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Learning to Reason in an Informal Math After-School Program

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This research was conducted during an after-school partnership between a University and school district in an economically depressed, urban area. The school population consists of 99% African American and Latino students. During an the informal after-school math program, a group of 24 6th-grade students from a low socioeconomic community worked collaboratively on open-ended problems involving fractions. The students, in their problem solving discussions, coconstructed arguments and provided justifications for their solutions. In the process, they questioned, corrected, and built on each other's ideas. This paper describes the types of student reasoning that emerged in the process of justifying solutions to the problems posed. It illustrates how the students' arguments developed over time. The findings of this study indicate that, within an environment that invites exploration and collaboration, students can be engaged in defending their reasoning in both their small groups and within the larger community. In the process of justifying, they naturally build arguments that take the form of proof.

Generally, researchers concur that reasoning and proof form the foundation of mathematical understanding and that learning to reason and justify is crucial for growth in mathematical knowledge (Hanna & Jahnke, 1996; Polya, 1981; Stylianides, 2007; Hanna, 2000; Maher, 2005). For example, Ball and Bass (2003) identify mathematical reasoning as a basic mathematical skill and indicate that mathematical understanding depends on reasoning. The ability to reason is not only fundamental to learning new mathematics, but is also critical to applying that mathematical knowledge to other situations. Reasoning is a process that enables the revisiting and reconstruction of previous knowledge in order to build new arguments. Hence the ability to reason contributes to one's knowledge growth.

Maher and Davis (1995) indicate that different forms of reasoning coexist within a community of learners. In making one's argument public, input can be provided by others in the community. In a collaborative setting, students may question each other's ideas and help refine them. In this way, arguments can be shared and new, revised arguments can be coconstructed. Researchers also maintain that participating in discussions about mathematical ideas in a community of learners leads to mathematical reasoning (Balacheff, 1991; Cobb, Stephan, McClain, & Gravemeijer, 2001;

Maher, Powell, Weber, & Lee, 2006). Cobb (2000) stresses the relationship between an individual student's ways of thinking and the overall classroom practices in building students' mathematical understanding. There is increasingly more evidence that students successfully can create justifications, make and refute claims, and engage in mathematical reasoning given a supportive environment (Maher & Martino, 1996; Yackel & Hanna; 2003, Maher, 2005). However, reasoning that leads to proof is a relatively new finding. Examination of the conditions that tend to evoke proof-like arguments suggests that this does not happen without inviting students to justify their solutions. Students' arguments vary in completeness and elegance. The teacher has a key role in requesting that students explain and give evidence of their claims. Teachers are key in encouraging students to make their ideas public, and in offering arguments that are convincing to their classmates (Yackel & Hanna, 2003).

There is increasing evidence that students can be successful in reasoning and justifying their solutions to problems under certain conditions that invite sharing and collaboration (Bulgar, 2002; Francisco, 2005; Francisco & Maher, 2005; Maher, 2002, 2005; Maher & Martino, 1996; Mueller, 2007; Powell, 2003; Reynolds, 2005; Steencken, 2001). Students' success in justifying their ideas and in engaging in thoughtful mathematical activity can be underestimated by teachers whose expectations are that students follow certain procedures and apply those procedures to solve routine problems. Under these conditions, important opportunities are missed for students to learn to rely on their own resources and learn from each other in the building and sharing of arguments offered to justify solutions to problems.

The current study adds to the body of work on reasoning by extending the research to an informal, after-school setting with a population of students where expectations for thoughtful learning have been traditionally low. The three-year study investigates the forms of reasoning used by urban, middle-school minority students who worked collaboratively in an informal, after-school program to construct and justify solutions to problems. Because there has not been comparable research with this population of students under the informal conditions of the study, our research fills an important gap.

The participants were 24 urban, sixth-grade students, all volunteers from a generic after-school program in which students received general home-work assistance from school personnel or were involved in athletic activities. Students were recruited from the after-school program to participate in the mathematical problem-solving sessions.[1]

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[1] The sessions discussed in this paper were part of the "Informal Mathematical

Our paper is organised by episodes from sessions in the mathematics after-school program. We share transcribed data captured by videos to demonstrate the process by which students' mathematical reasoning develops. The informal, after-school setting was designed to provide a supportive environment for the children as they worked on strands of open-ended tasks. On the basis of our findings, we offer suggestions for practice for establishing classroom conditions that encourage student collaboration, risk-taking, and learning to reason.

The following questions guided our study:

(1) What forms of reasoning are used by student in justifying their solutions to problems posed?
(2) How does the input of other students contribute to building and justifying arguments? and
(3) How do the findings from this study compare to other, related work?

Theoretical Framework

The Role of Community in Reasoning and Proof

Research has shown that young children can support their reasoning with proof-like arguments (Maher & Martino, 1996; Maher, 2005; Yackel & Hanna, 2003). In settings where learners are encouraged to use each other as resources, arguments are first built from early, intuitive ideas and later extended to more general, formal forms of reasoning.

Reasoning. Skemp (1979) identified three different types of reasoning: instrumental understanding, relational understanding, and formal or logical understanding. He described instrumental understanding as occurring when one used rules or procedures without a conceptual understanding, and relational understanding as transpiring when one was able to deduce rules and procedures from general relationships. Finally, Skemp noted logical understanding when the learner was able to form strands of logical reasoning by connecting mathematical symbols with conceptual ideas. Skemp suggested that logical understanding occurs when the learner uses his or her relational understanding to explain their reasoning to others or to convince others in a community. With his research into the different levels of understanding, Skemp laid the framework for the importance of community

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in the role of individual understanding.

Thompson (1996) defines mathematical reasoning as "purposeful inference, deduction, induction, and association in the areas of quantity and structure" (p. 267). Yackel and Hanna (2003) extend this definition to recognise the social aspects of reasoning and describe it as a communal activity in which learners participate as they interact with one another to solve mathematical problems. Yackel and Hanna stress that given a supportive environment, all students, as early as elementary school, can and do make and refute claims and participate in inductive and deductive reasoning. At the same time, Yackel and Hanna state that creating a classroom atmosphere that supports mathematical reasoning is difficult and requires time and effort.

Justifications leading to proof. Stylianides (2007) defines proof as a *mathematical argument* that builds upon statements or facts that are accepted by the community as true, utilises various forms of reasoning shared by the community and within their conceptual reach, and is communicated by a shared meaning of discourse. Stylianides suggests that the notion of a valid mathematical proof in elementary school differs from what is accepted in high school based upon the terms or forms of reasoning used. According to Stylianides, students are afforded the opportunity to reason mathematically when they are allowed to use forms of argumentation to the best of their cognitive ability. Stylianides suggests that the notion of proof is dependent upon the community in which it emerges, indicating that as students engage in reasoning and justifying and communicate their reasoning to others, they begin to develop proofs that are appropriate to that community.

Francisco and Maher (2005) concur and suggest that in elementary mathematics classes the justifications of solutions that are convincing to students be emphasised in contrast to providing formal proofs. They suggest that in promoting the use of informal justification, students will be afforded opportunities to engage in proof-like activities before having access to formal notation. A further benefit is that students can learn to grow accustomed to the practice of convincing others of the validity of their ideas such that "proof-making" can become a fundamental part of problem solving.

Other researchers stress the role of discourse in the mathematics classroom in reasoning and proof (Balacheff, 1991; Hanna, 1991; Maher, 1995, 2008). Balacheff (1991) differentiates between argumentation and mathematical proof by describing argumentation as a mathematical conversation intended to convince another student of the validity of an argument, and proof as an explanation that is accepted by a the larger community. In contrast, he defines mathematical proof as meeting "the requirement for the use of some knowledge taken from a common body of knowledge on which people (mathematicians) agree" (p. 189). Balacheff suggests that argumentation naturally develops into proof as children experience the efficiency of argumentation in social interactions with other children and adults. Balacheff (1988) also makes a distinction between what he calls "proof" and "mathematical proof." The latter is that used in the larger mathematical community, while the former is the more informal lines of reasoning used by students in a classroom community. From this perspective, it might be argued that in a mathematical community, justification leads to proof.

The Role of Collaboration and Discourse in Mathematical Communities

Vygotsky (1978) hypothesised that students internalise the discussions that occur in group contexts. He suggests that what learners "can do with the assistance of others is more indicative of their mental development than what they can do on their own" (p. 85). His work suggests strong benefits for collaboration in learning. Cobb, Wood, and Yackel (1992) build on this claim by stressing the role of social interaction in the construction of mathematical knowledge and calling attention to the role of discussions in a mathematical community.

One difference between formal and informal settings in the development of knowledge may be the formation of a classroom community that supports the sharing of ideas and knowledge. Communication with others in learning communities provides other opportunities for learning. Goos (2004) describes mathematical communities as communities of mathematical inquiry in which students learn to talk and work mathematically by participating in mathematical discussions, proposing and defending arguments, and responding to the ideas and conjectures of their peers. In building such communities the teacher models desirable behaviors, establishes expectations and sociomathematical norms, and engages students through thoughtful tasks and careful questioning (McCrone, 2005; Yackel & Cobb, 1996).

While one view of collaboration involves learners supporting each other by offering missing pieces of information needed to solve the problem, a more powerful form of collaborative work involves group members relying on each other to generate, challenge, refine, and pursue new ideas (Francisco & Maher, 2005). With this type of collaboration, rather than piecing together their individual knowledge, the students build new ideas and ways of thinking as a group.

Martin, Towers, and Pirie (2006) stress the importance of the social context of the learning environment in cultivating what they call *collective* *mathematical understanding*, that is, understanding that occurs when a group of learners work together on a mathematical task. They suggest that in supporting developing coactions, that is, actions carried out by an individual while being dependent on the actions of others in the group, learning is facilitated. They differentiate coactions from student interactions to indicate the importance of mutually acting with the ideas and actions of others.

That the classroom micro culture or community in which learning occurs has a major influence on the meanings that students construct is well established. When students engage in thoughtful mathematical discussions in a rich social environment, student ideas are made public and shared, and sometimes modified and agreed upon. These activities are fundamental to the building of a successful and active mathematical community of learners.

Conditions for Promoting Mathematical Reasoning

Several years ago, Yackel and Hanna (2003) emphasised that we were only beginning to understand how students' mathematical reasoning develops and learn about what type of environments support this development. Over the years, however, progress has been made. Through extensive analysis of data from cross sectional studies and a longitudinal study, now in its 23rd year, Francisco and Maher (2005) found that certain conditions are necessary in promoting mathematical reasoning. Those conditions were implemented in the study on which we report here: the posing of strands of challenging, open-ended tasks, establishing student ownership of their ideas and mathematical activity, inviting collaboration, and requiring justification of solutions to problems. Recognising that these conditions need to be in place to promote an environment for student reasoning, we describe each of these factors in more detail as follows:

Tasks. Francisco and Maher (2005) stress the crucial role of the given task in sustaining student engagement in problem solving and promoting sense-making and mathematical reasoning. Presenting students with complex tasks, rather than scaffolding a series of simple tasks, stimulates mathematical reasoning and leads to the building of mathematical understanding. Francisco and Maher suggest that in addition to using challenging tasks, the tasks should be presented to students as strands of related problems that can be revisited over time and in different contexts, thus enhancing students' opportunities to overcome obstacles in problem solving. Doerr and English (2006) suggest that tasks be designed to engage students in important mathematical problem situations such that their representations and justifications offer insight into their mathematical thinking. In addition, they indicate that tasks should allow for students to self-evaluate their solutions and reflect on their own reasoning.

Supportive environment. Seating students in small, heterogeneous groups affords them the opportunity to build ideas collaboratively, test out their own conjectures with a non-threatening audience and hear the ideas and justifications of others. Offering students ample time to explore allows them to internalise their own ideas and those of others and test out alternative theories when necessary. Without time pressures, students can lead their own explorations and formulate their own challenges. Inviting students to share ideas with the whole class allows them to hear other ideas and strategies and make connections with their own thoughts. Finally, requiring students to be the arbitrators of what makes sense gives them the responsibility for evaluating their own justifications and those of their peers and ultimately leads to mathematical autonomy.

The role of the facilitator. Maher (1998) identified characteristics of teachers/researchers in facilitating sessions that promote problem solving and reasoning. One is to encourage students to interact with one another. Another is to elicit from students representations of their mathematical ideas and offer tools for multiple representations. The importance of eliciting from students their explanations and encouraging students to provide justifications cannot be overemphasised. By building a classroom climate that promotes student discourse and providing time and flexibility for the sharing of ideas, teachers allow students to share representations among each other. Providing opportunities for students to revisit ideas and connect these to new ideas facilitates transfer in learning.

Manipulatives as tools for model building. Making manipulative tools available to students for building physical models in problem solving encourages and promotes the exploration, representation, and communication of mathematical ideas (National Council of Teachers of Mathematics [NCTM], 2000). These tools allow students to manifest their thinking and share their representations with others. They also support the building of multiple representations and justifications of the ideas represented. Manipulatives as tools for model building can further support students' presentation of ideas and act as a "prop" in communicating students' developing reasoning. In this study Cuisenaire rods were available as tools for students to build models of solutions to the problem tasks.

A combination of student, teacher, task, and environment promotes understanding in the mathematics classroom. A teacher works regularly to facilitate an environment that offers students time for exploration and reinvention. In this study, working to keep intact the conditions described above, we document the forms of reasoning students used to convince each other and the researchers of the reasonableness of their solutions. We report on the development of reasoning that emerged as students provided justifications for their solutions over five sessions, each about an hour and a half in duration.

Method

Participants

The 24 participants for this study consisted of volunteers who were beginning sixth grade. They represented a wide range of math backgrounds, ranging from those who were enrolled in remedial mathematics to those who succeeded in school math.[2] Eight students, four males and four females, were the focus group for the study**.** All eight participants scored in the *below proficient* category on the eighth-grade state assessment exam.

Data Source and Setting

During the four week period, for five, 60-75 minute sessions, the participants worked in small groups on a strand of tasks dealing with fractions. All sessions were videotaped with four different camera views. Three of the cameras focused on two groups of students each and one of the cameras, the "roving" camera, followed the facilitators as they moved from table to table and also captured the presentations at the overhead projector. Video recordings and the transcripts were analysed using the analytical model outlined by Powell, Francisco, and Maher (2003). The video data were described at frequent intervals; critical events (episodes of reasoning and coconstruction of arguments) were identified and transcribed, and codes were developed for flagging solutions offered by students and the justifications given to support these solutions. To ensure dependability and validity another researcher verified coding of the data. If necessary a third researcher was asked to resolve disagreements. Researcher observation notes were used to supplement the transcripts and assist in constructing a story line. A researcher/graduate assistant was assigned to each table to take field notes. At the end of each session the facilitators, researchers, and graduate students met to discuss what transpired during the session, share researcher field notes, and plan for the next session. In addition, triangulation was achieved by using student work and explanations to make sense of the models that they created and written accounts of their reasoning. Students were asked to record all work on transparencies and

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2 The group of students who volunteered was representative of the overall population of sixth graders of that school. However, the research team deliberately chose not to identify students according to earlier success in school mathematics.

these were collected at the end of each session.

Tools

Cuisenaire rods were available for students to build models of their solutions to the tasks. A set of Cuisenaire rods, as shown in Figure 1, contains 10 coloured wooden or plastic rods that increase in length by increments of one centimetre. For these activities, the rods have variable number names and fixed colour names. As part of introducing students to working with the rods, the researchers explained that the rods are given permanent colour names. These names, along with the rods' respective lengths, are: white (1 cm); red (2cm); light green (3 cm); purple (4 cm); yellow (5 cm); dark green (6 cm); black (7 cm); brown (8 cm); blue (9 cm); orange (10 cm). Students were introduced to rods of different lengths by placing rods along side each other and making a "train". Figure 2 shows a "train" of a purple and an orange rod.

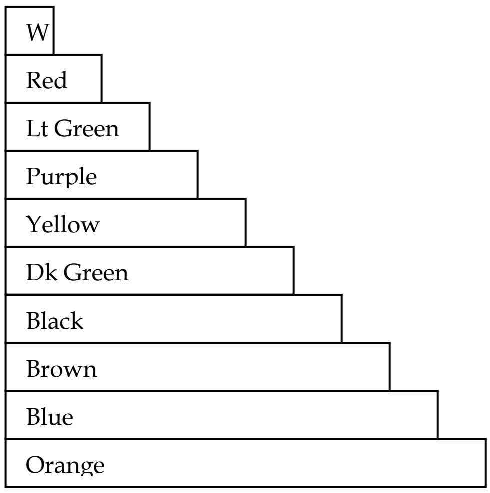

Figure 1. Staircase model of rods.

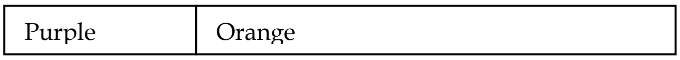

Figure 2. A train of a purple rod and an orange rod.

Tasks

Students were offered time to explore and discuss problem tasks. The tasks investigated by the students were developed from earlier research with fourth- and fifth-grade students. These tasks were found to invite collaboration and elicit justifications of solutions to problems, generating a variety of forms of reasoning.[3] Consequently, a set of tasks from this strand was used as an introduction to the IML program.

Unlike the students from the earlier study, who investigated fraction tasks before they were introduced formally in the school curriculum, the sixth-grade students in the study we report here were introduced to fraction rules and procedures as part of their school mathematics in Grade five. The research team was aware that the students had little, if any, conceptual understanding of the fraction ideas and operations that they were taught the previous year.

During each session, problems from the strand were presented to the entire group of 24 participants. For example, in Session 2, the following problem was presented:

What number name would you give to the dark green rod if the light green rod is called one? Discuss the answer with your group. (Maher, 2002.)

Groups were then provided time to investigate their solutions and make claims, first in their small groups and then with the whole class. Once each group had completed the task, they were invited to the overhead projector to share their findings with the larger group. During these whole group discussions, students discussed their ideas, challenged each other, and had opportunity to reflect on and revise earlier solutions. Table 1 outlines the tasks that were posed during each session.

[3] See Steencken and Maher (2002, 2003).

Table 1
Challenges Posed During the First 5 Sessions

Date	Tasks
11/12/03	1. If I gave the yellow rod the number name five, what number name would I give to the orange rod?
	2. Suppose I gave the orange rod the number name four, what number name would I give to the yellow rod?
	3. If I call the orange rod one, what number name would I give to the yellow rod?
	4. If I call the white rod two, what number name would I give to all of the other rods?
11/13/03	1. Suppose I called the dark green rod one, what number name would I give to the light green rod?
	2. Someone told me that the red rod is half as long as the yellow rod, what do you think?
	3. If I call the blue rod one, I want each of you to find me a rod that would have a number name one-half.
11/19/03	1. Convince us that there is not a rod that is half the length of the blue rod.
	2. Is 0.3 another name for the light green rod?
	3. If I call the blue rod one, what number name would I give to the white rod? What name would I give to the red rod?
11/20/03	1. If I called the blue rod one, then what number name would I give to the red rod? What name would I give to the light green rod?
	2. If I called the blue rod one, what number names would I give to the rest of the rods?
12/3/03	1. If I called the orange rod one, what number name would I give to the white rod? What name would I give to the red rod?
	2. If I called the orange rod ten (fifty), what number name would I give to the white rod?
	3. I want to know which is bigger, one-half or one-third and by how much.

Analysis

The following dimensions were used to code for arguments and justifications: form of reasoning (e.g., direct; by contradiction; by cases; by upper and lower bounds) and validity (e.g., warrants; valid inference; counter argument). Each form of reasoning was defined for the purpose of this study; these definitions are provided in the following section. While coding, it was noted that, at times, students exhibited faulty reasoning and offered incomplete arguments. Consequently, these arguments were included as subcodes. At times students would offer an argument that addressed a classmate's faulty argument rather than the original question. In this case, the arguments were flagged and coded as counter arguments.

Direct reasoning. Direct reasoning is used to establish the truth (or falsehood) of a given statement based on a combination of established facts. In this model, it is first assumed that p is true, and then steps are taken to arrive at the conclusion that q is true (Fletcher & Patty, 1995). A direct proof takes on the form: "If p then q." Often, in a direct proof of the statement $p \rightarrow q$, the transitive nature of implication is employed: if $p \rightarrow r$ and $r \rightarrow q$ then it follows that $p \rightarrow q$.

Reasoning by contradiction. Reasoning by contradiction, also known as the indirect method, is based on the agreement that whenever a statement is true, its contrapositive is also true; or that a statement is logically equivalent to its contrapositive. For example, $p \rightarrow q$ is equivalent to (not q) $\rightarrow$ (not p); so if (not q) $\rightarrow$ (not p) is true, then $p \rightarrow q$ is also true (Cupillari, 2005).

Reasoning by cases. In the five sessions, students often considered different cases with the Cuisenaire rods when forming an argument. For the purpose of this study, justifications were coded as reasoning by cases when students defended an argument by defending separate instances. They defended an implication in the form $p \rightarrow q$ by identifying propositions p_1, p_2, …p_n and established each of the implications: $p_1 \rightarrow q$, $p_2 \rightarrow q$, $p_n \rightarrow q$.

Reasoning using upper and lower bounds. Students sometimes create upper and lower bounds to argue about fraction lengths and equivalences. An *upper bound* of a subset is an element which is greater than or equal to every element in that set. A *lower bound* of a subset is an element which is less than or equal to every element in the set ("Upper and lower bounds", 2008). In addition, when reasoning using a lower and upper bound argument, it must be also established that there are not elements in the set in between the bounds. In using upper and lower bounds, a student defines the upper and lower boundaries or limits of a class of numbers or mathematical objects. For example, for the set of numbers $\{1<x<4\}$, the upper bound of the set is 4 and the lower bound is 1, since all the numbers in the set are contained within the two bounds. After these bounds have been defined, the student reasons about the objects between these bounds. This form of reasoning is often used to show that the set that is defined as all objects between the two identified bounds is empty (Yankelewitz, 2009).

Arguments were coded to indicate to what extent, if at all, the reasoning that was exhibited built upon or challenged previous students' arguments. The codes that emerged in this category were organised into three subsets: building on each other's ideas, questioning each other, and correcting each other. As the data were analysed, subcodes emerged. It was noted that in building on each other's ideas, students expanded upon, redefined, and reiterated each other's arguments.

Results

In order to address the research questions, the results section is divided into two categories: justifying solutions and community influences. In each of these sections we present selected episodes from the five sessions of the after-school program as evidence of students' presentation of ideas and argumentation.

Justifying Solutions

Episode 1, Session 2, finding a rod named one-half when the blue rod is named one. While attending to the task of finding a rod whose length was half of that of the blue rod the students produced arguments that represented four forms of reasoning. Each of these arguments is outlined below.

While working in their small group, Michael and Shirelle each built a model of a purple rod and a yellow rod lined up next to the blue rod and both students used *direct reasoning* to show that the purple and yellow rods were equivalent to half of the length of the blue rod. Dante and Chanel offered counter arguments to this invalid reasoning by showing that the purple rod was too short and the yellow rod was too long. While attempting to convince Michael and Shirelle that a rod named one-half did not exist given that the blue rod was named one, Dante and Chanel formulated a justification based on *upper and lower bounds* (described below).

At an adjacent table, Chris, Jeffrey, Brittany, and Danielle worked on the same task. While his group members began the task by building models of rod combinations equivalent to the length of the blue rod, Chris explained that one-third of the blue rod could be shown, but not one-half. Chris reasoned using *contradiction* to show that the blue rod did not have a rod equivalent to half of its length. He lined up a train of nine white rods next to the blue rod and explained to his group, "There is not a rod that is half of the blue rod because, there's nine little white rods you can't really divide that into a half so you can't really divide by two because you get a decimal or remainder so there is really no half, no half of blue because of the white rods." He built the model shown in Figure 3 to illustrate his justification.

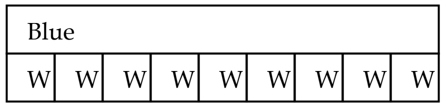

Figure 3. Chris's model of nine white rods lined up next to the blue rod.

As his group members offered different arguments, Chris refined his contradiction four times before presenting it to the whole class. Chris posed his arguments as follows: (a) "If you take out four that's an even number but if you put the four back, that's not a half because it's nine and nine is an odd number", (b) "You can't find a half of the blue one because if you put all white you only have nine so for nine you can't really do it", (c) "Overall you can't do it because if you use a white one it is an odd number so you can't divide by two", (d) "The thing we should say is that since we put the white cubes and we got an odd number then if you have an odd number you can't divide by two so you get one-half so you get a decimal or a remainder so you can't really divide it, right?" During the whole class discussion Chris explained, "There's nine little white rods you can't really divide that into a half so you can't really divide by two because you get a decimal or remainder so there is really no half, no half of blue because of the white rods."

Following Chris, Justina explained that her strategy of showing that the blue rod does not have a rod that is equivalent to half of its length was to instead find all of the rods that do have a rod equal to half of their length. Justina presented the diagram show in Figure 4. Justina used a *case approach* to justify her solutions and drew all of the rods that have a half next to the two rods that make up the half, for example, two yellow rods lined up next to an orange rod. Justina explained that all of the rods in her diagram had a rod that was equivalent to half of their length. She listed all of the cases of these rod combinations and named them "singles". Justina explained, "I was just making half of the color rods, I just made this picture, so like um, half of the orange was yellow, half of the brown was purple, half of dark green was light green, and the same for those two."

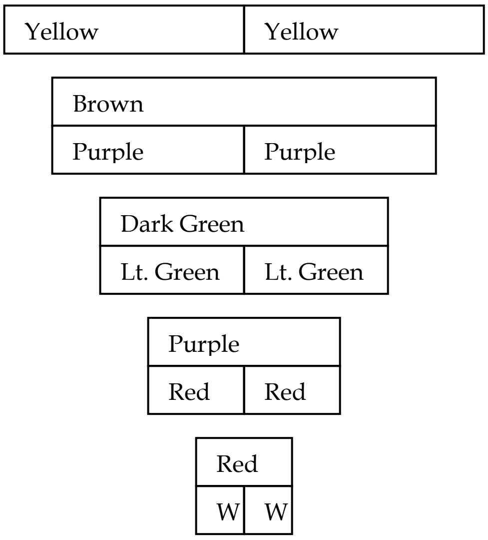

Figure 4. Justina's models of "singles."

In response to Chris's argument, Dante explained that instead of using the model of nine white rods lined up next to the blue rod he used a model of a purple rod and a yellow rod. He used the model illustrated in Figure 5 to show that the purple rod could not be considered to be half of the blue rod because the combination of two purple rods was not equivalent to the length of the blue rod (they were too short). Likewise, the yellow rod could not be named half of the blue rod because the combination of two yellow rods was not equivalent in length to the blue rod. He explained that the yellow rod is one white rod too long to be a half the length of the blue rod and the purple rod is one white rod too short. When asked why this persuaded him that there was not another rod whose length was half of the blue rod, Dante responded, "Because we tried all we can because if usually for the blue piece, it would usually be purple or yellow but yellow would be one um one white piece over it and the pink would be, I mean purple would be one white piece under it."

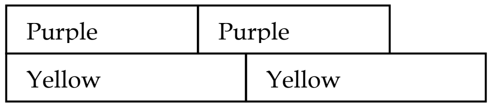

Figure 5. Dante's upper and lower bound model.

Chanel backed up Dante's justification and displayed the model illustrated in Figure 6, which shows the discrepancy of one white rod, using two yellow rods as an upper bound and two purple rods as a lower bound., indicating: "this is blue and the yellow is a little, the yellow is a little bit more than a half and the purple is shorter than a half."

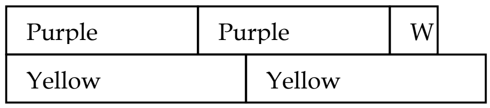

Figure 6. Chanel's upper and lower bounds model.

Analysis. This task elicited multiple forms of reasoning. It required that students show that the proposed rod did not exist. Although Michael and Shirelle attempted to use the direct relationship between the length of the yellow rod and purple rods and the blue rod, they seemed to not take into account the precise meaning of one half. Their use of *direct reasoning* was based on a *faulty* premise. Dante and Chanel used the upper bound of yellow and the lower bound of purple to counter this faulty argument. During whole-class presentations they completed this upper and lower bounds argument by stating that there was not a rod whose length was in-between the purple and yellow rods and showed the difference as being one white rod (the smallest rod in the set).

Chris's group members attempted to find all of the rods that did have a rod equivalent to half of its length as an exhaustive strategy. While they were building these models Chris explained his model based on the nine white rods representing an odd number using reasoning by *contradiction*. It seemed to be important to Chris that his partners shared his reasoning and therefore he restated his strategy five times, each time strengthening his original argument. Justina's group took a different approach by focusing on the rods that did have a rod equivalent to half of their length. They used the fact that the blue rod did not fit into this category as proof that when the blue rod was named one there was not a rod whose name was one-half and built an argument based on *cases*.

While working on the task and later during whole-class sharing the students did not seem concerned that they had built different models and formed justifications that differed from those of their peers. Rather, they used input from the other arguments to strengthen their own.

Episode 2, Session 4, naming all of the rods when the blue rod is named one. As students worked on the task of naming all of the rods, many were challenged with the naming of the orange rod as it resulted in an "improper" fraction. After working on the problem for a length of time students were asked to come up to the overhead to share their solutions. Lorrin named the orange rod by incrementally increasing by one-ninth and explained, "Before, we thought that because we knew that the numerator would be larger than the denominator and we thought that the denominator always had to be larger but we found out that that was not true. Because two yellow rods equal five-ninths and five-ninths plus five-ninths equal ten-ninths." She used direct reasoning to show that the length of two yellow rods was equivalent to the length of the orange rod. She explained that a yellow rod was named five-ninths so two yellow rods (five-ninths plus five-ninths) would be called ten-ninths (an orange rod).

Kia-Lynn also used direct reasoning and explained that when a white rod was attached to a blue rod, the length was equivalent to an orange rod. She built a train of a blue rod and a white rod next to an orange rod, shown in Figure 7, and named the orange rod ten-ninths. Kia-Lynn explained, "If you have one white rod and you add it to the blue, it's one-ninth plus one is one and one-ninth and so if the blue rod and one white. If you put them together then this means that it's ten-ninths also known as one and one-ninth."

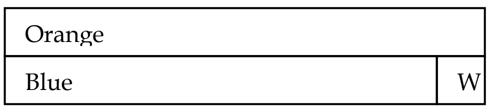

Figure 7. Kia-Lynn's model for naming the orange rod one and one-ninth.

Finally, Dante shared his strategy of using two purple rods and a red rod to name the blue rod ten-ninths by adding four-ninths, four-ninths, and two-ninths.

Analysis. Although all of the presenters in this episode used direct reasoning, they each built a different, but equivalent, model on which to base their arguments. Lorrin used the relationship between the blue rod and the two yellow rods to show that five-ninths plus five-ninths is equivalent to ten-ninths. Kia-Lyn and Kori built a model with a train of a blue rod and a white rod lined up next to the orange rod and directly reasoned that one-ninth plus nine-ninths is equivalent to ten-ninths. They also used this relationship to establish the other name for ten-ninths or one and one-ninth. Finally, Dante presented the model discussed above, reasoning that four-ninths plus four-ninths plus two-ninths is equivalent to ten-ninths.

Episode 3, Session 5, naming the red rod when the orange rod is named one. While students were working on this task, the researcher pointed out that some students were naming the red rod one-fifth, some were calling it two-tenths, and one student reported that the white rod was equivalent to half of the red rod. The researcher asked the class if some, none, or all of these statements were true. Chris reported that the statements were true. He used direct reasoning to show that the length of ten white rods was equal to the length of an orange rod, and the length of five red rods was equal to the length of an orange rod. He concluded that the red rod would be named one-fifth.

Chris: It's true because if ten white ones equal an orange one, five red ones equal an orange one and the red one is one-fifth.

Dante then named the red rod two-tenths. He explained using direct reasoning: "I think that red would be two-tenths because two times; you need five reds to get to make the orange one, so two times five would equal ten." The model that Dante used to explain his reasoning is shown in Figure 8. The researcher asked Dante to explain his idea to the class.

Dante: I think this would be two-tenths, that red would be two–tenths because two times, cuz you need five reds to get to make the orange one so two times five would equal ten so that's why I think it would be two-tenths.

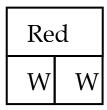

Figure 8. Dante's model used to name a red rod two-ninths.

Analysis. During whole-class presentations Chris used direct reasoning to name the red rod one-fifth. He created a model depicting the five red rods lined up next to the orange rod and explained that since five red rods were the same length as the orange rod, one red rod would be named one-fifth. Although he included white rods in his model, he did not use them in his argument. Dante also used direct reasoning to name the red rod two-tenths, although his argument was structured differently than Chris' direct argument. Dante established that five red rods were the same length as the orange rod and the red rod was equivalent to two white rods; he then used multiplication to establish the denominator of ten. Dante's model is shown in figure 3.

The Influence of the Community

Episode 4, Session 2, finding a rod named one-half when the blue rod is named one. As described previously, during the second session of the after-school program the students were asked to find a rod whose number name was one-half when the blue rods was named one. Michael and Shirelle each built a model of a purple rod and a yellow rod lined up next to the blue rod and both students attempted to use *direct reasoning* to name the yellow rod and purple rod one-half. Dante and Chanel offered counter arguments to this invalid reasoning by showing that the purple rod was too short and the yellow rod was too long. Dante argued, "If you put two purple together it's still smaller than the other, than the blue." He then explained, "The yellow rod takes up more space than the purple rod and to be halves they should be the same." Finally, he said, "Purple is smaller than yellow so it can't, this is not a half, yellow might be a half to orange but it's not a half to blue, purple is not a half either to blue." Based on these counter arguments Dante and Chanel formulated a justification based on *upper and lower bounds* (described above).

Analysis. Michael and Shirelle used the direct relationship between the length of the yellow rod and purple rods and the blue rod to name these rods one-half. They based this argument on the fact that the two rods were equivalent in length to the blue rod ignoring the part of the definition of one-half that states that the two "halves" must be equivalent. Thus they used direct reasoning based on a faulty premise. Dante and Chanel countered this faulty reasoning using the definition of one-half in an attempt to convince the two students that neither the yellow rod nor purple rod fit the definition of one-half of blue. As they justified this counter argument they described the upper and lower bounds of the rod that could be half of the blue rod.

Episode 5, Session 4, naming all of the rods when the blue rod is named one. In order to represent the relationship, Chanel created a staircase of rods of increasing lengths and named them beginning with one-ninth (white rod) and building incrementally by ninths. After naming the blue rod nine-ninths, she stopped at the orange rod and said she had to think about it and asked Dante what he thought the orange rod would be called. He immediately replied ten-ninths but then changed his mind and called it one. He then said it would start a "new one." Michael called the orange rod a whole and Dante named it one-tenth.

As the students were drawing the staircase of rods, Dante told the group that he heard students at other tables calling the orange rod ten-ninths. Michael insisted that they were incorrect, since the orange is equivalent in length to ten white rods. Chanel agreed and said that the denominator cannot be smaller than the numerator and Dante concurred. When asked to explain what he was thinking about the orange rod, Dante tried to convince one of the facilitators that the orange rod "starts a new one" and would therefore be named one-ninth. When reminded that the white rod was named one-ninth, Dante used the model of the staircase to name the orange rod ten-ninths. He explained that the length of ten white rods is equivalent to the length of an orange rod and since a white rod is called one-ninth the orange rod will be called ten-ninths. Even after physically building his model and seeing that the orange rod would be named ten-ninths, Dante asked, "But how can the numerator be bigger than the denominator?"

Later during the same session, students were asked to share their findings with the whole class. After a few other groups shared their strategies, Dante shared his model. He reported that he began with the white rod and placed the white rods up to the orange rod, using a similar strategy as one of his classmates. He said that he then found a different way to name the orange rod by using two purple rods and a red rod, as shown in Figure 9, and explained, "Since four and four are eight so which will make it eight-ninths right here and then plus two to make it ten-ninths."

<table>
<tr><td colspan="3">Orange</td></tr>
<tr><td>Purple</td><td>Purple</td><td>Red</td></tr>
</table>

Figure 9. Dante's model for naming the orange rod ten-ninths.

Analysis. In this episode there was a conflict between what the Dante built (a correct solution) and what he claimed to have remembered from his fifth-grade fraction learning. In initiating a discussion with his group he suggested that an alternative solution might exist. His partners discounted this idea and Dante seemed to agree. When the researcher asked Dante what number name he gave to the white rod (one-ninth), Dante used his model and direct reasoning based on the incremental increase of one-ninth to name the orange rod ten-ninths. After listening to the other presenters explaining their solutions and models, Dante shared his solution. Instead of presenting the argument he built with his group (increasing from one-ninth to ten-ninths), Dante built an alternative model using two purple rods and a red rod and using direct reasoning to explain that the sum of four-ninths (purple rod) plus four-ninths plus two-ninths (red rod) is equivalent to ten-ninths. By being able to explain using a different model, Dante solidified his own understanding and convinced himself as he was attempting to share his justification with classmates.

Episode 6, Session 5, naming the white rod when the orange rod is named one. Herman initiated the task by naming the white rod ten and using direct, although faulty, reasoning to explain that the orange rod was equivalent to ten white rods and therefore the white rod would be called ten. The researcher asked the class if they agreed with Herman. Dante challenged Herman and compared the task to that of naming the rods when the blue rod was named one.

Dante: Because I thought, since, like we did with the last one as blue as one, as blue as one and the white one was uh [he comes to the overhead projector (OH)] cuz we used to say that the blue was one so I thought that if the, if the that we called the white one one-ninth, why can't we still call it with the orange one, one-tenth though, cuz even though the orange one is one white one bigger than it … we should still….so like this is nine …this is ten ..that's why I think it should be one-tenth, I think it should be called one-tenth.

When Herman disagreed, Dante said that he could "prove it" and used the model, shown in Figure 10, that Herman had built at the overhead projector.

Dante I can prove…[R1 asks everyone to listen to Dante's proof] you have it up here already… because he already has it up here – see you need ten of these (white) to equal one orange rod. If we take nine of them away [Dante does this on the overhead projector] which will leave you with one-tenth and then if you keep if you add another one it will be two-tenths, three-tenths all the way up to ten. Which is one whole.

R1 So let me see if I understand what Dante is saying, you're saying that this is one-tenth and then if you add another one it would be one-tenth

Orange									
W	W	W	W	W	W	W	W	W	W

Figure 10. Herman's model for naming the white rod when the orange rod was named one.

The researcher again asked Dante if he agreed and Herman agreed with Dante and explained why the white rod would be named one-tenth in his own words.

R1	Ten-tenths is another name for one. What do you think about that Herman?
Herman	I agree
R1	Why? What changed your mind?
Herman	Because, because each one of these (white) equals one-tenth and if this (orange) is one whole.

Analysis. Herman built a model of ten white rods lined up next to the orange rod and used direct, although faulty, reasoning to name the white rod ten. Dante challenged this solution and justified his claim using an analogy based on the exploration from the previous session (when the blue rod was named one and the white rod was named one-ninth). He explained that with this task nine white rods were the same length as the blue rod and therefore the white rod was named one-ninth. He then used direct reasoning and explained that the orange rod was one white rod longer than the blue rod and therefore the white rod would be named one-tenth. When Herman was still not convinced, Dante used the model (of ten white rods lined up next to the orange rod) that Herman had constructed on the overhead projector to justify his response. He explained that if he removed nine white rods from the model the one remaining rod would be named one-tenth. He then backed this up by showing the incremental increase in white rods by adding one-tenth each time he replaced a white rod until he reached ten-tenths. When asked to explain, Herman was able to explain Dante's solution in his own words.

Episode 7, Session 5, naming the red rod when the orange rod is named one. One of the students (as reported previously) named the red rod one-fifth using a model of ten white rods and five red rods lined up alongside the orange rod. The researcher asked the rest of the students if they agreed with this number name. Chanel stated that she agreed that the white rod was half of the red rod and explained the equivalence (the length of two white rods was equivalent to the length of the red rod). She stated that if one of the white rods was removed, the other white rod would be half of the red rod and named the red rod one-half. The researcher asked for the class's opinion about Chanel calling the red rod one-half (when the orange rod was called one) and Dante said that he disagreed. He explained that the length of the red rod could not be half of the length of the orange rod because the length of the yellow rod was half of the length of the orange rod. He asked, "How could red be a half of orange if it takes five of them instead of two?" Dante explained that he recognised Chanel's source of error and attempted to explain it to her using an analogy by comparing the relationship using whole numbers.

Analysis. Chanel accepted the relationship between the red and white rods but erroneously used this relationship to name the red rod one-half. Dante then used a counter argument to correct Chanel based on the previously accepted relationship between the yellow rod and the orange rod. He contradicted Chanel's argument by showing that if the length of the yellow rod was equivalent to half of the length of the orange rod then the smaller red rod could not share this characteristic.

Conclusion

Emerging Forms of Reasoning

During the five sessions, there is evidence that students naturally used different types of arguments in justifying their solutions to the problem solving tasks: direct reasoning, reasoning by contradiction, upper and lower bounds, and case-based reasoning correctly to support their solutions. Episode 1 highlights the four different forms of reasoning (direct, contradiction, cases, and upper and lower bounds) that the students used when attending to a single task. It seems as though students' personal styles of thinking played a role in the type of representations and arguments that they built. Based on their individual mode of sense-making the students built alternative models and offered alternate forms of reasoning. The nature of the task elicited these different forms of reasoning. In fact, the students were asked to find a solution that did not exist and therefore were engaged in the act of disproving. Students seemed eager and confident to share their various opposing arguments, first with their small groups and later with the whole class.

When a task elicited the same form of reasoning, students offered different arguments with different rod models as backing. For example, in Episode 3, when naming the red rod (when the orange rod was named one), Chris and Dante both used direct reasoning and built a similar model. However, they displayed rod models for these arguments differently and found two different number names for the red rod (one-fifth and two-tenths). Likewise, in Episode 2 when naming the orange rod ten-ninths, the three students presenting arguments all used direct reasoning, however, they built four different models and presented different rod models. Again, students' particular models influenced their fraction representations, introducing them to a way of thinking about equivalent fractions.

The Influence of the Community on Reasoning

Examining the data across five sessions, we found that sometimes a student's faulty argument was pivotal in building meaning and promoting deeper understanding. The act of presenting justifications to the community and listening to the arguments of others seemed to prompt students to challenge each other's assertions which in turn led to stronger arguments.

Revisiting previous misconceptions. In analysing the sessions we found evidence that listening to the arguments of others prompted students to revisit previous misconceptions and challenged previously learned "rules". In episode 5, Dante overheard another group naming the orange rod ten-ninths and this planted a seed in his head. He then needed to convince himself and his partners that this name was reasonable. Even after using the physical model and incrementally building the white rods by ninths, Dante was not fully convinced. However, after listening to his classmates present their arguments using two different explanations, Dante was able to justify the name ten-ninths using two models and justifications.

Confidence in challenging arguments. In Episode 4, Dante and Chanel challenged Michael and Shirelle using a counter argument in an attempt to convince them that the yellow and purple rods could not be named one-half when the blue rod was named one. This led Dante and Chanel to the creation of a complete upper and lower bounds justification. Chris challenged himself to convince his partners that his argument made the most sense and this led him to refine his argument five times.

In Episode 6, Herman named the white rod ten (when the orange rod was named one). Dante challenged this naming by first comparing the task to the prior relationship between the white rod and the blue rod (when the blue rod was named one the white rod was named one-ninth) and then using direct reasoning to show the incremental increase from one-tenth to ten-tenth using Herman's physical model. Later in the session, Episode 7, when Chanel named the red rod one-half, Dante corrected her by offering a counter-argument in the form of a question ("How could red be a half of orange if it takes five of them instead of two?"). He then created an analogy using whole numbers to explain the relationship; finally, he switched back to fractions and used multiplication.

Connections to Previous Studies

The eliciting of a variety forms of reasoning in justifying solutions to the tasks is consistent with the findings of previous studies in other contexts and with students of different ages. The sixth-grade students in this study, similar to the students in previous studies, displayed convincing arguments representing various forms of (Maher, 2005). However, this study differed from the previous studies in three ways. First, the students in the other studies had been working together for years in an established mathematical community while the students in this study worked together in fewer sessions time of approximately nine hours of contact together. Yet, as early as the first session, the students engaged in mathematical discourse and appeared to enjoy it, as evidenced by their continuation in this volunteer program. Second, the students in this study joined the study in Grade 6. All came from a disadvantaged, urban environment in which open-ended, group problem solving was not available in regular classrooms. It is noteworthy, also, that the students in the study were not previously successful in school mathematics. In fact, they were categorised as *below proficient* in performance on the state mathematics assessment. Nevertheless, in our study, they exhibited well-formulated, valid reasoning in their solutions to the problems. Third, this study was conducted after-school, in an informal, environment rather than in the context of the school mathematics classroom. In the after-school sessions, students were encouraged to share ideas, collaborate with each other and support their solutions. The opportunity for collaboration must be recognised as an important factor that reduced pressure to work alone and might be attributed as an important contribution to their success. Finally, it should be noted that students demonstrated their valid reasoning in a relatively short period of time, suggesting that there is a largely untapped potential for successful problem solving.

Implications for Further Study

The students who participated in the after-school program generated mathematical justifications using both direct and indirect forms of reasoning. It is interesting to note that the task of finding a rod whose length is one half when the blue rod had length one, generated all four types of reasoning. This can be explained, at least in part, to the task design that prompted creative thinking to show that no such rod exists for the set of rods provided). The strand of tasks that were offered to students afforded opportunities for building different, and often equivalent, models, offering more than a single convincing argument. Thus, students could offer their own arguments and take ownership of a justification while listening to alternative approaches. Learning to listen to how others approach a problem from problems in a related strand could also explain the success of the students. These results support Yackel and Hanna's (2003) claim that given a supportive environment, all students can and do make and refute claims and participate in inductive and deductive reasoning.

The research setting made possible multiple opportunities for students to talk about and represent their ideas and to collaborate with each other. Corrections and input from others led to revisions that became increasingly more elegant over time. The context of an informal, after-school environment made possible a relaxed atmosphere for testing ideas and making them public. The reasonableness of arguments was the measure for a student's success. What is noteworthy in this study is that, in a relatively short period of time, a culture evolved where sense-making and reasoning were exhibited in the problem solving of the students.

This study suggests that establishing certain conditions for collaborative problem solving, students can be successful. They include: (a) inviting students to represent their ideas, share them with others, and provide justifications for solutions; (b) providing strands of open-ended tasks that can be solved in more than one way; (c) making sufficient time available for all students; (d) inviting communication with others; (e) revisiting tasks and discussing previous and new ways of knowing (f) promoting listening to alternative ideas; (g) requiring students to explain and justify their reasoning; (h) having a variety of materials and tools so that students can select their personal representation; and, finally, (i) respecting individual contributions; and (j) ensuring a respectful, supportive environment where students and their ideas are welcome and valued.

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