LINEAR EQUATIONS: equivalence = success

Wendy Baratta
University of Melbourne
<wbaratta@ms.unimelb.edu.au>

Introduction

The ability to solve linear equations sets students up for success in many areas of mathematics and other disciplines requiring formula manipulations. There are many reasons why solving linear equations is a challenging skill for students to master. One major barrier for students is the inability to interpret the equals sign as anything other than a 'do something signal'. To succeed in algebra, students must transition from this operational view of the equals sign to an equivalence, or relational, view (see Table 1 for definitions).

Table 1. Interpretations of the equals sign.

Interpretation of equals	Definition
Operational	The equals sign is interpreted as the 'do something' signal or as 'now find the answer'. For example, 33 + 5 =
Relational/equivalence	The equals sign is interpreted as the 'left expression is equivalent to the right expression'. For example, 3 + 5 = 8 or 8 = 3 + 5

In this paper we look at two possible approaches for assisting students to overcome this barrier. We first consider manual strategies and a related concrete model, and then discuss suitable uses of technology.

The key questions addressed in this paper are:

- How do we equip students with the knowledge and skills to successfully solve linear equations?
- What role does technology play in assisting students with the development of these skills and knowledge?

Although our main focus is on linear equations of the form $Ax + B = Cx + D$, where A, B, C, D are integers, some simpler cases are discussed.

What does the literature say?

The ability to view the equals sign as a sign of equivalence can be classified as a threshold concept of algebra. A threshold concept is defined by Mayer and Land (2005) as "a new way of understanding, interpreting, or viewing something" (p. 1) without which the learner could not progress.

Although students without a relational view of the equals sign cannot adequately interpret the linear equations, they can still experience some success. In Linsell's (2009) study on students' strategies for solving linear equations it was found that strategies requiring only an operational view of equals, for example counting techniques and working backwards, were used effectively by low to middle attaining students, on one-step and two-step problems, such as $4 + n = 7$ and $3n + 2 = 14$ respectively. Unfortunately these strategies were not able to be generalised to solve more complex equations of the form $Ax + B = Cx + D$, where only the high attaining students were successful and used either a guess and check strategy or transformations ("same to both sides").

Table 2. Examples of strategies.

Strategy	**Example**
Counting technique	$4 + n = 7$: 4, 5, 6, 7 requires the count of three numbers so $n = 3$
Working backwards	$3n + 2 = 14$ $n \Rightarrow 3n \Rightarrow 3n + 2$ $4 \Leftarrow 12 \Leftarrow 14$
Guess, check and improve	Keep improving guesses until the two sides of the equation are equal
Transformation	$2n + 4 = 6n - 28$ $\Rightarrow 2n + 4 + 28 = 6n - 28 + 28$ $\Rightarrow 2n + 32 = 6n$ $\Rightarrow 2n + 32 - 2n = 6n - 2n$ $\Rightarrow 32 = 4n$ $\Rightarrow 32 \div 4 = 4n \div 4$ $\Rightarrow 8 = n$

One further difference between operational techniques, such as counting techniques and working backwards, and transformations is that operational techniques involve students operating on numbers and unknowns, while the transformation strategy requires students to operate on the equation itself (Filloy & Rojano, 1989). It is therefore natural to associate the operational view with operations acting on components of the equation and the relational view with operations acting on equations themselves.

An effective way to develop students' understandings of the relational view of equals, and therefore their ability to operate on equations, is through the balance model (see Figure 1). Filloy and Rojano (1989) found that the model was effective in giving meaning to the abstract ideas. However, they stressed the importance of maintaining balance (pardon the pun) between the use of concrete model and the corresponding algebraic expressions to ensure students do not become reliant on the model. They also acknowledge the difficulty in using the balance model to represent equations containing negative numbers or resulting in a negative solution.

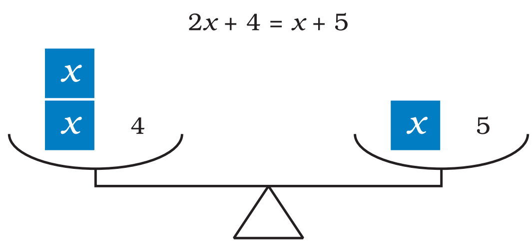

Figure 1. Balance model.

Another approach to developing a relational view is through the use of technology (Ball & Stacey, 2001), in particular with the use of spreadsheets and graphs (see Figure 2). Ball and Stacey believe that graphical and numerical approaches, for example spreadsheets, "give students a better understanding of what a solution of an equation means," (2001, p. 4) and prevents them from losing sight of the purpose of solving linear equations.

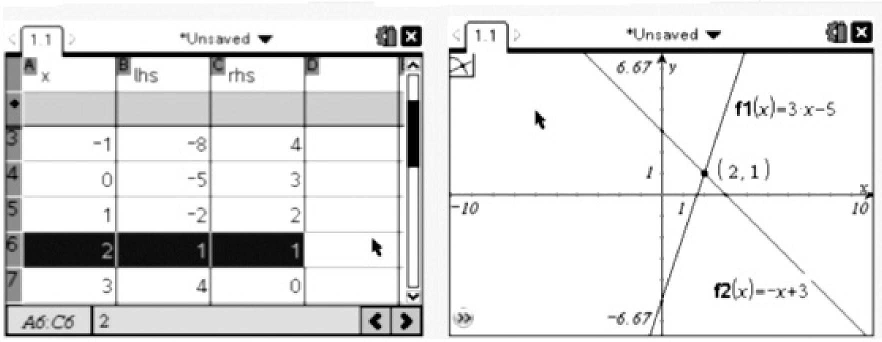

Figure 2. Numerical and graphical solution for 3x – 5 = –x + 3.

In addition to numerical and graphical techniques, standard calculators used by upper secondary students have the capacity to provide exact solutions of linear (and non-linear) equations (see Figure 3). Ball and Stacey (2001) acknowledged that with such technology available it is necessary to reconsider how we approach the teaching of equation solving. However, they stated that due to the "intellectual importance" and "centrality to mathematics" of the skill, students still require a sound knowledge of the fundamental principles of solving equations.

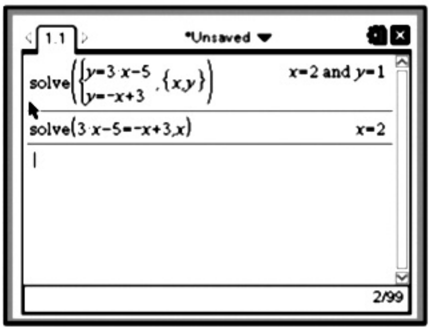

Figure 3. Solution for 3x – 5 = –x + 3 using a CAS calculator

Implications for the classroom

Due to the dominant use of the equals sign it is not surprising that the majority of students view it as an operation. Before beginning the topic of linear equations it is essential to identify students who have solely an operational view of the equals sign. A suitable diagnostic test is provided in the Appendix. Lesson ideas for moving beyond the operational view of the equals sign can be found on the Mathematics Developmental Continuum (DEECD, 2010a) under "Equivalence in number sentences." The lesson ideas are based in the arithmetic world making them accessible to pre-algebra students.

Although the long term aim is to enable students to solve more complex problems using the transformations, our teaching (and consequently students' learning) is greatly benefited by acknowledging students' existing, perhaps operational, strategies. For example, although the less sophisticated strategy of working backwards does not enable students to solve linear equations of the form $Ax + B = Cx + D$, it still plays an important role in the development of students' algebraic proficiency. The key idea in working backwards is the notion of an opposite operation. Without such knowledge students would not be able to progress to the more complex strategy of transformations. Approaches to teaching the working backwards strategy can be found on the Mathematics Developmental Continuum (DEECD, 2010b) under "Structure of algebraic expressions."

To further cement the notion of equivalence and introduce students to the idea of operating on equations we recommend the use of the balance model, shown previously in Figure 1. Pictures are effective, however a fantastic applet is available for free from the National Library of Virtual Manipulatives (Cannon, Dorward, Duffin & Heal, 2010). The applet contains a pictorial image of the problem and allows the student to work towards the solution by successively choosing operations to act on the equation. Note that the applet uses helium filled balloons to represent negative numbers and can thus represent equations containing negative numbers or resulting in a negative solution, overcoming one of the models original short falling identified by Filloy and Rojano (1989). I would also recommend using the applet before moving to drawn diagrams as with the applet students are only required to identify the step, for example divide both sides by 4, they are not required to carry out the computation. This is most helpful as students can observe the effects and make meaning of operating on the equation before progressing to doing both procedures on their own. For further details on effects of this type of computer-aided scaffolding refer to (Robson, Abell, & Boustead, 2009).

It is important to acknowledge the warnings of Filloy and Rojano, and ensure that students can eventually make sense of the abstract equations without the aid of the model. One approach to achieving this is to encourage students to work concurrently in two columns. Using the first column for the concrete representation, the second for the algebraic.

In addition to the technologies available on the internet, calculators can be used very effectively to show the relational nature of the equals sign. The following word problem can be solved three ways using a CAS calculator (see Figure 4).

> The events manager at the MCG (Melbourne Cricket Ground) employs you to walk around the crowd and sell pizza at the upcoming St Kilda versus Geelong game. He offers you two alternative pay structures. The first has a flat rate of \$30 for the game plus \$0.50 per pizza sold, the second has a flat rate of \$26 for the game plus \$0.75 per pizza sold. How many pizzas do you need to sell so that these two pay structures give exactly the same pay?

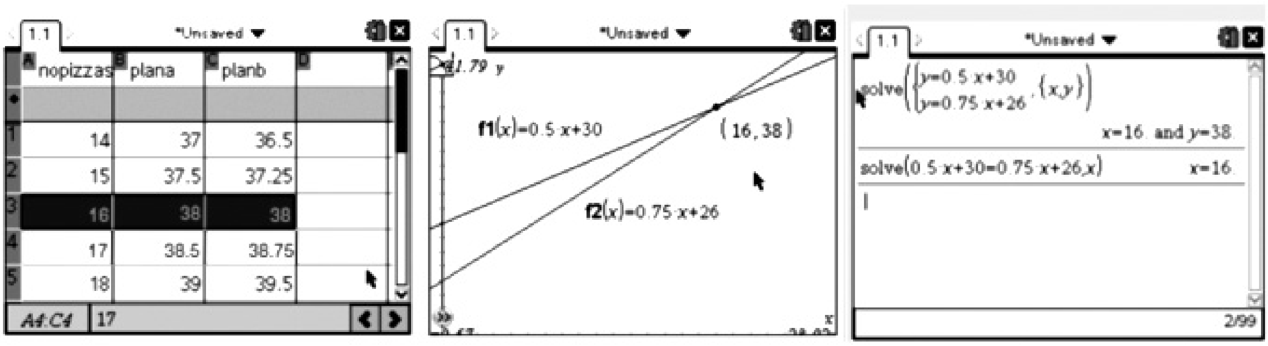

Figure 4. Screen dumps showing numerical, graphical and algebraic solutions of 0.5x + 30 = 0.75x + 26.

The table computes the left and right hand sides of the equation individually and helps students identify the value of x for which both sides are equal. The graph provides a visual representation of the problem. To ensure that this approach assists students' understanding of equals as an equivalence, students must appreciate that the intersection occurs when $0.5x + 30 = 0.75x + 26$. The algebraic approaches only provide an answer to the problem and consequently do not assist students' understanding of the equals sign as an equivalence. However, the algebraic approaches do show students the capabilities of the technology, and are therefore worth including.

The following aspects of the pizza sales problem are beyond the scope of this paper (some references are provided for the interested readers):

- generating algebraic equations from the given information (Mason, Graham & Johnston-Wilder, 2005);
- the difference between unknowns and variables (Küchemann, 1978);
- the benefits of using problems of this style to facilitate an appreciation of the purpose of learning to solve linear equations.

Conclusion

To conclude we summarise our paper by explicitly answering our key questions.

How do we equip students with the knowledge and skills to successfully solve linear equations?

The key to teaching students how to solve linear equations is in extending students' notion of equals from an operation to an equivalence. By first exposing students to this viewpoint in arithmetic, they can start converting their understanding before being introduced to the more complex world of algebra. Models can be used to support this awareness and furthermore assist students' understanding of operating on equations. However, as in the case with any use of models, care must be taken to ensure that the model is only used as a temporary support for developing understanding of the abstract ideas.

What role does technology play in assisting students with the development of these skills and knowledge?

Applets available on the internet and students' calculators can be used to effectively develop students' understanding of the equals sign as an equivalence. Graphical and numerical methods show students both sides of the equation separately and allow meaning to be made of the solution value.

In this paper we have shown how to build students' understanding of a concept (equals sign as an equivalence) to assist their understanding and development of a procedural skill (solving linear equations). This approach can be employed when teaching any procedural skill. Such an approach is more comprehensive and is likely to result is a higher quality of student learning.

References

Ball, L. & Stacey, K. (2001). New literacies for mathematics: A new view of solving equations. *The Mathematics Educator, 6*(1), 55–62.

Cannon, L. O., Dorward, J., Duffin, J. & Heal, E. R. (2010). *National Library of Virtual Manipulatives*. Accessed 23 March 2011 at http://nlvm.usu.edu/en/nav/topic_t_2.html

DEECD. (2010a). *Mathematics Developmental Continuum P10*. Accessed on March 23, 2011http://www.education.vic.gov.au/studentlearning/teachingresources/maths/maths continuum/structure/ST40004P.htm

DEECD. (2010b). *Mathematics Developmental Continuum P10*. Accessed 23 March 2011 at http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathsconti nuum/structure/St45003P.htm

Filloy, E. & Rojano, T. (1989). Solving equation: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19–25.

Knuth, E. J., Alibali, M.W., McNeil, N.M., Weinberg, A. & Stephens, A.C. (2005). Middle school students' understanding of the core algebraic concepts: Equivalence and variable. *ZDM, 37*(1), 68–76.

Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics is School, 7*(4), 23–26.

Linsell, C. (2009). A hierarchy of strategies for solving linear equations. In R. Hunter, B. Bicknell & T. Burgess (Eds), *Crossing divides. Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia*. Palmerston North, NZ: MERGA.

Mason, J., Graham, A. & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London: Sage.

Mayer, J. & Land, R. (2005). Threshold concepts and troublesome knowledge (2): Epistemological considerations and a conceptual framework for teaching and learning. *Higher Education, 49*(3), 373–388.

Robson, D., Abell, W. & Boustead, T. (2009). Scaffolding for learning equation solving. In R. Hunter, B. Bicknell & T. Burgess (Eds), *Crossing divides. Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia*. Palmerston North, NZ: MERGA.

Appendix

The following is from Knuth, Alibali, McNeil, Weinberg and Stephens (2005).

Task 1: Interpreting the equals sign

The following question asks about this statement:

$$3 + 4 = 7$$
$$\uparrow$$

a) The arrow above points to a symbol. What is the name of the symbol?

b) What does the symbol mean?

c) Can the symbol mean anything else? If yes, please explain.

Task 2: Using the concept of mathematical equivalence.

Is the number that goes in the [ ] the same in the following two equations? Explain your reasoning.

$$2 \times [\ \ ] + 15 = 31 \qquad 2 \times [\ \ ] + 15 - 9 = 31 - 9$$