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Sense Making as Motivation in Doing Mathematics: Results from Two Studies

Mary Mueller, Dina Yankelewitz, & Carolyn Maher

In this article, we present episodes from two qualitative research studies. The studies focus on students of different ages and populations and their work on different mathematical tasks. We examine the commonalities in environment, tools, and teacher-student interactions that are key influences on the positive dispositions engendered in the students and their interest and engagement in mathematics. In addition, we hypothesize that these positive dispositions in mathematics lead to student reasoning and, thus, mathematical understanding. The resulting framework is supported by other educational research and suggests ways that the standards can be implemented in diverse classrooms in order to achieve optimal student engagement and learning.

The National Council of Teachers of Mathematics (NCTM, 2000) describes a vision for mathematics education focusing on conceptual understanding. This vision includes students engaged in hands-on activities that incorporate problem solving, reasoning and proof, real-world connections, multiple representations, and mathematical communication. NCTM and others have prepared multiple documents and resources (e.g., Chambers, 2002; Germain-McCarthy, 2001; NCTM, 2000; Stiff & Curcio, 1999) to support teachers in achieving this vision and putting the standards into practice. However, differences in age, gender, ethnicity, and school culture often impede the implementation of successful teaching practice in mathematics classrooms and prevent students from taking ownership of mathematical ideas in the ways that have been envisioned.

While NCTM addresses factors such as classroom environment and mathematical tasks, this provides an incomplete picture of how to build students' conceptual understanding. For example, motivation to learn is pivotal in students' attainment of understanding in all content areas (Middleton & Spanias, 1999), but the NCTM vision does not explicate how to help students experience motivation as they learn mathematics. We have developed a framework for mathematics teaching and learning that provides this missing link. It provides teachers and researchers with a conceptual tool that explains how students build the positive attitudes (motivation, autonomy, self-efficacy, and positive dispositions) towards mathematics that are necessary to engage in mathematical reasoning. We believe that this approach that can be implemented across the spectrum of mathematics classrooms in the US.

Our research focuses on students who are working collaboratively as they engage in mathematical problem solving. We videotaped students as they engaged in mathematical tasks and then analyzed the reasoning that occurred as they worked to formulate strategies and defend their solutions. We have found that, although the demographics of the groups of students and the tasks may be different, the reasoning and subsequent understanding that occurs is quite similar.

In this article, we present two episodes from our research, focusing on students of different ages and populations as they work on different mathematical tasks. We then examine the commonalities in environment, tools, and teacher-student interactions that are key influences both on the positive attitudes towards mathematics engendered in the students and on their engagement in mathematics. We hypothesize that these positive attitudes towards mathematics lead to student reasoning and, thus, mathematical understanding. Based on our research, we created a

Mary Mueller is an Associate Professor in the Department of Educational Studies at Seton Hall University. Her research interests include the development of mathematical ideas and reasoning over time.

Dina Yankelewitz is an Assistant Professor in the School of General Studies at the Richard Stockton College of New Jersey. Her research interests include the development of mathematical thinking and the identification and development of mathematical reasoning in students and teachers of mathematics.

Carolyn Maher is a Professor of mathematics education in the Graduate School of Education at Rutgers University. Her research interests include the development of mathematical thinking in students, mathematical reasoning, justification and proof making in mathematics, and the development of a model for analyzing videotape data.

framework for teaching and learning that identifies the key factors in encouraging positive attitudes in the mathematics classroom as well as their role in enabling student reasoning and understanding. We support this framework using the extensive literature base centering on students' motivation in the mathematics classroom. The resulting framework suggests ways that the standards can be implemented in diverse classrooms in order to achieve optimal student engagement and learning. Although the development of our framework began with our data and then was supported by the literature, we begin by presenting the supporting literature in order to give the readers a background for the framework.

The Role of Intrinsic Motivation

In our framework, there are four factors that mediate between elements in the classroom environment, such as tasks, and the development of conceptual understanding through mathematical reasoning. These four factors are autonomy, instrinsic motivation, self-efficacy and positive dispositions towards mathematics. Because the literature concerning all four of these factors is interrelated, we have picked one factor, intrinsic motivation, to organize our discussion around.

All students must be motivated in some way to engage in mathematical activity, however, the nature of that motivation largely determines the success of their endeavor. In particular, students' motivations can be divided into two distinct types: extrinsic motivation and intrinsic motivation. Extrinsically motivated students engage in learning for external rewards, such as teacher and peer approval and good grades. These students do not necessarily acquire a sense of ownership of the mathematics that they study; instead they focus on praise from teachers, parents and peers and avoiding punishment or negative feedback (Middleton & Spanias, 1999). In contrast, students who are intrinsically motivated to learn mathematics are driven by their own pursuit of knowledge and understanding (Middleton & Spanias, 1999). They engage in tasks due to a sense of accomplishment and enjoyment and view learning as impacting their self-images (Middleton, 1995). Intrinsically motivated students, therefore, focus on understanding concepts. Thus, intrinsic, rather than extrinsic, motivation benefits students in the process and results of mathematical activities.

Sources of Intrinsic Motivation

Researchers (Deci & Ryan, 1985; Hidi, 2000; Renninger, 2000) have found that sources of intrinsic motivation include perceptions of autonomy, interests in given tasks, and the need for competence. Brophy (1999) concurs and notes that a supportive social context, challenging activities, and student interest and value in learning are crucial to the development of intrinsic motivation.

Autonomous students, in attending to problem situations mathematically, rely on their own mathematical facilities and use their own resources to make decisions and make sense of their strategies (Kamii,1985; Yackel & Cobb, 1996). Autonomy promotes persistence on tasks and thus leads to higher levels of intrinsic motivation (Deci, Nezdik, & Sheinman, 1981; Deci & Ryan, 1987; Stefanou, Perencevich, DiCinti, & Turner, 2004). Furthermore, through participation in classroom activities, mathematically autonomous students begin to rely on their own reasoning rather than on that of the teacher (Cobb, Stephan, McClain, & Gravemeijer, 2001; Forman, 2003) and thus become arbitrators of what makes sense.

Studies show that teacher support and classroom environments play a crucial role in the development of another source of intrinsic motivation, namely, positive (or negative) dispositions toward mathematics (Bransford, Hasselbring, Barron, Kulewicz, Littlefield, & Goin, 1988; Cobb, Wood, Yackel, & Perlwitz, 1992; Middleton, 1995; Middleton & Spanias, 1999). According to NCTM (2000), "More than just a physical setting … the classroom environment communicates subtle messages about what is valued in learning and doing mathematics (p. 18). The document then describes the implementation of challenging tasks that challenge students intellectually and motivate them through real-world connections and multiple solution paths (NCTM, 2000). Stein, Smith, Henningsen, and Silver (2000) stress that teachers need be thoughtful about the tasks that they present to students and use care to present and sustain cognitively complex tasks. They suggest that during the problem solving implementation phase, teachers often reduce the cognitive complexity of tasks. Overall, when students are presented with meaningful, relevant, and challenging tasks; offered the opportunity to act autonomously and develop self-control over learning; encouraged to focus on the process rather than the product; and provided with constructive feedback, they become intrinsically motivated to succeed (Urdan & Turner, 2005).

Effects of Intrinsic Motivation

Intrinsic motivation leads to self-efficacy, an individual's beliefs about their own ability to perform specific tasks in specific situations (Bandura, 1986; Pajares, 1996). Students' self-efficacy beliefs often predict their ability to succeed in a particular situation (Bandura, 1986). Specifically, in mathematics, research has shown that self-efficacy is a clear predictor of students' academic performance (Mousoulides & Philippou, 2005; Pintrich & De Groot, 1990). Furthermore, studies suggest that students with highly developed self-efficacy beliefs utilize cognitive and metacognitive learning strategies more vigorously while being more aware of their own motivational beliefs (Mousoulides & Philippou, 2005; Pintrich, 1999).

Unlike sources of extrinsic motivation, which need to be constantly reinforced, research shows that the common sources of intrinsic motivation are reinforced when students are encouraged to develop their self-efficacy (Urdan & Turner, 2005), For example, intrinsic motivation helps students succeed at a given learning objective, thereby further developing students' self-efficacy.In general, students are more likely to engage and persist in tasks when they believe they have the ability to succeed (Urdan & Turner, 2005). Therefore, intrinsic motivation can lead to an increased willingness to engage in reasoning activities.

In summary, research shows that when students are intrinsically motivated to learn mathematics, they spend more time on-task, tend to be more persistent, and are confident in using different, or more challenging, strategies to solve mathematical problems (Lepper, 1988; Lepper & Henderlong, 2000). These qualities of mathematical learners better enable them to actualize the recommendations put forth by NCTM (2000) and to master key mathematical processes in their pursuit of understanding mathematics. Intrinsic motivation, then, is correlated with self-efficacy and positive dispositions towards a conceptual understanding of mathematics, whereas extrinsic motivation results in merely a superficial grasp of the information presented.

Results from Two Studies

Through a combination of cross-cultural and longitudinal studies we have observed that a mixture of factors contribute to students' motivation to participate in mathematics and their dispositions towards mathematics (for details on our methodologies, see Mueller, 2007; Mueller & Maher, 2010; Mueller, Yankelewitz, & Maher, 2010; Yankelewitz, 2009; Yankelewitz, Mueller, & Maher, 2010). These include classroom environment, teacher questioning that evokes meaningful support of conjectures, and well-designed tasks. Together, these factors positively influence the establishment of favorable dispositions towards learning mathematics. In their quest to make sense of appropriately challenging tasks, students enjoy the pursuit of meaning and thereby become intrinsically motivated to engage in mathematics.

In this paper we present results from two research studies investigating students' mathematics learning. In particular, we present specific examples of elementary and middle school students who demonstrated sense making and higher order reasoning when working on mathematical tasks. In these episodes, the students were engaged, motivated, and, importantly, confident in their ability to offer and defend mathematical solutions; they demonstrated positive dispositions towards mathematics. We identified student behaviors that indicated confidence in mathematics and a high level of engagement. These behaviors include perseverance; the ability to consider more challenging, alternative solutions; and the length of time spent of the task. In the discussion that follows, we analyze the commonalities in the two teaching experiments, and consider how these commonalities may have positively influenced the level of motivation and confidence that students exhibited as they worked on mathematical tasks. In the discussion, we use our findings to define a framework that can be used to inform a teaching practice that will motivate students and encourage student engagement and mathematical understanding.

Data Analysis and Results

The episodes presented below come from two data sets. Data from the first study is drawn from sessions during an informal after-school mathematics program in which 24 sixth-grade students from a low socioeconomic urban community worked on open-ended tasks involving fractions. The students represented a wide range of abilities and thus their mathematical levels ranged from those who were enrolled in remedial mathematics to those who were successful in regular mathematics classrooms. The present discussion focuses on one table of four students, two boys and two girls.

The second source of data includes segments from sessions in which fourth and fifth grade students from a suburban school investigated problems in counting and combinatorics. This data is drawn from a longitudinal study of children's mathematical thinking. As part of the students' regular school day, researchers led the students in exploring open-ended tasks during which students were expected to justify their solutions to the satisfaction of their peers. These strands of tasks were separate from the school-mandated curriculum. Because of space limitations, we give examples of one task from each data set, one involving fractions and the other focusing on combinatorics.

Episode 1: Reasoning about Fractions in the Sixth Grade

The students in the first study worked collaboratively on tasks involving fraction relationships. Cuisenaire® rods (see Figure 1) were available and students were encouraged to build models. A set of Cuisenaire rods contains 10 colored wooden or plastic rods that increase in length by increments of one centimeter. For these activities, the rods have variable number names and fixed color names. The colors increased incrementally as follows: white, light green, purple, yellow dark green, black brown, blue, and orange.

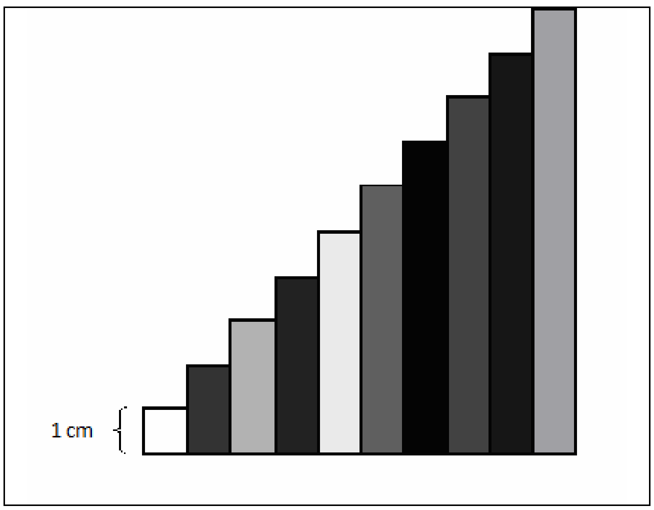

Figure 1. “Staircase” Model of Cuisenaire Rods.

Students were encouraged to build models to represent fraction tasks. For example, in one task, the blue rod was given the number name *one* and students initially worked on naming the red rod (*two-ninths*) and the light green rod (*three-ninths* or *one-third*). When the group completed this task, they initiated their own task of naming all of the rods in the set, given that the blue rod was named one.

Chanel used the staircase model (shown in Figure 1) to incrementally name the remainder of the rods beginning with naming the white rod one-ninth. As she was working, she said the names of all of the rods, “One-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine ninths, ten..– wow, oh, I gotta think about that one ….. nine-tenths”.

Disequilibrium. The teacher/researcher encouraged Chanel to share her problem with Dante. Chanel showed Dante her strategy of using the staircase to name the rods and explained the dilemma of naming the orange rod, “See this is One-ninth, two-ninths, three-ninths, four-ninths, five ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths - what’s this one?” Dante replied, “That would be ten-ninths. Actually that should be one. That would start the new one (one-tenth)”. Chanel and Michael then named the blue rod “a whole”. The students worked for a few more minutes and then Dante explained that he had overhead another table naming the rods.

Dante: Why are they calling it ten-ninths and [it] ends at ninths?

Michael: Not the orange one. The orange one’s a whole.

Dante: But I’m hearing from the other group from over here, they calling it ten-ninths.

Michael: Don’t listen to them! The orange one is a whole because it takes ten of these to make one.

Dante: I’m hearing it because they speaking out loud. They’re calling it ten-ninths

Michael: They might be wrong! …

Chanel: Let me tell you something, how can they call it ten-ninths if the denominator is smaller than the numerator?

Dante: Yeah, how is the numerator bigger than the denominator? It ends at the denominator and starts a new one. See you making me lose my brain.

A teacher/researcher joined the group and asked what the students were working on. Dante presented his argument of naming the orange rod one-tenth and explained that “it starts a new one”. The teacher/researcher reminded him that the white rod was named one-ninth and that this fact could not change. Again she asked him for the name of the blue rod and he stated, “It would probably be ten-ninths”. When prompted, Dante explained that the length of ten white rods was equivalent to the length of an orange rod.

The teacher/researcher asked Dante to convince his partners that this was true.

Chanel: No, because I don’t believe you because–

Michael: I thought it was a whole.

Dante: But how can the numerator be bigger than the denominator?

T/R: It can. It is. This is an example of where the numerator is bigger than the denominator.

Chanel: But the denominator can't be bigger than the numerator, I thought.

Michael: That's the law of facts.

T/R: Who told you that?

Chanel: My teacher.

Dante: One of our teachers.

Direct reasoning. The students continued working on the task. At the end of the session students were asked to share their work. Another student explained that she named the orange rod using a model of two yellow rods, "We found out the denominator doesn't have to be larger than the numerator because we found out that two yellows [each] equal five-ninths so five-ninths plus five-ninths equals ten-ninths." Another student explained that the orange rod could also be named one and one-ninth and used a model of a train of a blue rod and a white rod lined up next to the orange rod to explain (see Figure 2), "If you put them together then this means that it's ten-ninths also known as one and one-ninth."

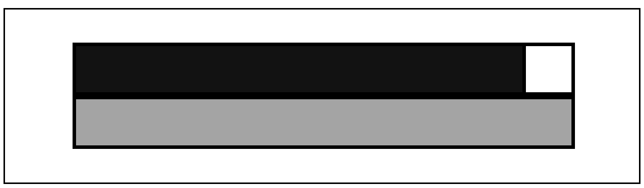

Figure 2. A train of rods to show that $\frac{9}{9}+\frac{1}{9}=\frac{10}{9}$ or $1\frac{1}{9}$

Finally, Dante came to the front of the class and explained that he found a different way to name the orange rod. Building a model of an orange rod lined up next to two purple rods and a red rod (see Figure 3), he explained that the purple rods were each named four-ninths and therefore together they were eight-ninths; the red rod was named two-ninths and therefore the total was eight-ninths plus two-ninths or ten-ninths, "four and four are eight so which will make it eight-ninths right here and then plus two to make it ten-ninths."

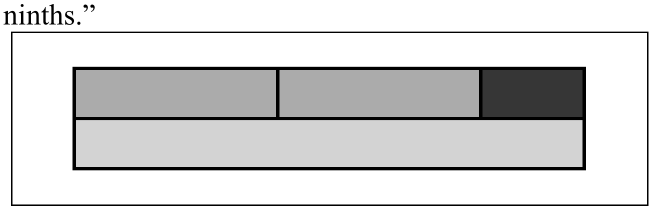

Figure 3. A train of rods to show that $\frac{4}{9}+\frac{4}{9}+\frac{2}{9}=\frac{10}{9}$.

In the beginning of the session described above, Dante and his partners were convinced that a fraction's numerator could not be greater than its denominator. At some point it seems that they were taught about *improper* fractions and may have internalized this to mean *incorrect* fractions. The children referred to this rule as "the law of facts" and, when presented with the task, although they visually saw that the orange rod was equivalent to ten white rods (or ten-ninths), they resisted using this nomenclature. We highlight this episode to show that the students did not simply accept the rule that they recalled and move on to the next task. Instead they heard another group naming the orange rod ten-ninths and grappled with the discrepancy between this name and their rule. Remaining engaged in the task, the students focused on sense making; they were motivated to make sense of the models they built and in doing so exhibited confidence in their solutions. For over an hour, Dante attempted to make sense of his solution by building alternative models, sharing his ideas, conjectures, and solutions, questioning the teacher, and revisiting the problem. When faced with a discrepancy between what he had previously learned and the concrete model that he built, Dante relied on reasoning, rather than memorized facts, to convince himself and others of what made sense. In particular, he relied on his understanding of the model that he had constructed to make sense of the fraction relationships.

This quest for sense making triggered the use of a variety of strategies, and the success of meaning-building led to persistence and flexibility in thinking, which, as described by Lepper and Henderlong (2000), are positively correlated with self-efficacy. Dante's self-efficacy gave him the confidence and autonomy to move beyond his erroneous understanding that was based on previous memorized facts. Similarly to discussions about autonomy from Kamii (1985) and Yackel and Cobb (1996), this autonomy encouraged Dante to believe in his own mathematical ability and use his own resources to make sense of his model. This autonomy, coupled with his positive dispositions toward mathematics, allowed him to use reasoning to make sense of and fully understand the mathematics inherent in the problem.

Episode 2: Reasoning about Combinatorics in the Fourth and Fifth Grades

In the second study, fourth- and fifth-grade students were introduced to combinatorial tasks. The students were given Unifix cubes and were asked to find all combinations of towers that were four tall when selecting from cubes of two colors. Over the course of the two years, students revisited the task in various settings. This provided multiple opportunities for them to think about and refine their thinking about the problem.

Stephanie, along with her partner, Dana, first constructed all possible towers four cubes tall by finding patterns of towers and searching for duplicates. After her first attempt to find all possible towers, Stephanie organized her groups of towers according to color categories (e.g., exactly one of a color and exactly two of a color adjacent to each other) in order to justify her count of 16 towers, thus she organized the towers by *cases* (see Figure 4). Stephanie then used this organization by cases to find all possible towers of heights three cubes tall, two cubes tall and one cube tall when selecting from two colors.

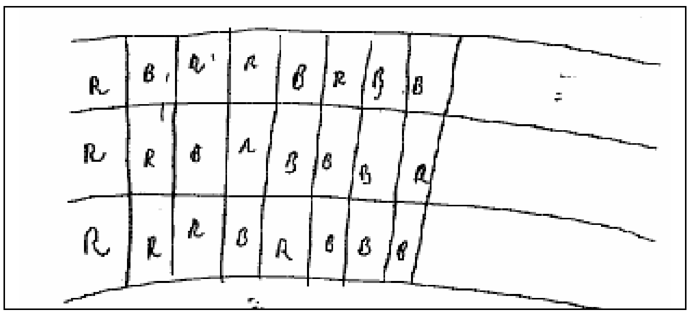

Figure 4. Stephanie's organization of towers by cases.

During further investigation, Stephanie noticed a pattern in the sequence of total number of towers for each height classification: "Two, four, eight, sixteen… that's weird! Look! Two times 2 is 4, 4 times 2 is 8, and 8 times 2 is 16. It goes like a pattern! You have the 2 times 2 equals the 4, the 4 times 2 equals the 8 and the 8 times 2 equals the 16." A few minutes later, Stephanie gave a rule to describe a method for generating towers, "all you have to do is take the last number that you had and multiply by two."

Stephanie's persistent attempts to make sense of the problem enabled her to think about the problem in flexible, yet durable, ways. She used multiple forms of reasoning to examine the problem from different angles and was confident in her findings. She was motivated by her own discoveries and the chance to create and share her own conjectures.

Milin also used cases to organize towers five cubes tall. He then went back to the problem and used simpler problems of towers four cubes tall and three cubes tall to build on to towers five cubes tall. While his partners based their arguments on number patterns and cases, Milin explained his solution using an *inductive argument*. Milin's explanation in each instance was based on adding on to a shorter tower to form exactly two towers that were one cube taller (see Figure 5). For example, when asked to explain why he created four towers from two towers, Milin explained:

> Milin: [pointing to his towers that were one cube high] Because – for each one of them, you could add … two more – because there's … a blue, and a red- … for red you put a black on top and a red on top – I mean a blue on top instead of a black. And blue – you put a blue on top and a red on top – and you keep doing that.

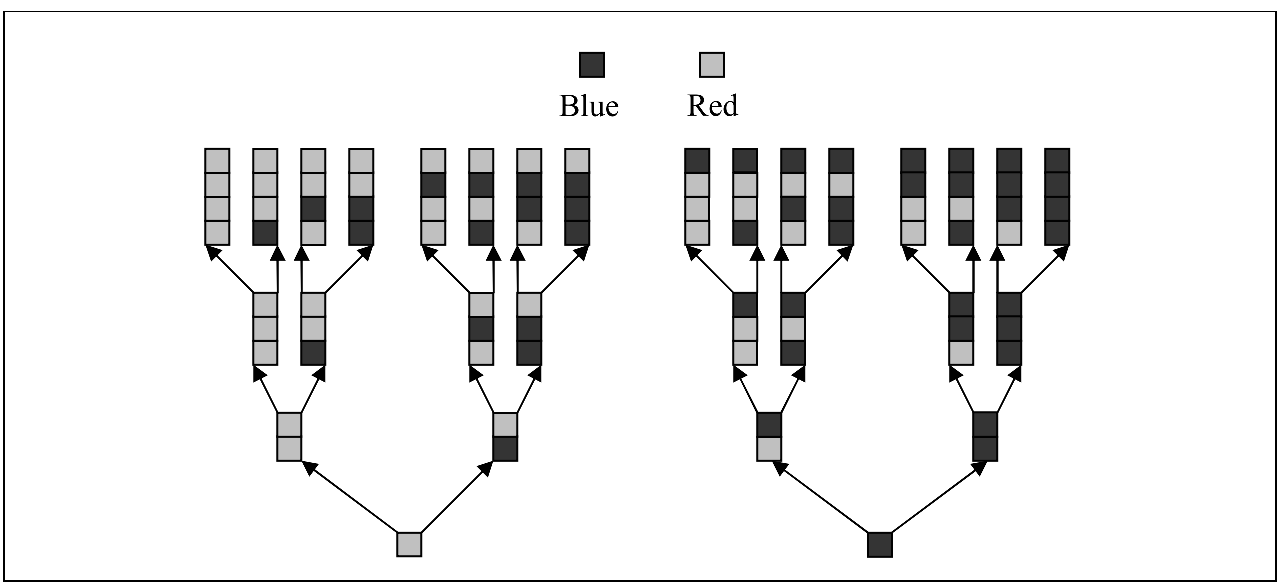

Figure 5. Milin's inductive method of generating and organizing towers.

Later in the year, four students participated in a group session, during which Stephanie and Milin presented their solutions to the towers problem. The next year, in the fifth grade, when the students again thought about this problem, Stephanie worked with Matt to find all tower combinations. Initially, they used trial and error to find as many combinations as they could. However, they only found twelve combinations. Stephanie remembered the pattern that she had discovered the year before.

Stephanie: Well a couple of us figured out a theory because we used to see a pattern forming. If you multiply the last problem by two, you get the answer for the next problem. But you have to get all the answers. See, this didn't work out because we don't have all the answers here.

Matt: I thought we did.

Stephanie: No. I mean all the answers, all the answers we can get . . . I don't know what happened! Because I am positive it works. I have my papers at home that say it works.

Persistence. Stephanie and Matt worked to find more tower combinations, but their search proved unsuccessful. Stephanie insisted that there were more combinations.

Stephanie: I don't know how it worked. I know it worked. I just don't know how to prove it because I'm stumped.

Matt: Steph! Maybe it didn't work!

Stephanie: Oh no. No. Because I'm pretty sure it would… I think we goofed because I'm still sticking with my two thing. I'm convinced that I goofed, that I messed up because I know that…

Flexible Thinking. The teacher/researcher encouraged Stephanie and Matt to discuss the problem with other students. Stephanie and Matt approached other groups to see how they had solved the problem. They visited Milin and Michelle, who had been discussing the inductive method of finding all tower combinations. After hearing Michelle's explanation of Milin's method, Matt adopted that method and told other students about it. Stephanie attempted to explain Milin's strategy to others, and, after the teacher/researcher questioned Stephanie about her explanation, she returned to her seat to work on refining her justification. Later in the session, the teacher/researcher again asked Stephanie to explain her original prediction of the number of four-tall towers using the inductive method. This time she demonstrated a newfound understanding and enthusiastically presented the solution to the class.

The motivation to make sense of the mathematical task and the confidence in the power of their own reasoning exhibited by this young group of students is evident from the transcripts and narrative above. In addition, the students exhibited characteristics that are correlated with intrinsic motivation (e.g., Lepper & Henderlong, 2000), including perseverance, the length of time spent on the task, and the students' flexibility of thought as they considered and adopted the ideas of others. Stephanie's investigations are especially interesting. Although she had previously solved the problem and was certain of her previous solution, Stephanie's autonomy motivated her to continue to work on the problem until she was convinced that her strategy made sense. Rather than accept the solutions of her classmates, Stephanie persisted in verifying her model in order to make sense of the mathematics. The episode described took a full class period, during which the students were actively engaged in solving the task. Stephanie insisted on rethinking the problem, eventually learning from Milin's explanation, and then she used her newfound knowledge to reason correctly about the task and verify her solution. Similar to Dante, she persisted in understanding why her solution worked and insisted on reasoning about the problem, thereby successfully solving and understanding the mathematical task.

Discussion

Both highlighted tasks, one dealing with fraction ideas and the other with combinatorics, engaged students in sense making. The students described in the above episodes demonstrated confidence in their own understanding as they justified their solutions in the presence of their peers, even as their partners offered alternate representations. It is important to note that the episodes described above are exemplars of numerous similar incidents involving many of the students. Students developed this confidence as they were encouraged to defend their solutions first in their small groups and then in the whole class setting. They relied on their own models and justifications and did not seek approval from an authority or guidance from the teacher/researchers for validation of their ideas. These findings correspond with Francisco and Maher's (2005) findings that certain classroom factors promote mathematical reasoning. The factors identified by Francisco and Maher include the posing of strands of challenging, open-ended tasks, establishing student ownership of their ideas and mathematical activity, inviting collaboration, and requiring justification of solutions to problems, all of which were present in the episodes above.

On the surface, the two classroom episodes seem quite different from one another. Specifically, the two classrooms were comprised of students of different ages and demographics. In addition, one of the highlighted tasks focuses on fractional relationships and the other on combinatorics. However, despite these dissimilarities, they share many characteristics that encouraged students to be intrinsically motivated by the mathematics that they learned.

In both episodes, an environment was created that facilitated an active, responsible, and engaged community of learners: Students were encouraged to share ideas and representations and to listen to, question, and convince one another of their solutions. The teacher/researchers facilitated learning while affording students the opportunity to create and defend their own justifications. The teacher/researchers employed careful questioning and support when needed, but the students were the arbitrators of what made sense, giving them a sense of autonomy. Students had opportunities to be successful in building understanding and in communicating that understanding through the arguments they constructed to support their solutions. The resulting discussions also required students to develop representations of their thinking in order to express their ideas with others.

In both groups, students used rich and varied forms of direct and indirect reasoning. The reasoning that emerged during these tasks may be explained, at least in part, by the open-ended nature of the two tasks: The tasks lent themselves to multiple strategies, and, hence, they elicited various forms of reasoning. The behaviors that were observed and the depth of reasoning exhibited can also be explained as a byproduct of intrinsic motivation. The students in both groups strove for conceptual understanding, were persistent in their endeavors, and displayed confidence in their final solutions.

Perhaps most importantly, in both episodes described, the students gained ownership of new mathematical ideas after being confronted with other students' differing understanding of challenging tasks. In accordance with other research (Deci, Nezdik, & Sheinman, 1981; Deci & Ryan, 1987; Stefanou et al., 2004), the students' autonomy led to their perseverance to find or defend their solutions and further increased their intrinsic motivation to make sense of the tasks at hand. Rather than accept the solutions of their classmates, both Stephanie and Dante verified their own strategies using the models they built and, thus, relied on their own reasoning to gain mathematical understanding. Dante and Stephanie were both motivated to rethink their understanding and justify their solutions after being exposed to the ideas of others and being challenged by the researchers to make sense of the task. Dante and Stephanie are representative of the other students we worked with, who displayed the ability to think about the solutions of others and use their own models to make sense of and acquire these solutions as their own. The consistency of these behaviors among our diverse sample suggests that, given the correct environment, all students can reason mathematically and succeed in engaging in mathematics.

Based on our analysis, we hypothesize that motivation and positive dispositions toward mathematics lead to mathematical reasoning, which, in turn, leads to understanding. Furthermore, we constructed a framework to show the relationship between contextual factors and the chain of events leading to conceptual understanding (Figure 6). Our framework begins with the posing of an open-ended, engaging, and challenging task that the students have the ability to solve. The task is supported by a carefully crafted learning environment, carefully planned facilitator roles and interventions, student collaboration, and the availability of mathematical tools.

In the episodes described above, both challenging tasks allowed students to deploy their own, personal solution strategy. Both tasks encouraged students to work collaboratively and utilize mathematical tools. In addition, the teacher/researcher adopted the role of facilitator and allowed the students to grapple with their own strategies as they listened to the strategies of their peers. Stephanie was given the opportunity to work on the problem independently and with a partner. She then listened to the strategies of others before refining her own solution strategy. Likewise, Dante was given the space and time to work through his misconception that the numerator of a fraction could not be larger than the denominator.

Due to the nature of the task and the environment, Dante and his peers were motivated to resolve the discrepancy and find a solution. As with Stephanie, after listening to the ideas of others, Dante worked to make sense of the problem himself and create his own justification. Both students spent over an hour developing their solutions. Their positive dispositions, coupled with intrinsic motivation, gave them the confidence and desire to find a solution. This is apparent in the amount of time that they spent developing their solutions. Both students persevered even after a classmate had offered a viable solution. In both episodes, the students' motivation to succeed at the tasks at hand led to feelings of self-efficacy and autonomy. Both Stephanie and Dante took the initiative to build several models and justifications in order to justify their solutions, first to themselves and then to the larger community.

The students relied on reasoning, rather than memorized facts or the solutions of others, to convince themselves and others of what made sense. This reasoning led to their mathematical understanding. In

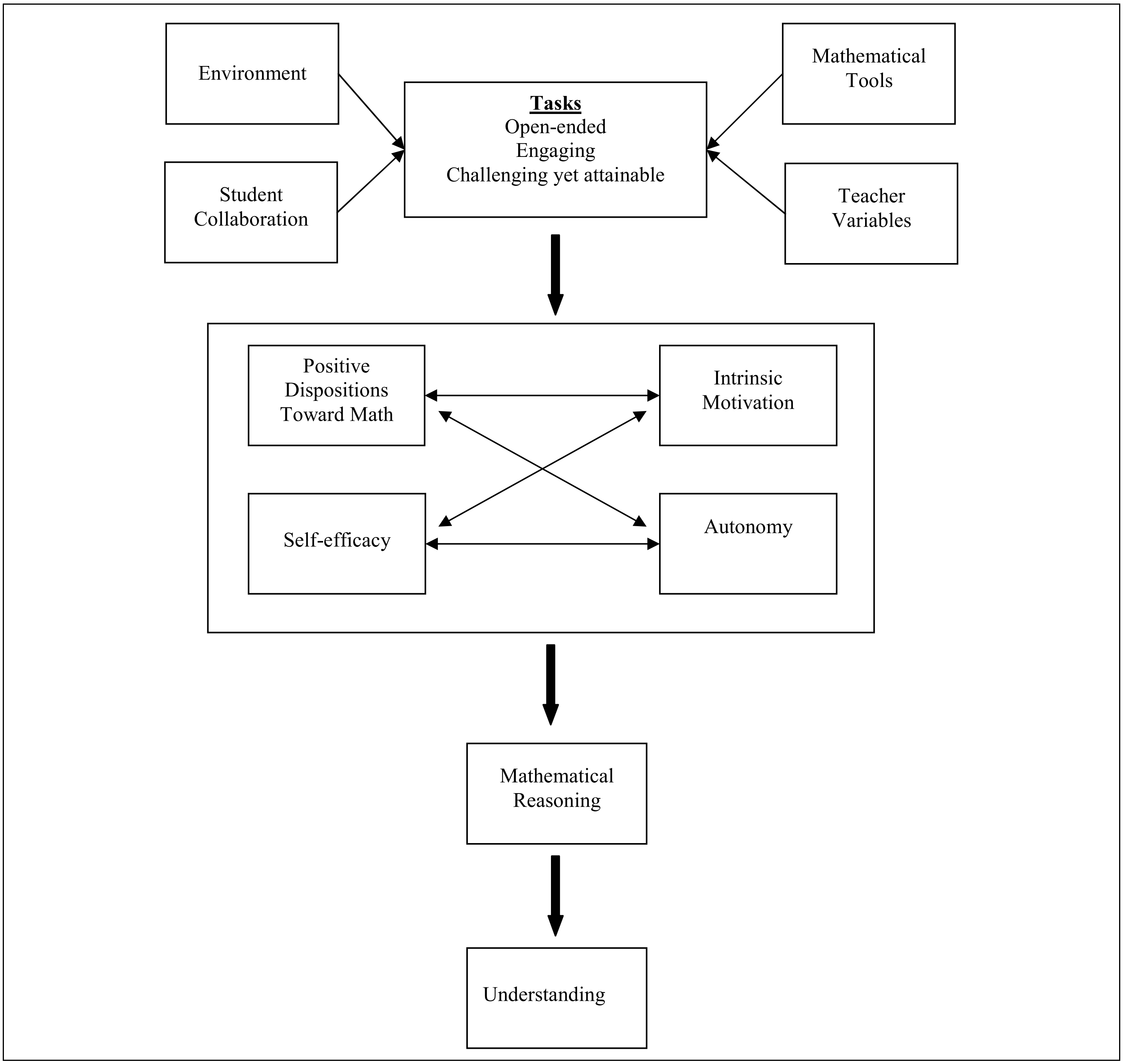

Figure 6. The relationship between contextual factors, motivation and other events leading to conceptual understanding.

particular, Dante proved to himself that 10/9 was a reasonable fraction and Stephanie was able to defend her doubling rule.

In summary, in such a learning environment, students are encouraged to communicate their understandings of the task, and their ideas are valued and respected. This respect engenders students' positive self-concepts in mathematics. At the same time, students become intrinsically motivated to succeed at mathematics. Intrinsic motivation fosters positive dispositions toward mathematics, which, in turn, encourage students to develop self-efficacy and mathematical autonomy as they discuss and share their understandings with their classmates. At the same time, students enjoy *doing* mathematics and develop ownership of their ideas. In such an environment and with such dispositions, students are more likely to engage in mathematical reasoning and, thus, acquire conceptual understanding.

Our framework and research suggest that with careful attention to developing appropriate and engaging tasks, a supportive mathematical environment, and timely teacher questioning, students can be encouraged to build positive dispositions towards mathematics in all mathematics classrooms. These positive dispositions towards mathematics, in turn, form the ideal conditions for achieving conceptual understandings of mathematics.

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