A middle-school classroom inquiry:

Estimating the height of a tree

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There is an old saying that "there is more than one way to skin a cat." Such is the case with finding the height of tall objects, a task that people have been approximating for centuries. Following an article in the *Australian Primary Mathematics Classroom* (APMC) with methods appropriate for primary students (Brown, Watson, Wright, & Skalicky, 2011) this article presents two more methods that are appropriate for middle school students who are beginning to learn about the trigonometric functions.

In *The Shape of the Australian Curriculum* (National Curriculum Board, 2009), the foundation for numeracy is seen to be built primarily in the mathematics curriculum but reinforced in other learning areas (p. 10). Measurement, one of the key areas of study in mathematics in the *Australian Curriculum: Mathematics* (ACARA, 2010), is the basis of two investigations presented here to estimate the height of a tree. The investigations involve students using ratio, proportional reasoning, properties of triangles, and the trigonometric function tangent (tan) to calculate an estimate of the height of something that they are unable to measure directly. In the *Australian Curriculum: Mathematics*, development of proportional reasoning features from Year 3 onwards (p. 5) and in Year 9, students are encouraged to "solve problems involving direct proportion" (p. 41). Congruence and similarity of triangles are introduced in mid secondary school (pp. 38-45) and trigonometric ratios are formally introduced in Year 9 when students "use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles" (ACARA, 2010, p. 42).

As background to the investigations presented here, the three upper primary investigations described by Brown et al. (2011) moved from: the informal measurement technique used by Native American Indians, in which they would bend over and look at the tree through their legs; to the more formal use of proportional reasoning through using the estimation technique of "chunking;" and using ratio in a comparison of the lengths of shadows cast by the tree and by a stick of measureable height.

The activities suggested in this article are intended for use with middle school students and it is important to check that students have the necessary prerequisite skills. Maxwell (2006) describes how a teacher in America introduced the use of a clinometer to estimate the height of trees to his middle school students by first reminding the students of important ideas about "units of measure, the process of counting discrete objects, the additive nature of measurement, and the relative precision of instruments" (p. 133). The first investigation presented here was used by the authors during a professional learning session with middle school teachers (in the ARC funded research project "Mathematics in an Australian Reform-Based Learning Environment" (MARBLE)). The second investigation was used with pre-service teachers as part of the Bachelor of Teaching program in the Faculty of Education at the University of Tasmania.

As well as being a hands-on activity for students, the methods used in these investigations have real-life relevance and are employed by architects, planners and surveyors who use the same principles to estimate the height of buildings, and/or land formations, often employing the use of a clinometer for accurate measurements. These methods are also used in the forestry industry where knowing the height of trees is necessary for safe and efficient felling and logging. Trees are used in these investigations as one would expect to see trees in and around most schools. The investigations can be modified so that students measure the height of buildings or other tall structures. For example, Quinlan (2006), with a class of Year 9 students, used the properties of triangles and rectangles, and ratio to measure the height of the classroom.

Investigation 1

Framing the activity

Using knowledge of triangles, estimate the height of a tree.

Data collection

Students need to work in pairs. Each pair needs a ruler and measuring tape.

- Both students stand at the base of the tree.
- Student α walks backwards, from the base of the tree, holding a ruler vertically until the length of the ruler matches the height of the tree (the bottom of the ruler matches the bottom of the tree, and the top matches the top of the tree), represented as Position 1 in Figure 1.
- Student α rotates the ruler 90 degrees so that it is parallel with the ground, noting that since the eye is above the ground the ruler will stick out a bit on one side of the trunk of the tree.
- Student β walks slowly sideways from the base of the tree at right angles to Student α stopping when Student α indicates that he/she is visually aligned with the "top" end of the ruler (refer to Position 2 in Figure 1).

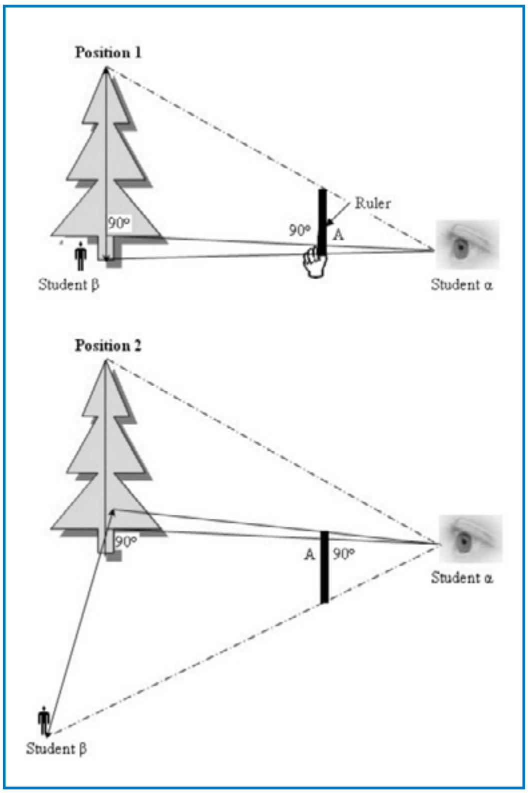

Figure 1. Pictorial representation of how to use the final position of Student β to estimate the height of a tree.

- Student α now measures the distance between the base of the tree and Student β. Adding this measurement to the height of Student α's eye above the ground estimates the height of the tree.

Repeat the process until all students in the class have an opportunity to estimate the height of the tree.

Data representation

Students can use a table to record their measurements.

Student	Distance of student from tree	Height of eye	Height of tree
Ingrid	11 m	1.5 m	12.5 m

Students can create a graph of their estimates and discuss reasons why they are not the same. Reasons might include errors in judgement of the 90° angle, twisting the ruler slightly, inaccuracy in measuring the distance of the student from the tree, or variable estimates of the centre of the tree.

Thinking about the mathematics

How does this method of measuring the height of a tree work?

The success of this method relies on the principle of similar triangles, where two triangles have the same shape but are of different sizes. In this investigation the ruler forms a triangle similar to the triangle made by the tree, as shown in Figure 1 (Position 1) and simplified in Figure 2, which shows two triangles, ABC and DEC, in which A represents the top of the tree, B the point on the tree equal to the height of the eye above the ground, C the observer's eye, D the top of the ruler, E the point where the ruler crosses the line from the eye to the tree, and G and F at ground level. The triangles are similar as angles A and D are the same, angles at B and E are the same (both are right angles), and both triangles share the same angle at C.

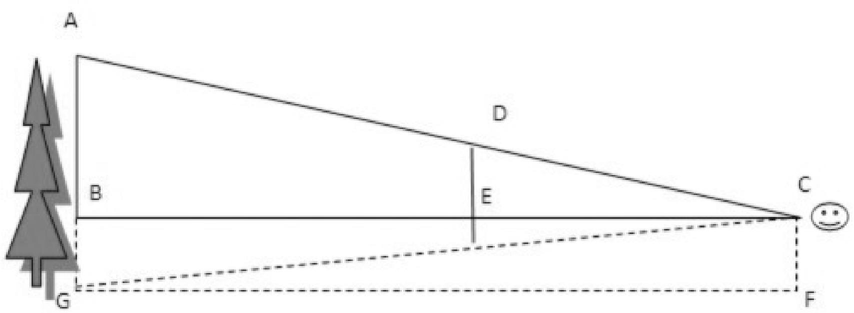

Figure 2. The use of similar triangles to estimate the height of a tree.

Once similar triangles have been established, to measure the height of the tree above the eye height of the person holding the ruler, compare the large triangle ABC with the large triangle formed when Student α turns the ruler at right angles and Student β walks away from the base of the tree (as shown in Figure 1, Position 2, and simplified as triangle BCX in Figure 3); again the triangles are similar. The triangle formed from Student β's final position (indicated as X in Figure 3), to the trunk of the tree and then to the observer (C) and the original triangle ABC both have a right angle at B. The side of the triangle from the observer (C) to the trunk of the tree (B) is shared between the two triangles, and the side B to A and from B to X are also equal. Using the Side Angle Side (SAS) rule, which states that if two triangles have two equal sides and an equal included angle then these triangles are congruent triangles, means that the lengths of all corresponding sides of the triangles will be the same (Rehill, 2010). Therefore, the distance from B to X will be the same as B to A and hence the height of the tree above the eye level of Student α is equal to the distance walked by Student β. All that is necessary is to add the height of the eye (CF = BG) to the distance from the tree. This gives the estimated height of the tree.

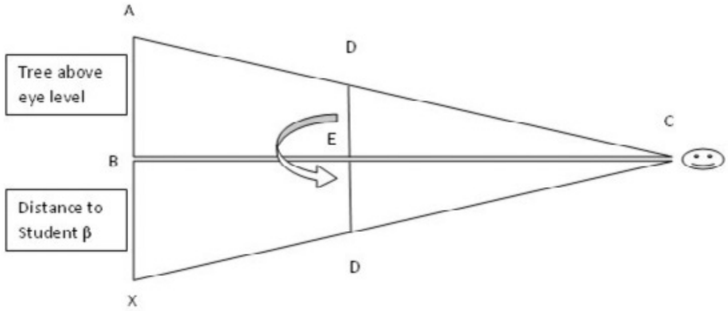

Figure 3. Congruent triangles formed by rotating the ruler 90° from A to X.

Investigation 2

Framing the activity

Using a clinometer, measure the height of a tree.

Collecting data

For this investigation, the authors used an Invicta clinometer ('Invicta' Plastics Limited, England) and instructions are specific to this instrument. The underlying principle is the same, however, regardless of the type of clinometer used, and if necessary students can make their own. Instructions on how to make a clinometer can be found online (for example, TeacherTube, http://www.teachertube.com/viewVideo.php?video_id=21956) or refer to Maxwell (2006).

Students work in pairs and need a clinometer and a tape measure or trundle wheel.

- Student α stands sufficiently far from the tree to be able to see the top.
- Student α holds the clinometer like a target pistol with the arm outstretched and the forefinger on the trigger (refer to Figure 4). The student points the clinometer at the top of the tree ensuring that the line of vision follows the line of the clinometer, and presses the trigger when the top of the tree is in the line of sight. The student keeps the trigger held pressed until the graduated disc stops moving.
- Student α then releases the trigger. Without touching the trigger again, the clinometer can be lowered to allow the student to read the angle of elevation

Figure 4. Using an Invicta clinometer.

(indicated by the "Read Here" arrow). Record the angle.

- Student β now measures the distance from the base of the tree to Student α. Record this measurement.
- Student β measures the height of Student α from the ground to eye-level only. Record this measurement.
- Refer to Figure 5 for a pictorial representation of the measurements needed to calculate the height of a tree, using the following formula:

Tree height = Distance from tree × tan(Angle) + Eye height

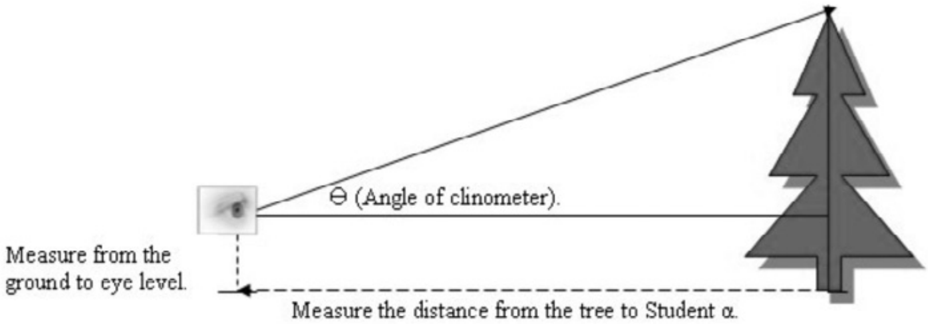

Figure 5. Pictorial representation of the measurements needed to calculate the height of the tree.

Data representation

Students can use a table to record their measurements and estimate of the tree's height. Again students can create a graph of their calculated heights and discuss the variation in their measurements. Are the estimates closer together this time?

Student	Distance from tree	Angle	Eye height	Tree height Distance from tree × tan(Angle) + Eye height
George	29.5 m	48°	1.6 m	29.5 × 1.11 + 1.6 = 34.4 m

Thinking about the mathematics

How does a clinometer measure the height of a tree? Why is the angle of elevation important?

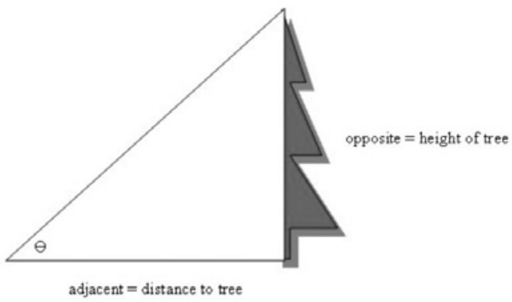

Figure 6. Angle looking at the height of a tree from the ground.

The method using a clinometer goes one step further than the earlier method in that it translates the proportional reasoning in the triangles into the trigonometric function tangent (tan). The tangent is defined for an acute angle of a right angle triangle as "opposite/adjacent." Figure 6 shows this in a simple form for the angle looking at the height of a tree from the ground. In this case, because

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{height of tree}}{\text{distance to tree}}$$

then

$$\begin{aligned}\text{opposite} &= \text{adjacent} \times \tan\theta \\ &= \text{distance to tree} \times \tan\theta.\end{aligned}$$

It is interesting to note that the first method from the North American Indians described by Brown et al. (2011) uses the same principle as shown in Figure 6. However, a clinometer is held at eye level so the distance of the eye above the ground needs to be taken into account, as shown in Figure 7. Hence the

Height of the tree = Distance to tree × tan θ + Height of eye from ground.

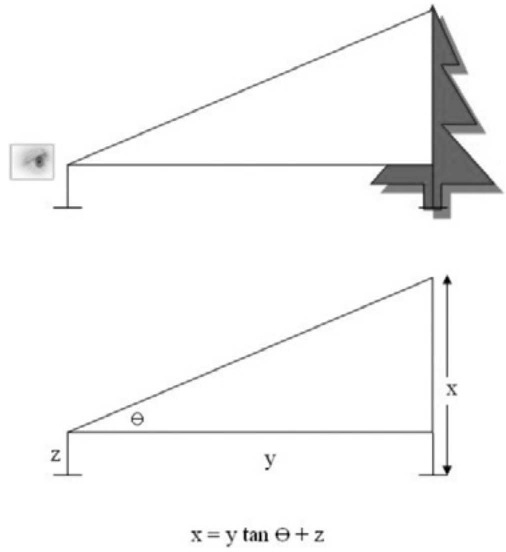

Figure 7. Measurements needed when using a clinometer from a standing position taking into consideration the height at eye level.

Variation and estimation in measurement

Within the *Australian Curriculum: Mathematics*, variation is introduced as early as Year 3, when students are encouraged to “identify the variation between trials” (ACARA, 2010, p. 22), and continues to be a major focus of the Statistics and Probability strand in upper primary school and throughout the secondary years of schooling (pp. 32, 40, 48). Either one or both of these investigations provide an excellent opportunity to discuss the variation that occurs when indirect measurements are made to estimate a fixed but inaccessible length (here, height). Linking the language of variation and estimation is important in the area of measurement. The estimated height of a tree is equal to the actual height plus the variation due to error in measurement. This variation can obviously be either positive or negative. Collecting many estimated measurements and calculating their average is likely to give a better estimate of the actual measurement unless there is systematic error. Systematic errors can lead to inaccurate although possibly precise measurement. Also, the variation observed in the estimates reflects the quality and precision of the method used. More variation is likely to reflect a less precise method. Even direct measurements of lengths are likely to show considerable relative variation as was shown in the activities described by Watson and Wright (2008) when many students measured the arm span of a single student.

Not only is it possible to consider the variation within the data collected for each method but also it is possible to consider the variation between the methods. One might expect less variation among the estimates of a tree’s height with a clinometer than with the method using a ruler. Considering the ranges of the estimates from the two methods and the “clumping” of values can be the basis of written reports.

Extension activity using technology

Using the educational software package *TinkerPlots* (Konold & Miller, 2005) it is possible for students to enter their measurements from each investigation, and that of their classmates, into data cards to investigate the degree of consistency between the two measurement techniques. Figure 8 shows the possible format of data cards and the *TinkerPlots* Formula functions that can be used to calculate the height of the tree for both Investigations 1 and 2 (differentiated in the data cards with the prefix I1 and I2 respectively).

The Invicta clinometer used in Investigation 2 provides a reading of elevation in degrees however the "tan" function in *TinkerPlots* returns the tangent of an angle in radians. Therefore, the *TinkerPlots* formula in Investigation 2 converts the angle measurement from degrees to radians.

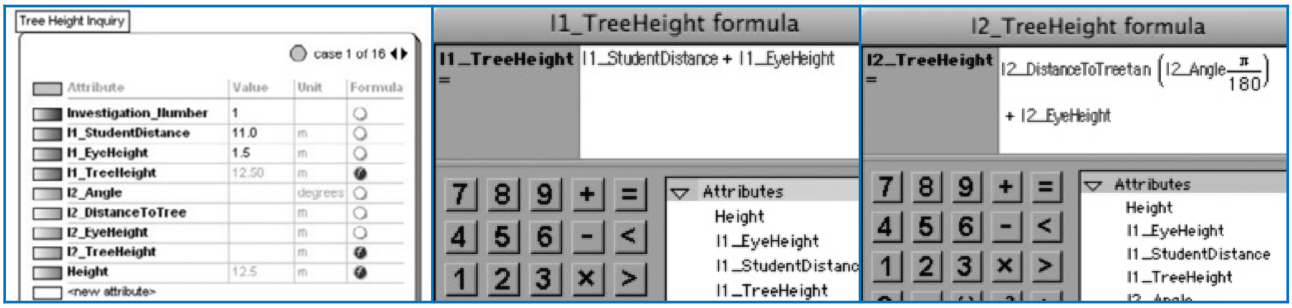

Figure 8. A *TinkerPlots* data card and formula boxes for calculating the heights of a tree.

This extension activity relies on some or all of the class measuring the same tree for each of the investigations. Once the data have been entered for each investigation, students can use the formula function and an "if statement" to create a single entry for the attribute Height, as shown in Figure 9. This attribute can then be plotted to compare the height measurements obtained from the two different investigations, as also seen in Figure 9. In *TinkerPlots* students can plot the mean and discuss how outliers may affect its value.

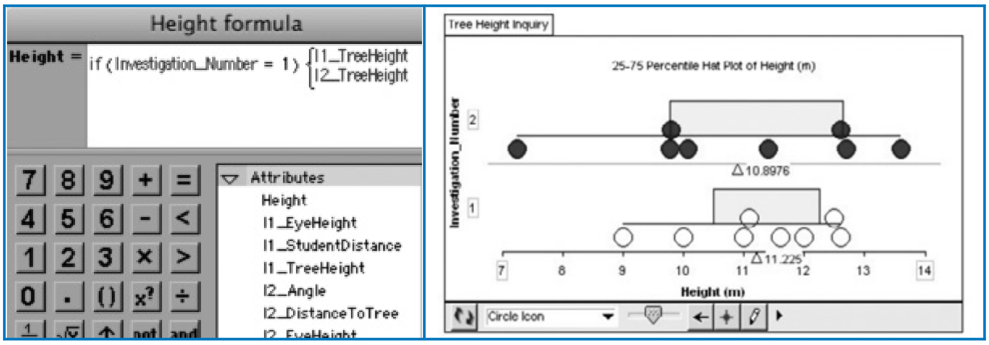

Figure 9. A formula box for the attribute Height and a plot showing the different height measurements for a single tree separated by the investigation number.

Discussion

The purpose of this article, in conjunction with the APMC article (Brown et al., 2011), has been to motivate teachers to present their students with meaningful investigations that lead to an appreciation and understanding of a variety of ways to estimate the height of an object that cannot be measured directly. Each of Brown et al.'s primary school investigations, and the middle school investigations presented here, require students to use computational estimation. Van de Walle (2004) describes this type of estimation as the process of "determining a number that is an approximation of a computation that we cannot or do not wish to determine exactly" (p. 229).

In these investigations, the tree's height is never directly measured because it is impractical to do so. However, this does not mean that the estimations are simply guesses. As Van de Walle (2004) says, students often confuse the idea of estimation with guessing, but computational estimation requires some form of computational reasoning and the implementation of a suitable computational strategy (p. 229). These hands-on investigations allow students to develop further their computational estimation skills in a measurement context, as well as to make connections between different mathematical concepts, such as ratio, proportionality, and the properties of triangles, and to use their prior mathematical knowledge for an unfamiliar and "non-routine" problem—ideals endorsed by the *Australian Curriculum: Mathematics* (ACARA, 2010, p. 5).

The use of technology in the extension activity enables students to explore the concept of variation in measurement and build upon their understanding in this area. This can be done without technology but use of *TinkerPlots* satisfies calls for the use of technology in the *Australian Curriculum: Mathematics* (ACARA, 2010, p. 9).

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From Helen Prochazka's Scrapbook

If we had managed to probe any other area of human endeavour or questioning to the depth that we have pushed pure maths, the world would be a far dandier place.

Adam Spencer, broadcaster