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An Alternative Method to Gauss-Jordan Elimination: Minimizing Fraction Arithmetic

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When solving systems of equations by using matrices, many teachers present a Gauss-Jordan elimination approach to row reducing matrices that can involve painfully tedious operations with fractions (which I will call the traditional method). In this essay, I present an alternative method to row reduce matrices that does not introduce additional fractions until the very last steps. The students in my classes seemed to appreciate the efficiency and accuracy that the alternative method offered. Freed from unnecessary computational demands, students were instead able to spend more time focusing on designing an appropriate system of equations for a given problem and interpreting the results of their calculations. I found that these students made relatively few arithmetic mistakes as compared to students I tutored in the traditional method, and many of these students who saw both approaches preferred the alternative method.

When solving systems of equations by using matrices, many teachers present a Gauss-Jordan elimination approach to row reducing matrices that can involve painfully tedious operations with fractions (which I will call the traditional method). In this essay, I present an alternative method to row reduce matrices that does not introduce additional fractions until the very last steps. As both a teacher using this alternative method and a tutor working with students instructed in the traditional method, I have some anecdotal experience with both. The students in my classes seemed to appreciate the efficiency and accuracy that the alternative method offered them. Since they were freed from unnecessary computational demands, they were instead able to spend more time focusing on designing an appropriate system of equations for a given problem and interpreting the results of their calculations. I found that these students made relatively few arithmetic mistakes as compared to students I tutored in the traditional method, and many of these students who saw both approaches preferred the alternative method. I find (and it is likely true for students) that it takes significantly less time to row reduce a matrix using the alternative approach than the traditional approach. Teachers are free to choose a preferred method (some may want to emphasize practice with fractions), but I believe this alternative method to be a strong alternative to the traditional method since students will perform significantly fewer computations and teachers can extend the technique to finding the inverse of matrices.

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Many students are not proficient at solving problems involving fractions, and this lack of proficiency is not restricted to any one grade band. For example, when Brown and Quinn (2006) studied 143 ninth graders enrolled in an elementary algebra course at an upper middle-class school, they found that many of the students had a lack of experience with both fraction concepts and computations. In their study, 52% of the students could not find the sum of 5/12 and 3/8, and 58% of the students could not find the product of 1/2 and 1/4. Unfortunately, students' difficulty with fractions can persist into postsecondary education. When studying elementary education majors at the University of Arizona, Larson and Choroszy (1985) found that roughly 25% of the 391 college students incorrectly added and subtracted mixed numbers when regrouping was involved. Hanson and Hogan (2000) studied the computational estimation skills of 77 college students who were majoring in a variety of disciplines; many of the students in their study struggled with problems that involved fractions and became frustrated with the process of finding common denominators. They noted that a few students in the lower performing groups added (or subtracted) the numerators and denominators and did not find common denominators. Commenting on the lack of understanding commonly associated with fractions, Steen (2007) observed that even many adults become confused if a problem requires anything but the simplest of fractions.

The use of matrices to solve systems of equations has long been a topic in high school and college advanced algebra and precalculus algebra courses. An increasing number of colleges and high schools teach Finite Mathematics, sometimes as a core course option. This means that increasing numbers of college and college-bound students are introduced to solving systems of equations by converting them into matrices and then row reducing them. For example, at the university where I teach, childhood education majors see this topic in a required core course. Fraction skills may be a reasonable requirement for all of these students, but I believe this is not the best context for practicing numerous fraction computations, particularly for students who are not typically math or science majors. Indeed, students' difficulties with fractions lead many instructors to carefully pick matrices that do not involve fractions during the intermediate steps of the traditional approach to row-reducing a matrix. However, the alternative method discussed below is similar to traditional Gauss-Jordan elimination but allows instructors to use any system of linear equations over the rational numbers because it prevents new fractions from appearing until the very last steps. Furthermore, the alternative method involves a similar number of computations as the traditional method, which decreases the likelihood of arithmetic mistakes.

When deciding which approach students should learn in order to row reduce matrices, teachers need to consider their motivation for showing students how to row reduce matrices. Typically, we want our students to be able to solve resource allocation problems, geometric problems, or other types of applications by finding the values of the variables in a system of equations and then correctly interpreting the results of their findings. In other words, we are interested in showing our students how to solve problems where row reduction of matrices is an appropriate strategy. Therefore, if we have two mathematically sound approaches for finding the values of the variables, one whose computational demands may distract from the main concept and the other that involves fewer computations and is less distracting, it seems reasonable to show students the method that will free them to focus on setting up the problem and interpreting the results rather than being immersed in the intermediate calculations. Such an instructional decision aligns with the National Council of Teachers of Mathematics (2000) teaching principle (2000) that advocates the skillful selection of teaching strategies to communicate mathematics.

The alternative method is not a new approach, but after reviewing many Finite Mathematics and Linear Algebra textbooks from a variety of publishers, I found that the vast majority of the texts do not clearly present to students with a method of solving a system of equations without incurring fractions in the intermediate steps (Goldstein, Schneider, & Siegel, 1998; Poole, 2003; Rolf, 2002; Uhlig, 2002; Young, Lee, & Long, 2004). Even the texts used at my university (Barnett, Ziegler, & Byleen, 2005; Lay, 2006) do not demonstrate the alternative method. Warner and Costernoble (2007), Shifrin and Adams, (2002), and Lial, Greenwell, and Ritchey (2008) were the only texts that I found that clearly presented the alternative method. In all of the aforementioned books no characteristics seemed to predict whether or not the alternative method was presented and they all covered roughly the same concepts that are traditionally presented in Finite Mathematics and Linear Algebra courses. For the benefit of students and teachers who have only been exposed to the traditional Gaussian methods of row-reduction, the remaining portion of the article develops the alternative technique. The following paragraphs describe operations with matrices of the type provided below (Figure 1).

$$\left[\begin{array}{ccc|c} a_{1,1} & a_{1,2} & a_{1,3} & k_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & k_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & k_3 \end{array}\right]$$

Figure 1. A typical 3×3 augmented matrix.

The most common method that students are taught Gauss-Jordan-elimination for solving systems of equations is first to establish a 1 in position $a_{1,1}$ and then secondly to create 0s in the entries in the rest of the first column. The student then performs the same process in column 2, but first a 1 is established in position $a_{2,2}$ followed secondly by creating 0s in the entries above and below. The process is repeated until the coefficient matrix (Figure 1) is transformed into the identity matrix, where 1s are along the main diagonal and 0s are in all other entries (Barnett, Ziegler & Byleen, 2005). Some teachers use a variation of Gauss-Jordan elimination called back-substitution that simplifies the process somewhat for solving systems of equations; however, back-substitution can not be used to find inverses of matrices.

The traditional approach of finding first the 1s for each of the diagonal entries and secondly finding the 0s for the remaining elements in each corresponding column becomes extremely cumbersome when fractions are involved. Students who are not comfortable or proficient with fractions may become frustrated with these types of problems. Asking instructors to teach students a method that they are only able to use to solve a limited class of problems does those students a disservice. The alternative Gaussian approach where 1s on the diagonal are not obtained until the very end of the problem is a nice alternative to the traditional method. In my opinion, the strength of this approach is that (a) no new fractions are introduced until the very last steps and (b) this process can still be implemented to find the inverse of a matrix (in contrast to the back-substitution method).

To set up this method, I review an approach for solving a system of two equations in two variables. For this smaller system, teachers commonly teach the addition method, which relies on multiplying each equation by the (sometimes oppositely signed) coefficients in the other equation and then adding the two equations to eliminate the target variable. Consider the following problem (Example 1 in Figure 2).

Step 1: We can choose to eliminate either the *x* or *y* variable. For this example, we will eliminate the *x* variable.

$$\begin{aligned} 3x+2y &= 8 \\ 2x-5y &= -1 \end{aligned}$$

Step 2: To eliminate the *x* variable, we will multiply the top row (R_1) by 2 and the bottom row (R_2) by -3. Then we will add the two equations together to create a new equation.

Note: We know that we are proceeding in the correct direction because we successfully eliminated the *x* variables when we added the equations together.

$$\begin{aligned} 3x+2y &= 8 \quad (2) \\ 2x-5y &= -1 \quad (-3) \end{aligned}$$

$$\begin{aligned} 6x+4y &= 16 \\ -6x+15y &= 3 \\ \hline 19y &= 19 \end{aligned}$$

Step 3: At this point, we simply solve for *y* and substitute our solution back into either equation to solve for *x*, checking both in the other equation.

$$y = 1$$
$$x = 2$$

Figure 2. Solving Example 1, a 2×2 linear system.

The process of eliminating the *x* variable in the above problem (Figure 2) by producing opposite coefficients of *x* is used in the alternative method for row-reducing matrices. Next, I show how to use the above idea to solve a typical system of *n* equations with *n* variables without incurring any fractions (Example 2 in Figures 3a and b).

Step 1: Recopy from the original system of equations into augmented matrix form.

$$\begin{aligned} 3x+4y+2z &= 5 \\ 2x-3y+z &= -7 \\ 4x+y-2z &= 12 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 2 & 5 \\ 2 & -3 & 1 & -7 \\ 4 & 1 & -2 & 12 \end{array}\right]$$

Step 2: Multiply R_1 and R_2 in such a way that you *create oppositely signed common multiples* in entries $a_{1,1}$ and $a_{2,1}$ as shown below.

$$\begin{array}{cccc l} 3 & 4 & 2 & 5 & (-2) \\ 2 & -3 & 1 & -7 & (3) \end{array}$$

Adding and then substituting the sum for row 2 results in a 0 in entry $a_{2,1}$.

$$\begin{array}{rrrrr} & -6 & -8 & -4 & -10 \\ + & 6 & -9 & 3 & -21 \\ \hline & 0 & -17 & -1 & -31 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 2 & 5 \\ 0 & -17 & -1 & -31 \\ 4 & 1 & -2 & 12 \end{array}\right]$$

Figure 3a. Step-by-step process for solving Example 2 using the alternative Gaussian approach.
Note: The process in the left column produces the matrix in the right column for each step.

Step 3: Multiply R_1 and R_3 in such a way that you *create oppositely signed common multiples* in entries $a_{1,1}$ and $a_{3,1}$.

$$\begin{array}{cccccc} 3 & 4 & 2 & 5 & & (-4) \\ 4 & 1 & -2 & 12 & & (3) \end{array}$$

Adding and then substituting the sum for row 3 results in a 0 in entry $a_{3,1}$.

$$\begin{array}{rrrrr} & -12 & -16 & -8 & -20 \\ + & 12 & 3 & -6 & 36 \\ \hline & 0 & -13 & -14 & 16 \end{array}$$

$$\left[\begin{array}{rrr|r} 3 & 4 & 2 & 5 \\ 0 & -17 & -1 & -31 \\ 0 & -13 & -14 & 16 \end{array}\right]$$

Figure 3b. Step-by-step process for solving Example 2 using the alternative Gaussian approach.

It is not important what values are produced on the main diagonal until the last step of this process. So, I will not divide the top row by 3 to get a value of 1 in position $a_{1,1}$ which would produce fractions in this intermediate step. Now, I will must establish 0s in the entries above and below $a_{2,2}$ (Figure 4).

Step 4: Multiply R_1 and R_2 in such a way that you *create oppositely signed common multiples* in entries $a_{1,2}$ and $a_{2,2}$.

$$\begin{array}{cccccc} 3 & 4 & 2 & 5 & & (17) \\ 0 & -17 & -1 & -31 & & (4) \end{array}$$

Adding and then substituting the sum for row 1 results in a 0 in entry $a_{1,2}$.

$$\begin{array}{rrrrr} & 51 & 68 & 34 & 85 \\ + & 0 & -68 & -4 & -124 \\ \hline & 51 & 0 & 30 & -39 \end{array}$$

$$\left[\begin{array}{rrr|r} 51 & 0 & 30 & -39 \\ 0 & -17 & -1 & -31 \\ 0 & -13 & -14 & 16 \end{array}\right]$$

Step 5: Multiply R_2 and R_3 in such a way that you *create oppositely signed common multiples* in entries $a_{2,2}$ and $a_{3,2}$.

$$\begin{array}{cccccc} 0 & -17 & -1 & -3\text{`} & & (-13) \\ 0 & -13 & -14 & 16 & & (17) \end{array}$$

Adding and then substituting the sum for row 3 results in a 0 in entry $a_{3,2}$.

$$\begin{array}{rrrrr} & 0 & 221 & 13 & 85 \\ + & 0 & -221 & -4 & -124 \\ \hline & 0 & 0 & 30 & -39 \end{array}$$

$$\left[\begin{array}{rrr|r} 51 & 0 & 30 & -39 \\ 0 & -17 & -1 & -31 \\ 0 & 0 & -225 & 675 \end{array}\right]$$

Figure 4. A continuation of the solution of Example 2 using the alternative Gaussian approach.

Having now established 0s in the appropriate positions in columns 1 and 2 (Figure 4), we repeat the process to establish 0s in column 3. However, it would be useful at this point to reduce the numbers in row 3 before we establish the last set of 0s (See optional step in Figure 5).

Optional Step: Since 675 is a multiple of -225, simplifying R_3 by dividing the entire row by "-225" (or multiplying by the reciprocal) will make the arithmetic easier from this point on:

$$0 \quad 0 \quad -225 \quad 675 \qquad (225^{-1}) \qquad \rightarrow \qquad 0 \quad 0 \quad 1 \quad -3$$

$$\left[\begin{array}{ccc|c} 51 & 0 & 30 & -39 \\ 0 & -17 & -1 & -31 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Note: Dividing a row by a common factor simplifies the arithmetic by producing smaller values for each entry.

Step 6: Multiply R_1 and R_3 in such a way that you *create oppositely signed common multiples* in entries $a_{1,3}$ and $a_{3,3}$.

$$\begin{array}{cccccc} 51 & 0 & 30 & -39 & & (1) \\ 0 & 0 & 1 & -3 & & (-30) \end{array}$$

Adding and then substituting the sum for R_1 results in a 0 in entry $a_{1,3}$.

$$\begin{array}{rrrrr} & 51 & 0 & 30 & -39 \\ + & 0 & 0 & -30 & 90 \\ \hline & 51 & 0 & 0 & 51 \end{array}$$

$$\left[\begin{array}{ccc|c} 51 & 0 & 0 & 51 \\ 0 & -17 & -1 & -31 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Step 7: Multiply R_2 and R_3 in such a way that you *create oppositely signed common multiples* in entries $a_{2,3}$ and $a_{3,3}$

$$\begin{array}{cccccc} 0 & -17 & -1 & -31 & & (1) \\ 0 & 0 & 1 & -3 & & (1) \end{array}$$

. And then substituting the answer in for R_2 results in a 0 in entry $a_{2,3}$.

$$\begin{array}{rrrrr} & 0 & -17 & -1 & -31 \\ + & 0 & 0 & 1 & -3 \\ \hline & 0 & -17 & 0 & -34 \end{array}$$

$$\left[\begin{array}{ccc|c} 51 & 0 & 0 & 51 \\ 0 & -17 & 0 & -34 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Final Step: The last step in this process is to divide each row by its first non-zero entry (multiply by its reciprocal), in this case the values on the main diagonal.

$$\begin{array}{cccccc} 51 & 0 & 0 & 51 & & (51^{-1}) \\ 0 & -17 & 0 & -34 & & (-17^{-1}) \\ 0 & 0 & 1 & -3 & & (1) \end{array}$$

$$\left[\begin{array}{ccc|c} 51 & 0 & 0 & 51 \\ 0 & -17 & 0 & -34 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Thus, $x = 1, y = 2, z = -3$.

Figure 5. Concluding steps for solving Example 2 using the alternative Gaussian approach.

Showing students how to solve systems of linear equations using the alternative version of Gaussian elimination allows them to avoid becoming inundated with fraction computations. For Example 2, if the operation between any two integers counts as one computation, then using the traditional method to solve the system of equations results in 58 computations; the alternative method results in 46 computations. Because the alternative method produced 21% fewer computations than the traditional method, students are less likely to get lost in the intermediate computations and are more able to focus on the overall purpose of the method.

Note again that the alternative method can be used for systems of rational equations and can be followed fairly mechanically for rational systems containing n equations with n variables. In the event that the system of equations has infinitely many solutions or no solution, the idea behind the alternative method is the same: get 0's for entries above and below the leading non-zero entry in each row, then divide each row by the value of this non-zero entry. The following example illustrates this point (Example 3 in Figure 6).

Step 1: Recopy from the original system of equations into augmented matrix form.

$$\begin{aligned} 6x+4y+13z &= 5 \\ 9x+6y \quad\quad &= 7 \\ 4x+8y-\quad z &= 12 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 6 & 4 & 13 & 5 \\ 9 & 6 & 0 & 7 \\ 12 & 8 & -1 & 12 \end{array}\right]$$

Step 2: Multiply R_1 and R_2 in such a way that you create oppositely signed common multiples in entries $a_{1,1}$ and $a_{2,1}$ as shown below.

$$\begin{array}{cccccc} 6 & 4 & 13 & 5 & & (-3) \\ 9 & 6 & 0 & 7 & & (2) \end{array}$$

Adding and then substituting the answer for R_2 results in a 0 in entry $a_{2,1}$.

$$\begin{array}{rrrrr} & -18 & -12 & -39 & -15 \\ + & 18 & 12 & 0 & 14 \\ \hline & 0 & 0 & -39 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 4 & 13 & 5 \\ 0 & 0 & -39 & -1 \\ 12 & 8 & -1 & 12 \end{array}\right]$$

Step 3: Multiply R_1 and R_3 in such a way that you *create oppositely signed common multiples* in positions $a_{1,1}$ and $a_{3,1}$.

$$\begin{array}{cccccc} 6 & 4 & 13 & 5 & & (-2) \\ 12 & 8 & -1 & 12 & & (1) \end{array}$$

Adding and then substituting the answer for R_3 results in a 0 in entry $a_{3,1}$.

$$\begin{array}{rrrrr} & -12 & -8 & -26 & -10 \\ + & 12 & 8 & -1 & 12 \\ \hline & 0 & 0 & -27 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 4 & 13 & 5 \\ 0 & 0 & -39 & -1 \\ 0 & 0 & -27 & 2 \end{array}\right]$$

Figure 6. Beginning steps of solution for Example 3.

Looking at the preceding matrix, we have a 0 in position $a_{2,2}$, so I cannot use it to eliminate the 4 in position $a_{1,2}$; and since I have a 0 in position $a_{3,2}$, I do not benefit from switching row 2 and row 3. Thus, I can focus our attention on -39 in position $a_{2,3}$. (I could also focus our attention on -27, but the end result would not change). The objective is still the same: get "0's" in the entries above and below -39 (Figures 7a and 7b).

Step 4: Multiply R_1 and R_2 in such a way that you *create oppositely signed common multiples* in positions $a_{1,3}$ and $a_{2,3}$.

$$\begin{array}{cccccc} 6 & 4 & 13 & 5 & & (3) \\ 0 & 0 & -39 & -1 & & (1) \end{array}$$

Adding and then substituting the answer in for R_1 results in a 0 in position $a_{1,3}$.

$$\begin{array}{rrrrr} & 18 & 12 & 39 & 15 \\ + & 0 & 0 & -39 & -1 \\ \hline & 18 & 12 & 0 & 14 \end{array}$$

$$\left[\begin{array}{ccc|c} 18 & 12 & 0 & 14 \\ 0 & 0 & -39 & -1 \\ 0 & 0 & -27 & 2 \end{array}\right]$$

Figure 7a. Continuation of solution for Example 3.

Step 5: Multiply R_2 and R_3 in such a way that you *create oppositely signed common multiples* in positions $a_{2,3}$ and $a_{3,3}$.

$$\begin{array}{cccccc} 0 & 0 & -39 & -1 & & (27) \\ 0 & 0 & -27 & 2 & & (-39) \end{array}$$

Adding and then substituting the answer in for R_3 results in a 0 in position $a_{3,3}$.

$$\begin{array}{ccccc} & 0 & 0 & -1053 & -27 \\ + & 0 & 0 & 1053 & -78 \\ \hline & 0 & 0 & 0 & -105 \end{array}$$

$$\left[\begin{array}{ccc|c} 18 & 12 & 0 & 14 \\ 0 & 0 & -39 & -1 \\ 0 & 0 & 0 & -105 \end{array}\right]$$

Figure 7b. Continuation of solution for Example 3.

Based on the previous matrix (Figures 7a and 7b) we can see that the system of equations does not have a solution since row 3 states that 0 = -105 (clearly a false statement). If we wanted to finish simplifying the matrix, we would divide rows 1 and 2 by the values of their leading non-zero entries to get the following (Figure 8).

Final Step:

$R_1 \div 18 \rightarrow R_1$

$R_2 \div -39 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & \frac{2}{3} & 0 & \frac{7}{9} \\ 0 & 0 & 1 & \frac{1}{39} \\ 0 & 0 & 0 & -105 \end{array}\right]$$

Figure 8. Final steps of solution for Example 3.

I hope that those who have not considered this alternative method will see the possible advantages for themselves and their students. First, this method may increase the accessibility of matrix material for students with weaknesses in fractions. Next, the method has the potential to increase the speed and accuracy of computations for students and teachers alike by the substitution of integer computations for rational number computations. I have found that some students avoid fractions by using decimal approximations, sacrificing precision. However, with this method, teachers can still require the precision of fractional solutions without the excessive mire of fractions, potentially encouraging more student effort and success. Finally, teachers who are wary of requiring extensive fractional computations may be freed by this method to have a greater flexibility in problem selection.

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