# Positive experiences with negative numbers: Building on students in and out of school experiences

**Kim Beswick**
University of Tasmania
<kim.beswick@utas.edu.au>

The introduction of negative numbers should mean that mathematics can be twice as much fun, but unfortunately they are a source of confusion for many students. As one teacher observed, "All of a sudden there are these concepts in maths where half the numbers are negatives and they've got no idea what it's all about…"

Despite the confusion that working with negative numbers can cause for middle school students, there is evidence that the concept of numbers less than zero is not difficult. In fact the National Council of Teachers of Mathematics (NCTM) (2000) recommended that "in grades 3–5 all students should explore numbers less than 0 by extending the number line and through familiar applications". Simple games that involve losing as well as gaining points or experiences with debt can facilitate quite young children believing that it is possible to have less than nothing and that such amounts are less than zero. In the Australian curriculum (Australian Curriculum Assessment and Reporting Authority (ACARA), 2011) although the continuation of number patterns is mentioned from Year 1 and could lead to the discovery of negative numbers they are not specifically mentioned until Year 6. In the middle years, particularly Years 7 and 8, students need to build on their intuitive understandings in order to use negative and positive integers to represent and compare quantities and extend number properties developed with positive integers to negative integers as well.

Difficulties occur in moving from such intuitive understandings to formal mathematical representations of operations with negative and positive integers. This paper describes a series of activities that were used with a group of 14 middle school students. The approach taken attempted to bridge the gap between students' intuitions, existing mathematical knowledge and recent experience, and the mathematical concepts of operations on negative and positive integers. The activities also needed to cater for the diversity of experience and ability in the group of participating students, 12 of whom were from composite Year 7/8 classes with a further two Year 5 students. The students had been identified by their teachers as likely to benefit from additional work in mathematics but not necessarily highly motivated. The lessons described here were part of a series of weekly classes of approximately one and a half hours, conducted over an 11 week school term, and planned and delivered by the author in conjunction with one of the Year 7/8 teachers.

Linchevski and Williams (1999) pointed out that useful contexts for building mathematical understandings in classrooms need to be familiar to students but must also be transferable in an authentic form to the classroom context. They acknowledged, however, that classrooms are unique in that the expectation of both the teacher and students is that learning mathematics is the over-arching purpose of activities. It is arguably not possible to evoke with complete authenticity in the classroom, non-classroom based experiences. Nevertheless, the challenge for teachers is to build on students' existing knowledge and intuitions as meaningfully as possible.

Figure 1. Thermometer.

## Familiar contexts

The broader community in which the school was set and the students lived provided a range of contexts that were useful in stimulating their initial thinking about numbers less than zero. These were:

1. Negative temperatures: the students were familiar with temperatures a few degrees below 0° C. The photograph shown in Figure 1 was taken in a place with a much colder climate but the students were able to locate familiar temperatures, above and below zero, on the thermometer scale shown.
2. Sea level: an island location made this a readily accessible context. Most students were familiar with heights above sea level (some recalled particular examples that were signposted on roads in the state), and could envisage depths below sea level in the contexts of fishing and diving.
3. Debt: many students were aware of phrases such as "in the red" referring to debt and all were aware of credit cards and hence the idea of bank balances of less than $0.
4. Underground mine levels: Underground mining is a major industry in the state in which these lessons were taught and hence familiar to students who were able to envisage ground level as a reference point or 'zero'. The photo shown in Figure 2 was not taken locally but students could connect it to these understandings.

Figure 2. Sign in lift.

Possible contexts that may be relevant to other students include golf scores and BC time. Although the students in this group had some knowledge of these contexts they did not resonate with them to the same extent as the four listed.

## Number lines

Although we were aware of alternative approaches that offer particular benefits it was decided to use number lines as the basis for the lessons because (1) most of the contexts described above are represented in ways analogous to number lines, and (2) the lessons preceding those on negative numbers had involved working with number lines in relation to rational number operations. Number lines were thus a model that connected well to students out of school experience and with which they were familiar in the context of mathematics lessons.

It is usual in mathematics classrooms to represent number lines horizontally, but at least three of the four contexts discussed above (i.e., 1, 2, and 4) lend themselves most naturally to vertical number lines. Horizontal number lines have advantages in terms of placing the full length in reach when displayed on a classroom wall or whiteboard but vertical number lines marked on floors or pavements or drawn on paper might arguably better evoke familiar contexts. We chose to use horizontal number lines because of the convenience of wall based lines and the fact that the students were already very familiar with horizontal number lines from recent mathematics lessons.

Number line representations of numbers can be ambiguous in that a number has a particular position at a point on the line and can also be thought of as a length, that is, a distance from zero. The students with whom we were working had spent quite a lot of time considering the positions of rational numbers on the number line and had conceptualised 'larger' as a position further to the right. They had also worked on tasks that involved identifying numbers between given pairs in order to build their understanding of the density of the number line (i.e. that in any given segment of the number line there are infinitely many numbers). In contrast to these activities that regard numbers as points on the number line, they had also used the length model of numbers for operations on rational numbers, for example by modelling $4 \times \frac{1}{2}$ as shown in Figure 3.

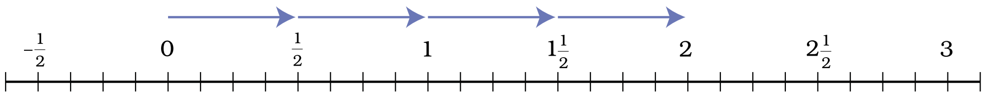



Figure 3. Number line representation of $4 \times \frac{1}{2}$.

## Representing difference on the number line

After considering contexts in which the students had encountered numbers less than zero we briefly reviewed recent number line work. To this end a number line was drawn on the whiteboard with '0' marked approximately in the middle. This was a small change from preceding lessons in which positive numbers had been the focus and '0' had been placed close to the left hand end of the line and was readily accepted by the students. The following problem was then posed:

> In a certain place the minimum temperature was 3° C and the maximum was 10° C. What was the difference between the highest and lowest temperatures?
>
> Show the difference on a number line.
>
> Write a number sentence that gives the difference?

Although the calculation involved was trivial, the task served to connect the operation of subtraction with the number line and the symbolic representations of the difference between the two temperatures, i.e., 10 – 3 = 7. Students repeated the exercise with other pairs of temperatures all 7° apart. These included 0° C and 7° C, –2° C and 5° C, and –4° C and 3° C. The students were able to suggest many others and appreciated that there were infinitely many pairs of temperatures with a difference of 7° C. The set of problems was designed to reinforce understandings already established for positive numbers of the relationship between addition and subtraction and extend these to negative numbers. In addition, we wanted to consolidate the students' conception of subtraction as finding a difference that could be located anywhere along a number line. We believed that such a conception of subtraction combined with a strong visual image of the number line would be helpful in thinking about subtraction involving numbers either side of zero.

## Operations with negative and positive numbers

Because of the students' recent experience, explicit teaching of addition and subtraction of negative and positive integers was designed to build upon existing understandings developed in the context of positive numbers. That is, (positive) numbers can be regarded as distances to the right of zero. Addition is movement in a forward direction and subtraction is movement in the opposite (backwards) direction. The result is the number represented by the distance from the beginning of the first number to the end of the second.

In keeping with ideas that were intuitively reasonable to the students, negative numbers were seen as distances in the opposite direction from positive numbers, that is, to the left. Students were able to appreciate and articulate that this made sense because, for example, subtracting or taking away a negative temperature would amount to increasing the temperature (e.g., The overnight minimum on Cradle Mountain was -5°C. On Mt. Ossa it was -8°C. How much warmer was Cradle Mountain than Mt. Ossa?). Before formalising the model or mandating any particular representations we gave the students the following series of problems:

a. The lift was on the 3rd floor. A man got in and wanted to go five floors higher than that. For which floor was he heading?

b. I had $3 in my wallet. My sister reminded me that I owed her $5. What is my overall balance?

c. The temperature was -6°C. If it got 2°C warmer, what would the temperature be?

d. I had 5 bracelets until I lost 2. How many do I have now?

e. The water level in the dam was 5 cm below its normal level (normal level is 0). As a result of hot weather it dropped a further 2 cm. What was the water level then?

f. I owed my little brother 7 jelly beans. He decided I didn't need to repay 4 of them. How many jelly beans do I have now?

We hoped that, in grappling with the problems, including how to show how they arrived at their solutions, the students would make connections between their existing in and out of class experiences with number lines and numbers less than zero.

Figure 4 shows how Bella (Year 8), used number lines to represent her answers to these problems. She correctly calculated the answers to each question but did not always write a number sentence that matched her answer. It is likely that she used intuitive understandings of the situations to answer the questions before attempting to produce a number line representation. For Question c, the number sentence that she wrote, 6° – 2° = 4°, suggests that she considered the problem in terms of the change in temperature rather than in relation to the particular numbers involved. She was, however, able to place the increase of 2° correctly on the number line and to identify the solution as –4.

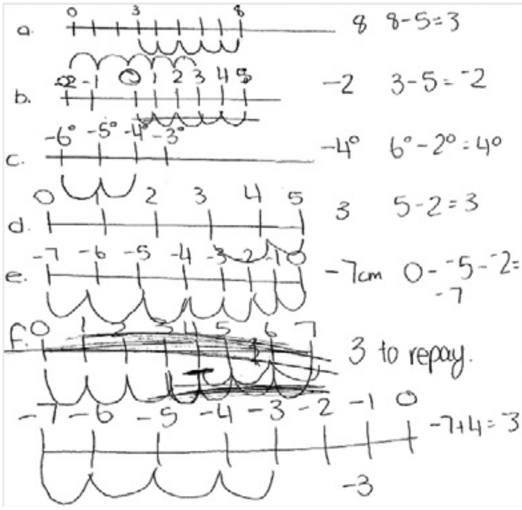


Figure 4. Bella's solutions to the problems.

Bella's number sentence for Question e is incorrect, even though her answer (–7) is correct. She appears to have interpreted "below" and "dropped" as negative numbers but also represented the operation as subtraction. This is the only example of the juxtaposition of an operator and a sign in these work samples and was used similarly rarely by other students in the group. In the vast majority of cases, students avoided the use of negative numbers by interpreting the situations in the problems in terms of addition and subtraction of positive integers. With the exception of Question e, Bella's number line representations all began at the position indicated by the first number in the calculation. That is, she treated this number as a point on the number line and the operation as a move.

In subsequent lessons the number line model for operations on positive numbers was extended more formally to account for negative numbers, and particularly to address the students' difficulty with juxtaposed operations and negative or positive signs. The model can be described succinctly as; positive numbers are distances to the right and negative numbers are distances to the left, addition is moving forward and subtraction is moving backward. The result is the distance from the beginning of the first number to the end of the second and the direction of this move determines its sign: to the right is positive and to the left is negative. Figure 5 shows how a range of calculations can be represented using this model. These diagrams follow the conventions described by Van de Walle (2004) who also illustrated how students can use much less elaborate versions of these diagrams.

We only rarely asked students to produce diagrams, and instead had the students physically step out their solutions in front of a number line drawn on the whiteboard or along a number line marked on the floor. The students were more willing to engage with the ideas modelled in this way rather than using pen and paper. For a positive number they faced right, for a negative left. Addition meant stepping in the forwards direction and subtraction stepping in the backwards direction. The students quite quickly dispensed with the need to do this but were able to articulate their thinking in terms

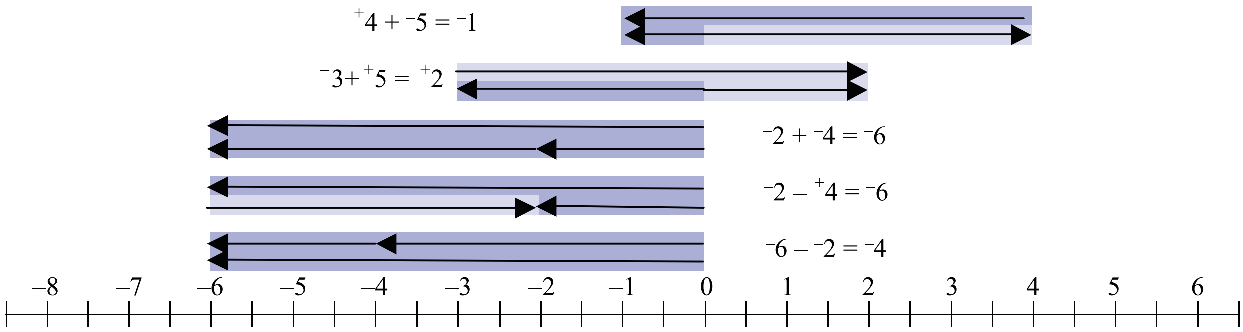



Figure 5. Examples of number line representations of calculations.

of steps on the number line and would occasionally get up and step to check their thinking.

The students also soon noticed that two different signs next to one another are equivalent to subtraction and two similar signs, addition. That is, they arrived at the rules for addition of subtraction of negative and positive numbers and became fluent in their use. Importantly, however, these rules were grounded in and linked to meaningful experiences and the students had strategies for thinking about problems that they could and did go back to when unsure.

## Multiplication and division

Multiplication of any number by an integral amount can be modelled as repeated addition. For multiplication of two non-integral amounts, however, this way of thinking is not helpful and we were aware that students need to conceive of multiplication as an operation distinct from addition (Siemon & Breed, 2006). Nevertheless, because the students were accustomed to thinking in terms of repeated addition (e.g., as shown in Figure 3) we maintained this model for multiplication involving positive and negative integers. Thus, a positive number multiplied by a positive number was modelled as repeated steps to the right thus yielding a positive result; a positive number multiplied by a negative number as repeated steps to the left thus yielding a negative result; and a negative number multiplied by a negative number as repeated backward steps facing left resulting in a distance to the right of zero and hence a positive result.

Division of integers was connected to the relationship between multiplication and division using the measurement understanding of division. That is, an expression such as 6 ÷ 2 was interpreted as “how many 2s make 6” and answered by counting the number of moves of +2 needed to get from 0 to +6. Because these moves are to the right the result is positive. Dividing a negative number by a positive number means finding out how many right facing moves are needed to get from 0 to a position left of 0. Because the moves will need to be backwards the answer is negative. Dividing a positive number by a negative number, requires left facing moves to be made backwards and so the result is negative. Finally, dividing a negative number by a negative number is asking how many left facing moves do I need to make to get to a position left of 0. These moves will be in the forwards direction and hence the result is positive. An example of each of these possibilities is shown in Figure 6.

There is a danger, of course that the rules for these representations can simply be learned and followed as mechanically as the traditional “same

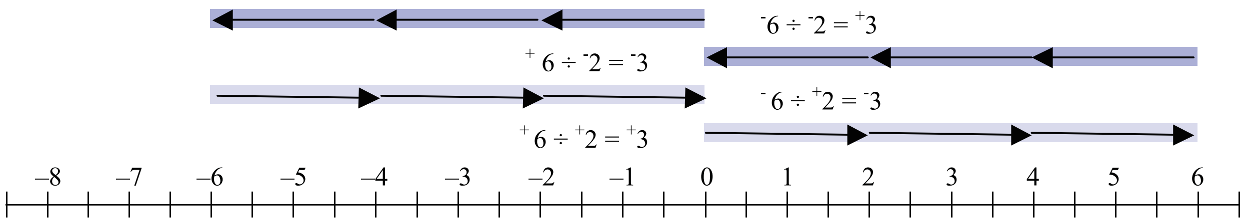


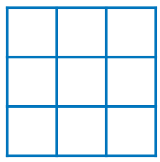

Figure 6. Examples of number line representations of division calculations.

signs make a positive" and "different signs makes a negative". To be of any benefit, students must be required constantly to articulate their reasoning and to justify their solutions. We found the number line model to be less obvious and hence less useful in the case of multiplication and division than for addition and subtraction.

## Some additional problems

Two quite challenging tasks that we offered the students were:

1. Using four –2s and whatever operations you like, write expressions that are equivalent to as many of the integers between 1 and 20 as you can. Some may be impossible.
2. Fill in each of the numbers –15, –12, –9, –6, –3, 0, 3, 6, and 9 so that every row, column and diagonal has the same sum.



The students seemed to enjoy the first task and the fact that it was open-ended meant that all could and did engage with it. The numbers and operations could, of course be varied endlessly to suit the needs and abilities of students. The magic square was particularly challenging. Some of the students took this on and worked on it out of class time. Others were less inclined to persevere.

The task shown below was used as part of an end of term assessment. It required knowledge of order of operations as well as of operations with integers. Nine of the 14 students completed it successfully or made just a minor error, usually in inserting brackets to correct an equation.

A class were asked to write expressions using three operations and at least one negative number that equalled 4. Here are some of the equations that they came up with:

$16 \div 2 \div -2 \times -1 = 4$
$8 \div -1 + -1 \times -1 = 4$
$-7 + 5 \times 2 + 1 = 4$
$-2(-2 + 7) - -2 \times 3 = 4$
$-2 \times -3 - -2 + 4 = 4$

1. Which of the equations are correct?
2. Some of the incorrect equations would be correct if brackets were added. Rewrite these expressions with the brackets that are needed so that they equal 4.

## Reflections on the time spent

The tasks offered and the approach used, were successful in engaging students with the concepts of negative numbers and operations on them. The students had strong intuitions built on experiences, from which they could draw. Although the students became proficient at performing calculations involving negative and positive integers we are reluctant to use this as a measure of success of the approach. As Van de Walle (2004) pointed out, that objective could more easily be achieved by simply providing the standard rules without justification or any attempt to provide them with conceptual meaning. We are also conscious of Linchevski and Williams's (1999) observation that models and the rules for their use can be every bit as unjustified and meaningless. Teaching negative numbers perhaps brings this dilemma into focus more than other topics because the contrast between the speed with which traditional rule giving and approaches aimed at concept development result in procedural efficiency is so great. Furthermore, it is a topic in relation to which conceptual understanding is perhaps less valued than for other topics in which such understanding more clearly forms a basis for further conceptual development and flexible application (understanding fractions as a basis for understanding and working with other forms of rational number would be an example). It would have been beneficial to have spent considerably more time helping students to build links for themselves between their informal understandings and the number line model. Nevertheless, we are confident that these students did not come to see operations with negative and positive numbers as "arbitrary and mysterious" (Van de Walle, 2004, p. 459) and hence the time was well spent.

## Alternative approaches

Other approaches are essentially of two kinds, (1) those based on discrete objects (e.g., counters) that 'cancel' each other out, and (2) those based on extending the mathematical structure of operations with positive numbers to negative numbers. Linchevski and Williams (1999) described two teaching experiments in which they used the first of these approaches. They described how using situations such as the numbers of people entering and leaving a disco recorded using a double abacus (one colour for people entering and another for people leaving), and dice games in which the double abacus is used to keep score can be used to facilitate students' gradual construction of negative numbers and operations on them. Van de Walle (2004) described an equivalent approach using counters of two colours. The same complexities that arise with the number line model in relation to multiplication and division also emerge using counter models and hence they are also less likely to be obvious and helpful for students in relation to multiplication and division than for addition and subtraction.

Liebeck (1990) set out to compare a counters approach, using a card game called 'Scores and Forfeit', with the number line model. She presented test data for two groups of students taught essentially the same lessons but with the different models. The results for the Scores and Forfeits group were significantly higher than for the number line group (Liebeck, 1990). Although there are many variables that are difficult to control in such a small scale comparison (e.g., there were different teachers with significantly differing amounts of experience) the fact that one group performed better on a test that did not require explicit evidence of conceptual understanding again raises the question of the purpose of using models at all. Van de Walle (2004) stressed that finding one or other model (number lines or counters) easier does not mean that that model is best.

Freudenthal (1987, cited in Linchevski &Williams, 1999) concluded that because neither counter nor number line models are entirely satisfactory for building understanding, the introduction of negative numbers should be delayed until students are able to follow and believe arguments based on mathematical structure. This is the second alternative approach. It relies on extending patterns observed in operations with positive integers. For subtraction, this is essentially what the activity described in this paper that was aimed at helping the students to conceptualise subtraction as difference was doing. Patterns leading to multiplication of a positive integer by a negative integer and a negative integer by a negative integer could be:

| | |
|---|---|
| $+3 \times +2 = +6$ | $+2 \times -1 = -2$ |
| $+3 \times +1 = +3$ | $+1 \times -1 = -1$ |
| $+3 \times 0 = 0$ | $0 \times -1 = 0$ |
| $+3 \times -1 = -3$ | $-1 \times -1 = 1$ |
| $+3 \times -2 = -6$ | $-2 \times -1 = +2$ |

Mason (2005) has developed freely downloadable interactive grids that can be used to explore these and other patterns in number and algebra. Division, particularly of a negative number by a negative number, is more challenging because we must avoid dividing by zero. It is best and certainly in keeping with a mathematical structure approach to use the inverse relationship between division and multiplication. For example, if $+3 \times -2 = -6$, then $-6 \div -2 = +3$. For division of a negative integer by a positive integer the following sequence could be used:

$+6 \div +2 = +3$
$+4 \div +2 = +2$
$+2 \div +2 = +1$
$0 \div +2 = 0$
$-2 \div +2 = -1$
$-4 \div +2 = -2$

Arguments based on mathematical structure rely, for their credibility on students believing that mathematics is about pattern and order.

Additionally, approaches that view numbers in terms of their relative sizes, building on the conception of addition and subtraction as concerned with difference to see multiplication and division in terms of stretching and shrinking, have the potential to avoid the need to view of these operations as repeated addition or subtraction.

## Conclusions

The number line approach to negative numbers described in this paper was chosen to suit the particular context, including the students' out of school and recent in school experiences. There is evidence that it enabled students to make some useful links between these experiences and operations with negative numbers and to develop a degree of understanding of the conceptual bases of procedures for performing these operations. Using a counter model could well have achieved these outcomes just as effectively. Regardless of the model, much more time devoted to very carefully helping students to construct links between their existing knowledge and the new ideas would have been beneficial.

In terms of which approach, Van de Walle's (2004) argument is persuasive. That is, students can benefit from multiple approaches particularly when connections between them are a focus of class discussion. A useful combination of approaches could involve either or both of the number line and counter models for addition and subtraction, and a mathematical structure approach for multiplication and division. Regardless of the approach or combination of approaches used, avoiding a situation in which negative numbers and operations with them appear 'magical' and 'meaningless' is of paramount importance.

## Acknowledgement

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From Helen Prochazka's
Scrapbook

Powers of Ten

In 1977 the American couple Charles and Ray Eames, the designers of the stackable plastic chair, made a short film called Powers of Ten.

The nine-minute film is about scale, the universe and our place in it. The camera zooms out from a one-metre-square image of a sleeping man at a picnic. Moving 10 times further every 10 seconds, it takes the viewer to the edge of the universe. The camera then returns to the picnic and zooms in on the man's hand, until the viewer is taken inside a carbon atom.

Powers of Ten Day is held on October 10. It aims to show that an understanding of scale allows us to organise our thinking in terms of size and this awareness will enable us to increase our understanding and tolerance.

If we zoom in on a group of humans to a factor of 10 000 or 104, the physical differences between us become invisible to our eyes. At this magnification, not even skin colour can be distinguished.