that a likely pattern had emerged as no new responses were identified. At this point the responses had been exhausted and the proportion of incorrect errors was similar to that for the whole cohort. A conclusion was made that the sample of 20% would provide a likely representation of the pattern of responses.

Responses for the ratio (scale) question from the random sample of test papers were analysed and compared to reveal patterns of answers and reasoning. These responses were grouped into nine categories: correct responses, six categories of common errors or misconceptions, various other answers, and no answer recorded. For each category the percentage of responses was recorded and the error was explained using a descriptor as shown in Table 3. The second category of Chick, Baker, et al. (2006) PCK Framework, *Content Knowledge in the Pedagogical Context*, was used to interpret pre-service teachers' thinking and to make sense of the range of methods and solutions this cohort used and the nature of mathematical content knowledge demonstrated.

Table 2.
First-year pre-service teachers' responses to Item 38 and Item 49 (N=297)

Item number	Question	Correct Responses *N* (%)	Incorrect Response *N* (%)	No Answer *N* (%)
Item 38	3200 square centimetres is the same as __ square metres	28 (9%)	266 (90%)	3 (1%)
Item 49		33 (11%)	244 (82%)	20 (7%)

Results and Discussion

Summary of Responses to the Most Difficult Test Items

Descriptive statistics were used to tally the responses to the Mathematical Competency, Skills and Knowledge Test that were completed by all first year pre-service teachers during first semester of 2008. For each item the total number of correct answers, incorrect answers and no responses were recorded. Ranking of the test data items indicated that Item 38, a ratio (area) question, and Item 49, a ratio (scale) question, were the two most difficult items for the cohort (*N*=297, see Table 2).

Overall, the cohort found Item 38 (ratio (area)) to be the most difficult test item since only 9% provided a correct response. For Item 38 the pre-service teachers had to convert square centimetres to square metres. To do this they had to recall, or calculate the ratio for units of area measurement and then find the missing value where one of the values in the ratio was provided. The pre-service teachers needed to know that one square metre is equivalent to a square measuring 100 centimetres by 100 centimetres, that is, 10000 square centimetres. They had to use this knowledge to solve a whole-to-whole ratio problem demonstrating that 3200 square centimetres is the same as 0.32 square metres.

The second most difficult question was Item 49 a ratio problem relating to the use of a scale. For Item 49 the problem required knowing how to find the *missing value*, where three of the four values in the ratio were provided (Lamon, 2007). Only a small proportion of the cohort, 11%, answered Item 49 correctly (Table 2). A similar amount, 10% of the sample of 62 pre-service teachers selected from the cohort for further analysis recorded the correct answer of 75 km (see Table 3). These pre-service teachers were able to use a *method of solution* for a whole-whole ratio problem relating to scaling on a map and were able to convert the distance of 6 centimetres to the real distance 75 kilometres. The scale used in this example was 1:1 250 000 that is 1cm on the map represents 1 250 000 cm (or 12.5 km) on the ground (Table 2).

A teacher is expected to have knowledge of the content they teach. If they demonstrate knowledge at the mathematical horizon more, not less knowledge of ratio should be demonstrated. To answer both these ratio items (Item 38 and Item 49) knowledge of number, measurement and working mathematically would be the domains a teacher would draw on. When comparing the data of the 62 pre-service teachers in the random sample selected for analysis there were only three pre-service teachers who answered both ratio questions correctly (Item 38 and Item 39). This indicates an overall difficulty of ratio knowledge and lack of connections when comparing two different problems for the same topic.

When engaging in problem solving to achieve the correct answer, mathematical knowledge and understanding is needed (Even & Tirosh, 1995). Working mathematically and problem solving were required to solve both ratio questions as the pre-service teachers needed to draw on their known concepts and processes to implement a procedure or derive a method of solution, especially for Item 49. Both items could also be identified as a multiple step problem (Siemon et al., 2001) as knowledge of both ratio and measurement were required to successfully answer them.

A teacher will draw on their mathematical content knowledge to assist their students in years 5 to 8 build breadth and depth for many topics across the mathematics curriculum. VELS (2007) provides statements of mathematical understanding students are expected to learn and when this should occur. Within Victorian schools VELS is used by teachers for planning teaching of the content their students should know. With reference to ratio, students will first learn to use decimals, ratios and percentages to find equivalent representations of common fractions, for example 3:4 equals 6:8 equals 75% (VELS Level 4 in Years 5/6). Then the students learn to find equivalent representations of ratios, for example a subset:set ratio of 4:9 can be expressed as 4/9. Next they develop

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understanding of ratio as both whole-whole comparisons and part-whole comparisons, finding integer proportions of these, including percentages, for example 4:6 equals 40%:60% (VELS Level 5, Years 7/8).

The results of these two ratio items suggest these first-year pre-service teachers lacked ratio content knowledge that is similar to students at Level 5. The majority of these pre-service teachers, most likely, would not be able to answer questions of a year six student working one VELS level above the expected level. They also lacked knowledge of the future understanding their primary students will need to learn. These pre-service teachers' do not demonstrate mathematical knowledge of post-primary year levels or knowledge at the mathematical horizon. In addition, many pre-service teachers were unable to think logically about their responses since they appear not to have used estimation or logical reasoning to think whether their responses were reasonable answers. These pre-service teachers would find it difficult to assist their students that are expected to be developing logical thinking and estimation in the middle years (Department of Education and Early Childhood Education, 2006).

When compared, more pre-service teachers recorded their working out for the ratio (scale) question, Item 49, than the ratio (area) question, Item 38. Therefore, the answers and common errors for the second most difficult test item, a ratio (scale) question, were analysed (Table 3) to identify content knowledge using the PCK framework of Chick, Baker, et al. (2006).

Correct Responses to the Ratio (Scale) Item

Figure 1 illustrates the most common correct method used for solving the ratio (scale) question (Item 49). In this response the pre-service teacher demonstrates knowledge of the common conversions of metric units by choosing to convert the centimetres to metres and then to convert metres to kilometres.

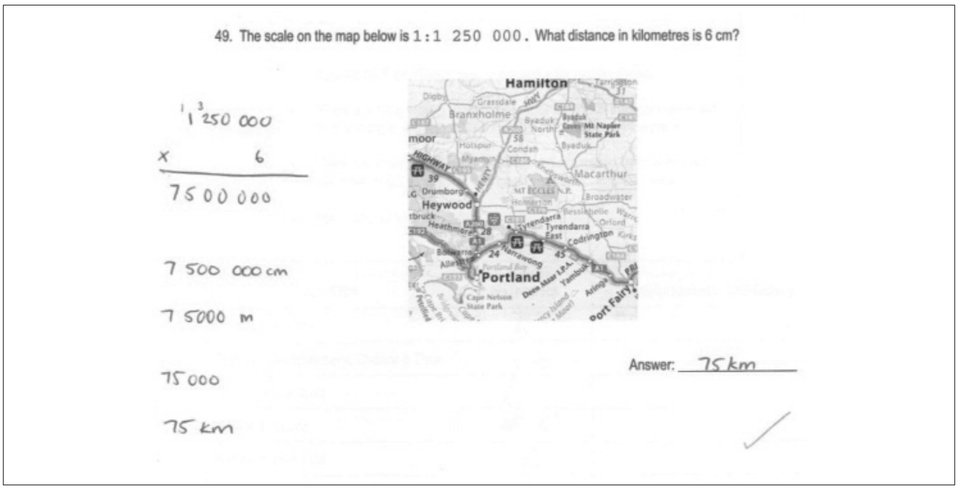

Figure 1. Common correct response demonstrating multiplication.

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Another correct method (Figure 2) used by one pre-service teacher was an algebraic method where the knowledge of an algebraic representation and procedure was used firstly to identify the missing part indicated as x. The pre-service teacher using a common method, the place value algorithm, where they multiplied by 6 to find the missing part and then converted the answer to the correct unit, then completed the remainder of the problem. Also included in this pre-service teacher's response was a graphic organiser used to convert kilometres to centimetres. There were other responses that showed pre-service teachers who drew a similar graphic

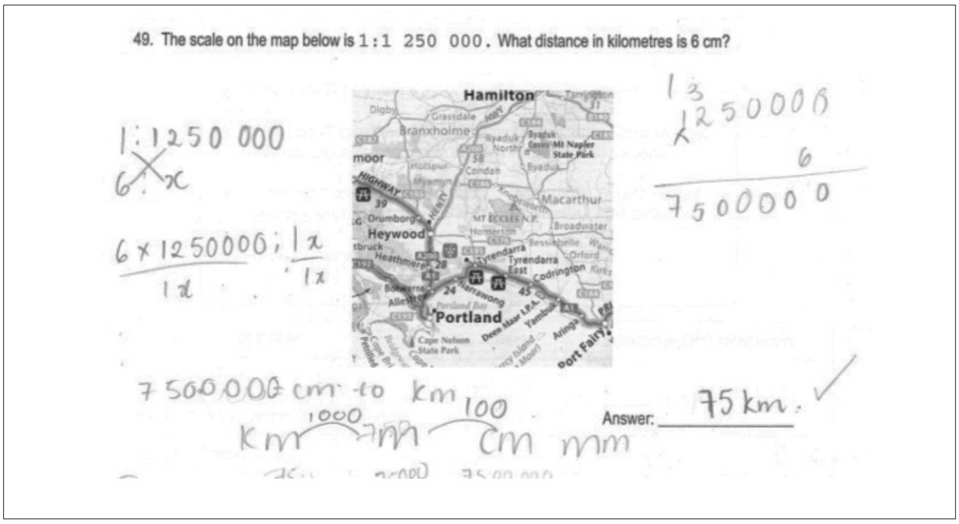

Figure 2. Correct response demonstrating knowledge of algebra, multiplication and measurement unit conversion.

Pre-service teachers who answered the ratio (scale) question correctly demonstrated knowledge of whole-whole ratio and they could complete the steps required to find the correct answer. These pre-service teachers showed conceptual understanding, not merely procedural knowledge, by connecting their knowledge of ratio and measurement. They demonstrated knowledge of measurement facts and the relationship between measurement units.

Incorrect Responses to the Ratio (Scale) Item

It is a concern that many pre-service teachers were unable to correctly answer the ratio (scale) question (89% of 297 students). The random sample of 20% (N=62) also indicated a proportion of 90% were unable to correctly answer this item (Table 3).

For the ratio (scale) question, many pre-service teachers recorded working out, providing the steps used for their method of solution. In the following discussion we analyse a sample of these written, responses to infer the level of mathematical knowledge, strategies demonstrated, errors or misconceptions evident for this question (Item 49). Table 3 provides the distribution of responses to the ratio (scale) question, identifying nine sub-headings to classify and describe the misconceptions. Eight common incorrect responses were identified from more than half (59%) of the sample (*N*=62) and are discussed in greater detail below.

Table 3
*Distribution of responses to ratio (scale) question for a random sample (*N*=62).*

Answer	Responses (%)	Category	Type of misconception
75	10%	Correct	Correct response (no misconception)
7 500 000	27%	Incomplete	Answer not converted from cm to km (correct multiplication used)
7.5	3%	Incorrect measurement conversion	Misconception with relationship between measurement units; some knowledge of ratio; able to multiply
75 000 000	3%	Incorrect measurement conversion	Misconception with relationship between measurement units; some knowledge of ratio; able to multiply
Various (e.g., 15 000 000)	18%	Ratio with large numbers	Misconception of recording of ratio (various answers 15000000, 1500000, 150, 15) and misconception with relationship between measurement units; able to multiply (13%), (e.g., 250 000 x 6 =15 000 000)
1250006	10%	Additive thinking	Additive thinking; no understanding of ratio to show relationship
Various (eg. 0.006)	6%	Common incorrect invented strategies	Common invented strategy of multiplying one by six; no understanding of ratio and measurement of units (various answers 0.00006,0.006,600,6000)
Other various	23%	Other incorrect invented strategies	Range of errors with little knowledge of ratio, multiplication or units of measurement (e.g., 25, 10000000, 1.25, 150666)
Blank	8%	No response	Unknown

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Correctly identify the missing part. Ten percent of the sample (N=62) recorded the correct answer, 75km to Item 49. A further 33% of the sample demonstrated some knowledge of whole-whole ratio. Three groups of incorrect answers 7 500 000 (27%), 75 000 000 (3%) and 7.5 (3%) demonstrated understanding that the problem involved a multiplicative relationship and demonstrated a correct multiplication procedure. Altogether, these responses (43%) showed knowledge of how to find the missing part given three parts. The misconception of 33% of the sample concerned knowledge of the relationship between the units of measurement, not knowing how to convert the answer to the required unit. Ratio knowledge was also demonstrated within other categories of incorrect responses.

Failure to complete multi-step problem: The most common error (7 500 000) was provided by a greater number of students (27%) than the correct response of 75 km (10%). The pre-service teachers who recorded 7 500 000 correctly multiplied but did not convert their answer from centimetres to kilometres. Their answer was incomplete rather than incorrect. These pre-service teachers correctly interpreted the ratio, understanding the representation and its multiplicative structure, and correctly used multiplication to find the equivalent ratio (6:7 500 000). However, they did not complete the problem indicating that they did not attempt to make sense of the solution and so ignored the connection with the real world context. Figure 3 provides an example of the error 7 500 000 and common working out. These pre-service teachers were unable to correctly interpret or decode the problem.

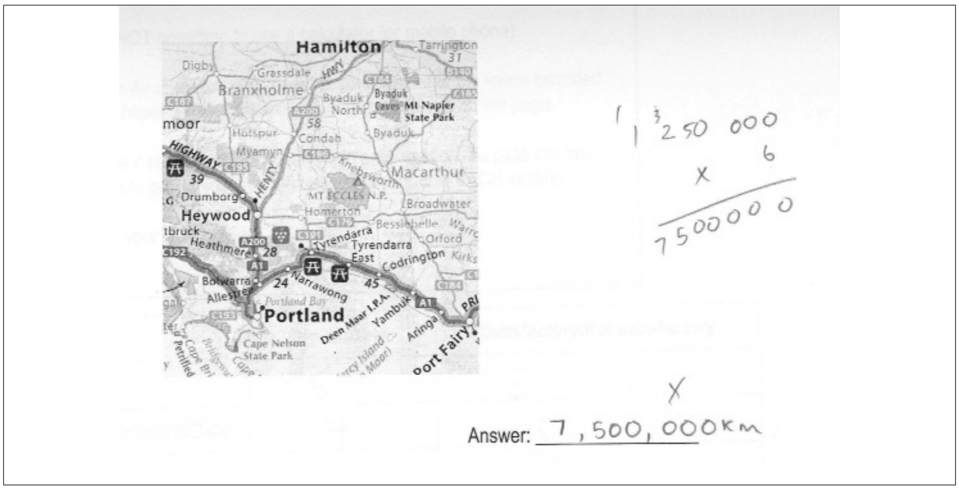

Figure 3. Incorrect answer 750 000 km

Working with formulae and solving multi-step problems has been identified as an area of concern and difficulty for Years 5 to 9 students in recent research (Siemon et al., 2001). Children find worded problems challenging and have difficulty transforming the calculation into numbers (Lawton, 2008). Cockburn (1999) found children can make mathematical errors when they are unable to understand the language used in the question. These pre-service teachers may also have made errors due to one or all of these reasons or due to other misconceptions. Their error may have been avoided had the question been worded more clearly: What distance in kilometres is represented by 6cm?

Repeated addition. Figure 4 shows one student who used a repeated addition method for multiplication, to reach the same incorrect answer (7 500 000). This was not a common method used by pre-service teachers. This method shows that while pre-service teachers may understand whole-whole ratio, they did not know the multiplicative facts or have confidence in using them to complete the multiplication algorithm correctly. This method should not be a preferred method for modelling the answer when working with students to develop understanding of ratio.

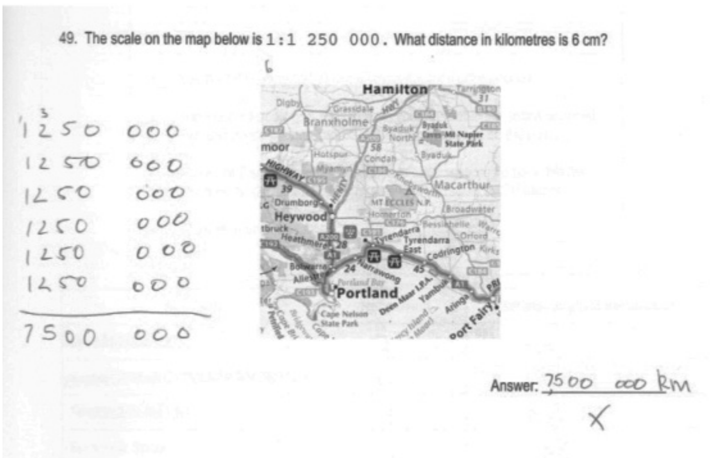

Figure 4. Incorrect response of 7 500 000 km.

Misinterpretation of ratio representation. The second largest grouping of incorrect responses (18%) included various answers. Two examples of responses in this category were 15 and 15 000 000 (Table 3). These pre-service teachers demonstrated correct knowledge of multiplication of 250 000 by 6 (1 500 000 demonstrated by 13%) but had a misconception of how ratio involving large numbers was represented. Pre-service teachers in this group used a range of methods to convert their answer to kilometres and most converted the units incorrectly. Those who gave 15 for their answer were able to correctly convert the units of measurement.

Figure 5 provides a common example of this misconception. It is difficult to ascertain from the test data why this error occurred. These pre-service teachers

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may have seen the ratio as 1:1 followed by 250 000, and made an error when interpreting the question or in their thinking related to proportional reasoning. Another source of error may have been due to a lack of understanding of place value of the digits for 1 250 000. Some students may not read the digits as one million and two hundred and fifty thousand. The spacing of the digits in the ratio question may have been confusing to some pre-service teachers who may be more familiar with commas when recording large digits. This is consistent with Cockburn (1999) who claims that errors can also occur if the presentation of the task is inappropriate.

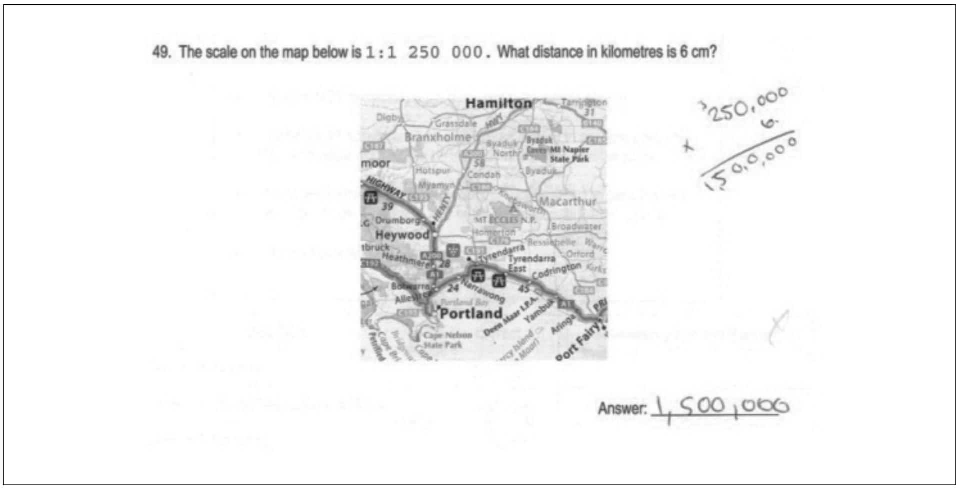

Figure 5. Interpretation of ratio representation: 1 500 000.

Misunderstanding of the structure of measurement units. There were two similar errors that a small percentage of the sample demonstrated, they were 7.5 (3%) and 75 000 000 (3%). In these responses the missing part using multiplication was completed correctly but an error occurred with the method for conversion of measurement. In Figure 6, the pre-service teacher converted the units in the ratio first before finding the missing part. They, like many others in this category, recorded 12 500 metres as equal to 1.25 kilometres rather than 12.5 kilometres. They did not correctly divide by 1000 and had difficulty with knowledge of conversion of measurements. Lawton (2008) suggested reasons children and adults find ratio questions difficult was because a ratio problem may relate to fractions, decimals, percentage or measurement problems. The results from this study support Lawton's theory as many pre-service teachers lacked knowledge of the relationship between measurement units.

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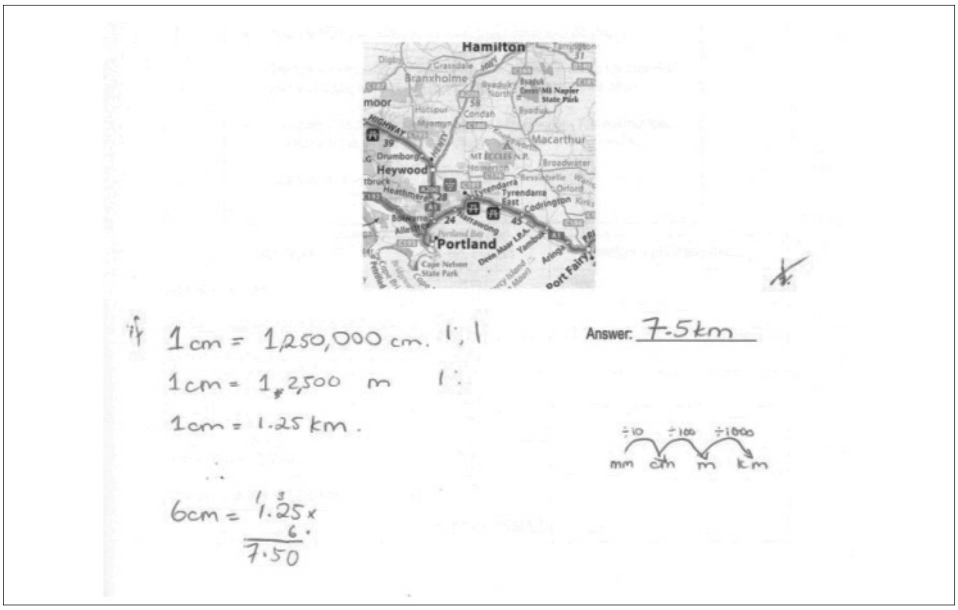

Figure 6. Incorrect response 7.5 km.

Additive thinking. Another grouping of misconceptions was additive thinking (2%), for example adding 6 to 1 250 000 (Figure 7). A common mistake in ratio is the use of additon or subtraction (Suggate et al., 2006). But, this error was not very common for this sample of pre-service teachers, although a misconception with significant implications for the depth and breadth of their knowledge of mathematics in the middle years.

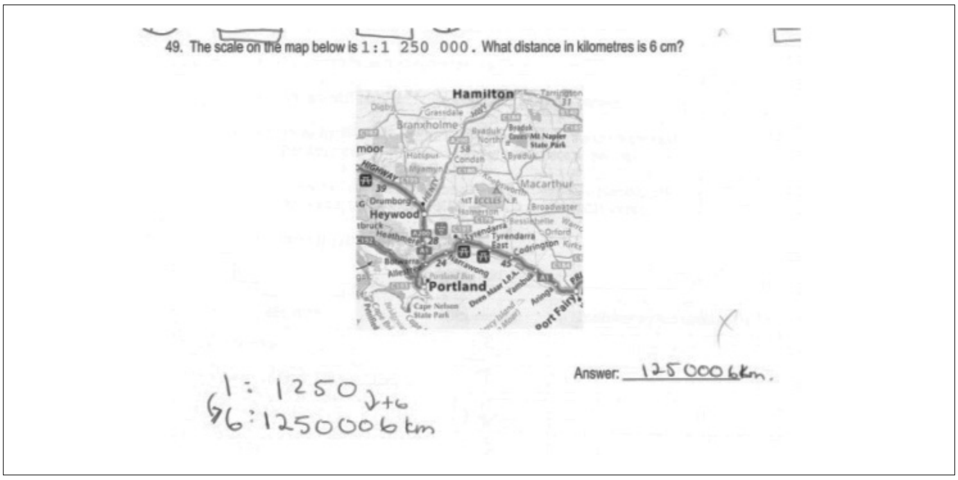

Figure 7. Additive strategy for ratio: 1250006 km.

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Common incorrect invented strategies. When investigating the range of errors a group of common invented strategies was identified for a small percentage of responses (8%). These invented strategies assumed the ratio was 1:1 and they multiplied one by six and then used a step (or steps) for converting the answer to kilometres providing a range of different responses such as 0.00006, 0.006, 600 and 6000. An example of the method that results in an answer of 600 is illustrated in Figure 8 where a graphic organiser was used for converting units of measurement, showing conversion of centimetres to metres as multiplication by 10 then metres to kilometres (k) as multiplication by 100. An answer of 600 km is recorded. This example illustrates little understanding of ratio or relationships between measurement units.

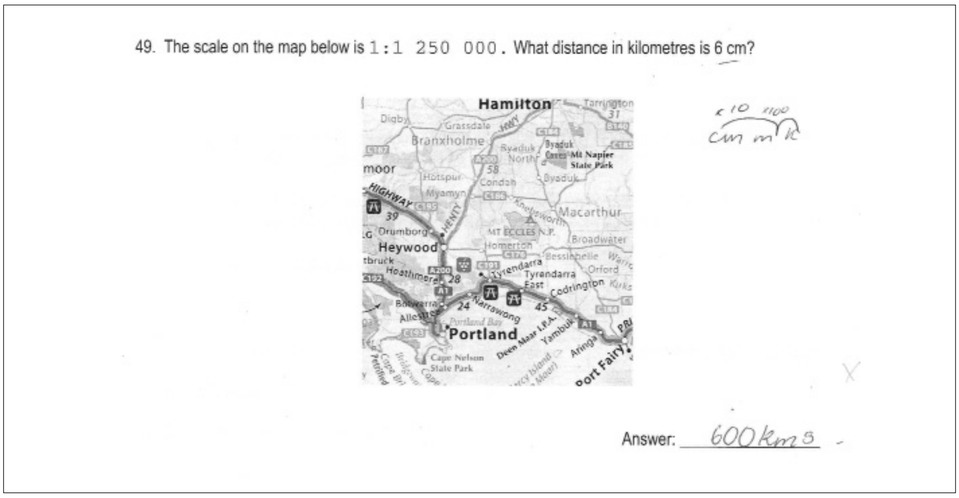

Figure 8. Common invented strategy, multiply one by six.

Other incorrect invented strategies. About one quarter (23%) provided various incorrect answers, such as 10 000 000 km, 250 000 km, 650 km and 25 km. Also, a small percentage (8%) of pre-service teachers recorded no response. These results were of concern, as nearly one third (31%) of pre-service teachers, demonstrated little or no mathematical content knowledge of the whole-whole ratio concept. Many of these answers used invented strategies and demonstrated a lack of both ratio and measurement knowledge.

There may be a combination of reasons for the high proportion of incorrect responses. These pre-service teachers may have misread the question, had difficulty interpreting how to solve the problem or they may not have known the mathematics needed for answering the question. It should also be noted that this question was the last test item and this may have affected the way the participants answered it. For example, if they thought they had failed the test they may not have given a lot of time to answering the last item and guessed the answer.

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Making sense. In the original printed format of the test, Item 49 was presented with the map represented to scale, that is, one centimetre on the map was one centimetre on the test page. On the map some distances were recorded, for example Heywood to Hamilton was 58 km and measured less than 6 cm, the distance that they needed to convert to solve the problem. Therefore a response of 7 500 000 km was not a reasonable answer. There was no thinking about what 7 500 000 km might look like in relation to the map that was shown. The distance across Australia from Melbourne to Perth is 3 424 km and highlights the absurdity of the answer 7 500 000 km. When thinking about problems the use of estimation with ratio can be used to find approximate answers (Reys, Lindquist, Lambdin, Smith, & Suydam, 2004). These pre-service teachers were not able to think about the reasonableness of their answer.

Interpretation of Responses with Reference to Content Knowledge in a Pedagogical Context

After marking the ratio (scale) question (Item 49), a lecturer could make the generalisation that these pre-service teachers had difficulty answering a ratio question and needed to consolidate their knowledge of ratio and proportion. Yet, about one third of the random sample group (33%) recorded an error with the conversion of measurement but demonstrated knowledge of this ratio situation. They could multiply by 6 to find the missing part when given three of the parts. This highlighted an important factor. An incorrect answer may provide evidence of ratio knowledge while the misconception relates to gaps in other mathematical knowledge or connected knowledge (Ma, 1999). This error demonstrated that pre-service teachers have not made connections between topics, they lacked understanding of *Mathematical Structure and Connections* and/or they lacked a *Method of Solution* since they did not know the conversion from centimetres to kilometres or could not derive a method to do so correctly.

Pre-service teachers who gave correct responses demonstrated *Procedural Knowledge* and/or *Methods of Solution* (Chick, Baker et al., 2006). Interviewing and asking the pre-service teachers to explain their answers and reasoning would have provided additional data for further classification of their mathematical content knowledge, errors and misconceptions. Some pre-service teachers, most likely those who correctly answered both Item 38 and Item 49, may have been able to demonstrate the other three descriptors of content knowledge; *Profound Understanding, Deconstructing Content*, and/or *Mathematical Structure and Connections* (Chick, Baker et al., 2006) but the data does not provide sufficient evidence.

Pre-service teachers who successfully answered Item 38 and Item 49 were able to interpret and solve multi-step whole-whole ratio problems suggesting the capacity to deconstruct content. If the pre-service teachers demonstrated more than one method to achieve their answer they would have demonstrated *Deconstructing Content to Key Components* and/or *Mathematical Structure and Connections* (Chick, Baker et al., 2006). Most of the pre-service teaches were not able to identify the various mathematical components within the ratio (scale) item. They did not understand the concept of ratio and or conversion of measurement units. They were not able to deal with the multi-step problem and were unable to demonstrate one correct method of solution.

Profound Understanding of Fundamental Mathematics would have been demonstrated if a wide variety of ratio examples were answered and explained using more than one method, demonstrating understanding of the three common situations of ratio: part-part, part-whole and whole-whole (Suggate et al., 2006). The structure of Item 49 and Item 38 discussed in this article did not provide the detail needed for analysing pre-service teachers' knowledge using all descriptors of Content Knowledge in a Pedagogical Context (Chick, Baker et al, 2006).

Conclusion

The research reported in his paper has provided information on first-year pre-service teachers' knowledge of whole-whole ratio in two contexts (Item 38 and Item 49). Both questions were identified as difficult and only three pre-service teachers (5%) from a random sample (N=62) were able to solve both whole-whole ratio items. They demonstrated a connection of their ratio mathematical content knowledge with measurement to answer both multi-step problems and were more likely to be working towards demonstrating knowledge of *Mathematical Structure and Connections*. Understanding ratio in these items suggests knowledge of mathematical structure since ratio describes the relationship between two objects. By completing both of the multi-step problems these pre-service teachers appreciated the possible contexts in which ratio has been used.

Further discussion focused on Item 49, a ratio (scale) question, 89% of first year pre-service teachers (*N*=297) answered incorrectly. A random sample was selected (*N*=62) for in-depth analyses of the range of incorrect responses. Only 10% of the sample of pre-service teachers answered the ratio (scale) question correctly. They were able to *deconstruct the problem* into its component parts. However, there were 43% of pre-service teachers who understood whole-whole ratio, even though they may not have completed this question correctly. This is suggested because the digits in their responses, such as 7.5 (3%) and 75 000 000 (3%), indicated correct use of multiplication and the misconception related to incorrect measurement knowledge of converting the answer from centimetres to kilometres. The most common incorrect response 7500 000 (27%) occurred because the missing part was left in centimetres. It was not converted to kilometres in order to answer the question fully and proof reading of the question may have identified this error.

More than half of the random sample (59%) of pre-service teachers recorded common misconceptions for Item 49. A further third (31%) recorded a range of other errors or misconceptions (23%) or provided no response (8%) to Item 49. A range of common misunderstandings were identified suggesting concerns for primary pre-service teachers' mathematical content knowledge. This range of incorrect responses provided evidence that the majority of pre-service teachers clearly lacked specialised mathematical content knowledge as well as common mathematical content knowledge of the concept of ratio and measurement units. A major concern for a small group of pre-service teachers was the attempted application of additive thinking as a method of solution (2%). These pre-service teachers were failing to demonstrate multiplicative understanding, a necessary foundation for understanding proportion and mathematical structure more generally.

The analysis of responses found that the vast majority of pre-service teachers in this study needed to develop their understanding of *mathematical structure* and their capacity to *deconstruct content to key components* and make connections between mathematical concepts within the problem context. The *Content Knowledge in a Pedagogical Context* category from the PCK Framework (Chick, Baker et al., 2006) assisted analysis of data from one written test item to identify the types of mathematical content knowledge pre-service teachers can demonstrate. The use of an assessment tool that includes the different categories of mathematical content knowledge within a PCK framework, such as Chick's et al. (2006), can assist teachers and teacher educators when focusing on the mathematical content knowledge needed for teaching. Identifying the categories within mathematical content knowledge will also provide pre-service teachers with understanding of their development of specialised mathematical content knowledge needed for teaching. It is hoped primary teachers develop their knowledge of content so they possess sufficient knowledge at *the mathematical horizon* to discuss problems and are aware of the range of strategies students will bring to tasks (Sullivan, Clarke, & Clarke, 2009).

Knowing and using mathematics for teaching entails making sense of methods and solutions, different from one's own (Ball et al., 2004). However, the majority of these pre-service teachers will need to first work on understanding the whole-whole ratio (scale) concept. Next they can develop their mathematical content knowledge of other ratio concepts as well as strengthen their knowledge of measurement units such as conversions for distance and area. They could also be encouraged to explore different methods for working out ratio problems. Solving and discussing similar problems and various methods of solution will help these pre-service teachers to practice solving ratio problems, strengthening and deepening understanding they can then use in their work as teachers.

Pre-service teachers can use their errors as a positive learning experience (Ryan & McCrae, 2005/2006). The incorrect responses may be shared with pre-service teachers during tutorials to identify misunderstandings and foster mathematical content knowledge. They can compare their misconceptions to those of children to identify when similar errors occurred. Identifying and correcting common misconceptions will firstly consolidate the pre-service teachers' content knowledge. Secondly awareness of their own misconceptions may assist pre-service teachers to develop knowledge of common misconceptions which is a key component of PCK (Chick, Baker et al., 2006) and *specialised mathematical content knowledge* (Ball et al., 2004).

To extend the findings from this study the first year pre-service teachers could be asked to explain their working out for Item 49 during an interview with the researchers. Other interview questions could include asking for a different method to check their response as well solving a selection of other ratio problems designed to identify the connected knowledge across the topic and evidence of *specialised content knowledge*. The problems should be designed to include the three common ratio comparisons of ratio, proportion and scaling. In particular items can be designed to further explore pre-service teachers' connections within ratio and how their mathematical content knowledge links to other mathematical topics. Consideration should also be given to researching tasks for these topics and their connections that provide pre-service teachers with an opportunity to demonstrate each of the dimensions of *Content Knowledge in a Pedagogical Context*.

As part of the larger study, further analysis will explore pre-service teachers' knowledge of a range of concepts, the connections and breadth of understanding using responses from a collection of Mathematical Skills and Knowledge Tests. The tests will be used to identify other topics pre-service teachers may have difficulties with. Further research of pre-service, primary teachers' knowledge of mathematics on the horizon, for example knowledge of concepts and skills in the middle years, will illuminate further the breadth and depth of their mathematics content knowledge for primary teaching.

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