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Growth Points in Students' Developing Understanding of Function in Equation Form

Erlina R. Ronda
University of the Philippines

This paper presents a research-based framework for analyzing and monitoring students' understanding of functions in equation form. The framework consists of *growth points* which describe 'big ideas' of students' understanding of the concept. The data were collected from Grades 8, 9, and 10 students using a set of tasks involving linear and quadratic relationships, and were used in order to identify and describe 'big ideas' in students' understanding of function in equation form. The framework also shows a typical learning path leading towards generality and abstraction in students' thinking about function.

Teachers' knowledge of students' thinking in acquiring concepts and procedures in a specific mathematical domain can be a powerful tool in informing instruction. This principle had been demonstrated by the results of such studies as the cognitively guided instruction project (Fennema et al., 1996) and the early numeracy research project (Clarke, 2001). These studies developed research-based models that describe learners' key cognitive processes in understanding specific domains of primary mathematics and that teachers can use to assess and to monitor students' understanding. These key cognitive processes are descriptions of students' 'big ideas' about a specific mathematical domain.

One can argue that the competencies provided in school curricula and other documents already provide teachers with a structure and direction in which to guide and monitor students' understanding. However, most of these competencies are stated in terms of outcomes. While these statements may be useful for teachers, they do not describe the strategies and thinking that students use (Horne & Lindberg, 2001). For example, one common competency for the topic on function is "to translate among the tabular, symbolic, and graphical representations of function" (National Council of Teachers of Mathematics [NCTM], 1989, p. 154). Students can do this in many ways, each involving different levels of abstraction and understanding of the concepts. The study presented in this paper developed a framework that describes students' strategies and thinking in their understanding of function in equation form. The framework is based on the premise that although teachers are aware of the differing levels of abstraction in students' thinking and reasoning, they are unlikely to be well equipped to design

with
is shown in the equation $2p = 6n$. If n = 5, what is s? Please show your solution.

4.1 Examine the two equations shown below. The specific values of $y = x^2 + 3x + 3$ is shown in the table on the left. Fill in the table on the right with values of $y = x^2 + 3x$. Please explain/show how you obtained your answer.

$y = x^2 + 3x + 3$

x	y
0	3
1	7
2	13
3	21
4	31

y = $x^2 + 3x$

x	y
0	
1	
2	
3	
4	

Explanation or solution:

appropriate pedagogy to lead students towards a deeper understanding of a specific mathematical concept.

The study was guided by the following questions:

What are the big ideas or growth points in students' understanding of function in equation form?

Is there a typical learning path for these growth points?

Theoretical Underpinnings

The framework was intended to provide a structure that describes secondary school students' understanding of function, and was based on the consideration of understanding as a growing network of conceptual nodes that is continuously being constructed and reorganised (Hiebert & Carpenter, 1992; Von Glasersfeld, 1987), and of the understanding of mathematics as a dynamic, multilevel process.

Describing Understanding

Understanding is a "never-ending process of consistent organization" (Von Glasersfeld, 1987, p. 5). It is not an all or none phenomenon, hence "it is more appropriate to think of understanding as emerging or developing rather than presuming that someone either does or does not understand a given topic, idea or process" (Carpenter & Lehrer, 1999, p. 20). The process of understanding is like building a network. Networks are built as new information is linked to existing networks or as new relationships are constructed (Hiebert & Carpenter, 1992). However, one not only links new mathematical knowledge to prior knowledge but also creates and integrates knowledge structures (Carpenter & Lehrer, 1999). These cognitive structures that students construct in the process of understanding a concept can be thought of as nodes or growth points (Clarke, Sullivan, Cheeseman, & Clarke, 2000). Growth points can be thought of as major conceptual nodes in the network of students' understanding of a mathematical concept. The notion of growth points is related to the concepts of schema (Marshall, 1990), theorems-in-action (Vergnaud, 1997), and key cognitive processes (Hiebert & Wearne, 1991).

In this study, each growth point describes students' 'big ideas' in terms of the strategies, knowledge, and procedures the students apply in working with tasks and problem situations. The phrase *growth point* was used to vividly reflect the essence of understanding as something that is growing and developing. The phrase is simple, less technical, and can easily become a part of teachers' everyday language (Clarke et al., 2000).

Describing Understanding of Function

Generalizing, formalizing, and abstracting are inherent to mathematics. Describing understanding of mathematical concepts in terms of the process-object theory highlights the formalizing and abstracting nature of mathematics. According to this theory, which some authors traced back to Piaget's theory of reflective abstraction, an individual starts by engaging in computational processes that lead them to a process conception, which later is encapsulated as a mental object (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Selden & Selden, 1992). Sfard (1991), using historical examples and in light of the schema theory, also argued that for most people, concepts are conceived as a process before they are conceived as a mathematical, mental object. According to the process-object theory, an object conception is attained generally after one has experience of performing actions on the concept. Freudenthal articulated this process: "My analysis of the mathematical learning process has unveiled levels in the learning process where mathematics acted out on one level becomes mathematics observed on the next" (1978, p. 33).

Early theoretical frameworks for analyzing students' understanding of function focused on various modes of representation and the translation between representations (see Janvier, 1987; Kaput, 1989). Later frameworks combine the process-object theory and the different representations of function (e.g., Moschkovich, Schoenfeld, & Arcavi, 1993; DeMarois & Tall, 1996).

Another route to objectification of the process conception of a mathematical concept can be through understanding its properties. Through experiences with various function exemplars and noting their properties, students could conceive function as objects either possessing or not possessing these properties (Slavit, 1997). Once the properties are identified, the student can 'see' a function as an object either with or without these functional properties.

The concept of function "was born as a result of a long search after a mathematical model for physical phenomena involving variable quantities" (Sfard, 1991, p. 14). In 1755, Euler (1707-1783, cited in Sfard, 1991, p. 15) elaborated on this conception of function as a dependence relation. He proposed that "a quantity should be called a function only if it depends on another quantity in such a way that if the latter is changed the former undergoes change itself". Seventy-five years later, Dirichlet (1805-1859) introduced the notion of function as an arbitrary correspondence between real numbers. Approximately one hundred years later, in 1932, with the rise of abstract algebra the Bourbaki generalised Dirichlet's definition. Thus, function came to be defined as a correspondence between two sets (Kieran, 1992). This formal set-theoretic definition is very different from the original definition. Function is no longer associated with numbers only and the notion of dependence between two varying quantities is now only implied (Markovits, Eylon, & Bruckheimer, 1986). The Direchlet-Bourbaki definition allows function to be conceived of as a mathematical object, which is the weakness of the early definition. However, the set-theoretic definition is too abstract for an initial introduction to students and is inconsistent with their experiences in the real world (Freudenthal, 1973; Leinhardt, Zaslavsky, & Stein, 1990; Sfard, 1992).

Textbooks, which often define function as a set of ordered pairs, usually start the discussion with relation and introduce function as a special kind of relation. However, relation is more abstract than function. Thus the supposed pedagogical value of having to learn relation first before one understands function is, in the opinion of Thorpe (1989), wrong. Freudenthal (1973) also expressed strongly that "to introduce function, relations can be dismissed" (p. 392). Thorpe went on to say that the use of the set-theoretic definition, which defines function as a set of ordered pairs, "was certainly one of the errors of the sixties and it is time that it were laid to rest" (p. 13). For this reason, the present study did not consider the relationship of function to relation in describing students' developing understanding of function.

Definitions are abstractions of the concept but knowing the definition of a concept does not necessarily translate to object conception. It is the experiences of performing actions on the concept or actions associated with the concept that enriches students' image of the concept. Furthermore, the first thing that is learned in understanding a concept is the experiences associated with it and not its definition (Vinner, 1992). Hence, to describe students' understanding of a mathematical concept, especially in describing initial understanding, the focus should be on the actions on the concept and its properties and representations, and not so much on the definition.

Function is not a simple concept. At least three representational systems are used to represent the concept of function in secondary schools: the tables (including ordered pairs), graphs, and formulae or equations. However, unlike students' early experiences with graphs and tables that are used to show relationships between two quantities, students' early experiences with equation involve equation not as a function representation but as a statement of condition of a single unknown quantity. Furthermore, the equal sign in the early grades was usually interpreted as "to do something" or perform an operation rather than denoting a relationship of equality between the quantities on both sides; this difference is reflective of the traditional divide between arithmetic and algebra (Stephens, 2003). When used as representation of function, equation takes an additional meaning: that of a representation of relationship between two varying quantities and not simply a statement of equality between two quantities on both sides of the equal symbol.

The equation representation of functions, like the graph, is versatile in the sense that it naturally lends itself to process and object interpretations alike. However, the equation has a feature (the coefficient of x) that conveys conceptual knowledge about the constancy of the relationship across allowable values of x and y, and parameters in equation aid the modelling process since they provide explicit conceptual entities with which to reason. For example, in $y = mx$, m represents rate (Kaput, 1989). Thus, the understanding of function in equation form is a major node or domain in the network in the understanding of function and teachers' knowledge of students' conception of equation as representation of function would be valuable information for designing tasks and instruction. Of course, since representational system has its own strengths and limitations in representing the concept, a full understanding of the concept of function necessitates not only an understanding and facility in working with each of these representations but also the flexibility to think of function in terms of the other representations.

Method

The objective of the study was to identify major nodes or growth points in students' understanding of function in equation form, and to identify a typical learning trajectory of these growth points. The research approach was interpretive and exploratory during the initial stages of analysis. The research then moved to a quantitative approach to identify typical patterns across the growth points, before returning to an interpretive phase in refining the growth points in light of these data. The following discussion describes in more detail the research approach adopted by the study to collect the data necessary for the identification of the growth points and a typical learning trajectory.

Initial Framework

To identify and describe students' understanding of function, the present study developed an initial framework (see Figure 1) based on the process-object and the property-oriented theoretical perspectives in students' developing understanding of function. These perspectives provided a theoretical base for identifying an initial list of growth points in key domains of the function concept. Hence, the initial framework of growth points served as a conceptual framework for the study. The initial framework was also made the basis for the development of assessment tasks that were used to collect the data for the study.

Assessment Tasks

To identify and describe the growth points, the study needed data from students' performance and strategies in solving problems. Hence, the main instrument used for collection of data was a set of assessment tasks that would highlight and draw out different levels of abstraction in students' strategies and understanding of function and its representations. The assessment tasks and students' solutions and justifications were then analysed and classified into *meaningful chunks* of information, which led to the identification of new growth points and refinement of the descriptions of existing growth points in the initial framework.

The students' responses in the tasks were documented and analysed, and the mathematical models, concepts, knowledge, skills, and strategies they used, and the explanations they gave, were noted. Interviews were also conducted with a limited number of students to gain more insight into their strategies and thinking.

The study involved generating data reflecting a wide range of students' strategies and thinking. Hence, some of the tasks included in the instrument could be answered only by very few students in the pilot studies, but results showed that these questions were clear and required a deeper understanding of the concept than that which most students possessed. Also included were tasks included that almost everyone could answer. This set of tasks was regarded as assessing the entry level in students' understanding in a particular domain while the former set was regarding as assessing higher-level understanding.

On the basis of the results of the pilot study, an easy version of a difficult task was added to assess if students could do the same task involving a less difficult analysis than that of the present task. Likewise, tasks demanding a higher level of analysis were added when many students could answer the present task easily and there was reason to believe that some of the students were capable of higher-level thinking. This was done in order to capture the range of strategies of which students were capable. For example, Task 3.1 (see Appendix) was designed when most of the Year 8 students in the pilot study had difficulty with Task 3. In total, nine tasks were used to assess and identify students' 'big ideas' for function in equation form.

The main data of the study were collected via the assessment tasks in a test-like environment, because the use of written assessment far outweighed the advantages afforded by use of interviews in terms of the objectives of the study. Written assessment could be administered to a large number of participants and required less time in administration. Furthermore, the study's aim of describing a typical map of students' developing understanding of function would need more than a few participants for study, not only for establishing typicality but for providing a range of students' performance and strategies. Time needed to work out the tasks and the nature of the tasks were also important considerations. The study needed to assess conceptual understanding in terms of students' strategies and thinking processes used in working with function tasks and not simply skills and knowledge. These types of tasks require time to think through. In addition, to gather a range of students' performance and strategies also meant more assessment tasks would be needed. It was also desirable that students be given the chance to try all the tasks, as this would enhance the validity of the research result. Administering the tasks in the form of a written test would also enable the students to select which tasks they wanted to do first.

Interview

The purpose of the interview during the pilot studies was to determine the clarity of the tasks and whether they were interpreted the way the researcher hoped they would be interpreted. During the main data collection, the interviews were focussed on gaining further insight into students' thinking in order to help the researcher make sense of the students' solutions and explanation in the written form. Interviewing students using the tasks gave the researcher more confidence in interpreting the written responses and in describing the growth points.

Participants in the Pilot Studies

During the pilot studies, data were collected from students with varied ability range, in order to cover a wider range of students' strategies and performance in function tasks, as well as assess the appropriateness and clarity of the tasks. The first two pilot tests were conducted in Melbourne. Two more pilot tests were conducted in the Philippines before the main data collection. The participants in the pilot studies comprised three levels, Years 8, 9, and 10, and represented low, average, and high performing students.

Participants for the Main Data Collection

The data for the main study were collected from Years 8, 9, and 10 students, aged 14, 15, and 16 years respectively, from public science high schools in the Philippines. The decision to consider only the students from science high schools for the main data collection was influenced by the results of the pilot studies, which showed that the majority of students from these schools were more likely than students from other schools to work out the tasks correctly and explain their answers or show their solutions. That students should be able to do this was important to the study, since the development of the framework depends on students' solutions, explanations, and some level of success. Thus, while the decision to consider the science high schools for the main data collection may have limited the scope to which the findings of the study could be generalised in terms of percentage of students at particular growth points, it also significantly increased the validity of the results because of the richness of data gathered.

Table 1 shows the number of students in each year level from the three schools selected for the main data collection.

Table 1
Number of Respondents from Each School

	School O	School B	School M	Total
Year 8	28 (1 class)	70 (2 classes)	51 (2 classes)	149
Year 9	58 (2 classes)	53 (2 classes)	41 (2 classes)	152
Year 10	53 (2 classes)	50 (2 classes)	40 (2 classes)	143
Total	139	173	132	444

In all three schools, Year 8 students were just starting their work with coordinate systems and doing some point-plotting activities when the first data collection occurred, but all had covered the topic on linear functions before the second data collection was undertaken. In Year 9, Schools O and B had completed quadratic equations and function and School M was about to start with quadratics when the second data collection was undertaken. Consequently, revisions involving linear relations had been done. The Year 10 students had theoretically studied all the content covered in the instrument (linear and quadratic relationships) in Year 9. Because the students study other families of function in Year 10, one could assume that they worked with linear and quadratic relationships as well during this time.

The data were collected twice from the same students in 5 months interval in order to determine students' movements in the growth points and the trend in the order of the growth points. The assessment tasks were also given to students from Years 8, 9, and 10 to cover a range of students' performance and strategies.

Identifying and Describing the Growth Points

After each pilot study, the skills and knowledge involved in tasks and students' responses were analysed in terms of the representation involved, that is, whether only point or single values, or an interval or a part of the representation, or the whole representation were involved. Because the focus was on both answers and strategies, a record sheet of students' responses and strategies was developed and used.

The analysis of the tasks and students' strategies and therefore the identification and description of the growth points were guided by the initial framework in Figure 1. The diagram shows that the tasks were classified according to the points of analysis involved: individual points versus the whole representation. Working with individual points is a manifestation of a process or procedure conception of function, while the latter points toward conceiving function as an object.

	Process	→	Object
Points of analysis:	Individual points	Set of points, interval	Whole representation, Relationship
Strategies: (action performed)	Perform series of same procedure (point-by-point)	Combination of points and whole representation	Perform general operation on the representation
Interpretati on of properties:	Local properties	Trends and patterns and local properties	Invariant properties

Figure 1. *Initial framework.*

Students' strategies used in working out the tasks were also classified according to the procedure performed on the representation: series of the same procedure versus performing a general operation. The former is a manifestation of conceiving function as a process and the latter shows understanding of function more as an object than a process. Strategies in between these conceptions included the use of trend and patterns; use of properties and individual points; and interpretations based on local properties.

Tasks involving properties were also classified according to the kind of property: local properties versus invariant properties of the function and how knowledge of the properties of the function was used in the tasks. Use of invariant properties in working with function tasks was considered as evidence that students conceived function as a permanent construct.

Correct answers with sufficient and insufficient explanations were both included in the analysis of the data. Students showing more than one solution or explanation were coded at the higher-level solution.

Investigating for Typical Learning Trajectories

The typical learning trajectory was determined by comparing the percentage of students coded at each of the growth points or by comparing the frequency of students moving from one growth point to the higher growth points between the two data collection periods. Although there is insufficient evidence that the three year levels are comparable, the achievements of students in each of the year levels were also examined.

Results and Discussion

Four growth points or 'big ideas' in students' understanding of function in equation form were identified based on the type of tasks and the strategies used in working out the tasks. The tasks/strategies were clustered as indicators of students' achievement of each 'big idea'. Empirical evidence regarding the general trend in the acquisition of these growth points and why they were considered as 'big ideas' are described later in this section. The growth points are described in the following section.

Growth Point 1 - Equations are Procedures for Generating Values

The Year 8 data showed that when students are introduced to function in equation form, the students are likely to solve tasks that can be analysed by individual points. The results also showed that for all the year levels, the most preferred solutions to the tasks involved evaluating the equation for specific values first even if this strategy is the most tedious way of completing the tasks. In Task 4, for example, students guessed an equation and then checked it against individual values in the table rather than examining the relationship between the two given tables. The same is true for Task 4.1. Task 4.1 may involve a quadratic expression but because it can be solved by working with individual points, the students found the task easier.

Preference for point-by-point analysis was further confirmed by the results of students who were given the tasks in interview form only. In the interview, students were asked first to notice the similarities and the differences between the two tables (in Task 4) and the between two equations for Task 4.1 before they were asked to work on the tasks. There were those who noticed the difference but they did not use that knowledge to complete the tasks.

The following excerpts show the results of the interview with a Year 9 student for Task 4 and 4.1. Task 4 was given first.

Int:	Look at these two tables, what do you notice about them? (Showed the two tables, covering the question first)
Stu:	I think in the first table, the first column of the first table or the x values is the same while the y variable in the table (pointing to the first table) are higher value than the variable in second table.
Int:	Any other relation between the y values?
Stu:	Umm… In Table 1, the y variable is greater than Table 2?
Int:	By how much?
Stu:	Two
Int:	Is it true to all?
Stu:	Yes. (Checks the values one by one.)
Int:	(Read Task 1.2 slowly.)
Stu:	Can I have trial-and-error?
Int:	Sure.
Stu:	(Student wrote y = $2x^2 - 2$ then checked by evaluating for $x = 1$. When it did not match the value in the table tried the following equations: $y = 2x^2 - 3$, $y = 2x^2 + 5$ and $y = -2x^2 + 5$ for $x = -1$, getting 3 for this last equation). I think I know it, that's my answer (wrote $y = -2x^2 + 5$).

The following excerpts show the results of the interview with the same Year 9 student for Task 4.1. The interview shows that the student did not have the confidence to treat the function holistically. She opted to use point-by-point analysis to complete the task.

Int:	(Showed the two tables and equations.) What do you notice about the two equations?
Stu:	The first equation, there's a 3 added.
Int:	(Read Task 4.1 slowly)
Stu:	I will just compute (evaluated the second equation for $x = 0$ to 4, showing the computations step by step)
Int:	Can you think of other ways of doing this?
Stu:	Here, I can get the y variables [values] minus 3.
Int:	Why did you not do it?
Stu:	I wasn't sure if I could get it that way, I might get others [values].

This interview shows that students could not just move from a 'pointwise' understanding of function to an understanding of it as relationship and that this relationship is true for all allowable values for x.

Growth Point 2 - Equations are Representations of Relationships

Few students solved Tasks 4 and 4.1 by examining the relationships among the representations involved. This strategy certainly reflects a different level of understanding. Two more tasks involving linear relationships were therefore used to ascertain the existence of this 'big idea'. These are Tasks 3 and 3.1. Task 3 requires that students consider the equation not just as a procedure or formula for generating values but as a mathematical object. In Task 3.1, the interpretation may involve only a single value but also requires an analysis of the relationship between the variables in the two equations, as well as the interpretation of the relationship between the equations. The interpretation does not simply involve a straightforward evaluation of equations. The difference between Growth Point (GP) 1 and GP 2 is that in GP 1, students generate values from the equation but they may not necessarily conceive of the equation as a statement of relationship between the varying quantities.

Growth Point 3 - Equations Describe Properties of Relationships

An indication of understanding, apart from the way one performs actions on the concept, is knowledge of the properties of the concept. Students coded at this growth point could interpret the properties of function such as the y-intercept and the slope from a given task. Task 1b requires interpreting the intercept. Students could either evaluate the given equation for $t = 0$ or recognize the meaning of the constant in the equation. Most of the responses involved evaluating the equation for $t = 0$.

Some students evaluated the equation for $t = 1$ and then reasoned that because the value is 9, which is too big, the container must not have been empty. This kind of answer, though correct, was not considered acceptable for GP 3. The response does not clearly reflect that students knew the concept of intercept or could interpret it from the equation.

Task 1a was difficult for the students. Students' correct solutions in Task 3a included evaluating the two equations for several values then comparing the increase or interpreting the slope as showing the rate of change. Not one of the students used their knowledge of the slope or interpreted the coefficient of t as the basis for their answer. This latter solution is the preferred strategy since it is more straightforward, and the interpretation of the parameters in the equation reflects understanding of the properties of function. Students had another chance to show if they could interpret rate from equations in Task 2.

Growth Point 4 - Functions are Objects that can be Manipulated and Transformed

Some students could conceive an equation as a representation of a relationship between two variables as well as conceive the function it is representing as a mathematical object. Students coded at this growth point were those who could perform an operation on the equation of function as if it were an object in itself and not just a process.

This strategy was evident in a solution in which the entire equation became the input itself. For example, Task 3 and Task 4 involved solutions that considered the relationships of the entries in the two tables and then linked this with the equation by subtracting 2 from the given equation, and in Task 5 an operation could be performed on the given equation based on an analysis of the set of values generated by the equation. At this level, the students no longer 'see' just the equation but also the relationship it is representing. This relationship is a mathematical object, an object that students can manipulate and transform.

A Typical Learning Trajectory

The order of the growth points from 1 to 4 is consistent with the process-object theory. This order was also supported empirically by the frequency of students who coded at the growth points (see Table 2). In Table 2, GP 0 stands for GP zero. Students coded at this growth point were those not coded in any of the other four growth points.

The results showed that during the first data collection, only about 66% of 149 Year 8 students were coded at GP 1. Five months later, at the second data collection, there was a large increase, to 93% in Year 8, in the percentage of students coded at GP 1. The high percentage of students at GP 1 indicates that this pointwise understanding of function represented by equations is easily understood even at Year 8 level. That GP 2 is the next growth point in students' understanding of function was indicated by the second largest percentage of students coded at GP 2. GP 2 is about interpretations based on relationship and involves pointwise and holistic interpretations.

Table 2
Percentages of Students Coded at the Growth Points

	Yr 8 (n = 149)		Yr 9 (n = 152)		Yr 10 (n = 143)	
Growth Points	D1	D2	D1	D2	D1	D2
GP 0:	33.6	7.4	2.0	0.0	0.0	0.0
GP 1: Equations are procedures for generating values	66.4	92.6	98.0	100.0	100.0	100.0
GP 2: Equations are representations of relationships	10.1	28.9	23.0	52.6	44.8	68.5
GP 3: Equations describe properties of relationships	5.4	20.1	14.5	27.6	24.5	55.2
GP 4: Functions are objects that can be manipulated and transformed	0.0	0.7	1.3	0.7	3.5	12.6

The description and position of GPs 1, 2, and 4 are reflective of Sfard's (1991) interiorization- condensation-reification stages toward conceiving a concept as a mental object, or Breidenbach, et. al.'s (1992) action-process-object levels.

It was expected that those students coded at GP 4 would also be coded at GP 2 and GP 1, since conditions satisfying GP 4 satisfy GP 2. Also, conditions satisfying GP 2 satisfy GP 1. Therefore, the manner in which GP 3 is related to the other growth points needs to be investigated. That is, could students coded at GP 3 also work in terms of GP 2, and do students coded at GP 4 show understanding in terms of GP 3?

Performance of students who were coded at GP 1 and GP 3 but not at GP 2 was investigated further. These students did not appear to completely miss the conditions required for GP 2. Most of them were not coded at GP 2 because they did not meet the requirement of at least a code of Strategy 2 which was about holistic interpretation of the relationship represented by the equations. The students completed at least one of the tasks, but used a point-by-point strategy. Therefore, students who were coded at GP 1 and GP 3 but not at GP 2 could interpret equations as representation of function, but could not do so in a more holistic way. This finding further implies that the growth points identified are not discrete.

Students had difficulty in interpreting the rate or growth property of the function from equations. This difficulty may be because interpretation of rate involves analyzing and comprehending the changes not only between xs and ys but also between y_m and y_{m+1} and x_m and x_{m+1} (see Confrey & Smith, 1994; Slavit, 1997). Sierpinska (1992) also argued that students have difficulty identifying the changing quantities in a functional relationship. The fact that rate is one of these changing quantities explains students' difficulty with the concept.

Nearly all those who were coded as being at GP 3 used substituting individual values to the given equation, a very tedious approach, instead of interpreting the parameter m. This tendency may be because a parameter demands thinking at a general, abstract level. It is a higher-level variable; a change in its value affects not just one value of the function but the entire function itself (Drijvers, 2001). GP 3 therefore can be further subdivided into two growth points. GP 3a would involve interpretation of properties of function by point-by-point, and GP 3b would involve an understanding of the properties of function by interpreting the parameters in the equation. The latter reflects an understanding of function as an object.

No Year 8 student was coded at GP 4. Data for Year 9 showed that two of the 152 students were coded at GP 4 but were not coded at GP 3. During the second data collection period one student in Year 9 was coded at GP 4 and at GP 3. A similar proportion was found with Year 10 students in the first data collection period. However, in the second data collection period, 17 of the Year 10 students were coded at GP 4 and 15 of these were coded at GP 3 as well. This finding indicates that students who could work in terms of GP 4 may be expected to work in terms of GP 3 as well. Of course, achieving both GP 4 and GP 3 does not necessarily imply that understanding of GP 4 builds on an understanding of GP 3. However, because of the limited data, one might suggest that the assessment tasks be given to Year 11 students or those equivalent to first year college students in the Philippines to further confirm this observation.

The Growth Points as 'Big Ideas'

The large difference in the percentages of students who were working at each growth point within the same year level confirmed that the identified growth points were indeed 'big ideas'. For example, during the second data collection period, all of the Year 9 students could work in terms of GP 1, only half could work in terms of GP 2, about a quarter were at GP 3, and almost none were at GP 4. However, results of the second data collection period show that the difference in the percentage becomes smaller in Year 10. This change is to be expected since the Grade 10 students have considerable experience working with function.

It seemed a big step for the majority of the students working with the equation representation of function to move from individual point interpretation to a more holistic interpretation. To help students appreciate holistic interpretation, they could be given tasks that make point-by-point interpretation tedious and cumbersome. Linking the different representations of function, especially between graphs and equations, could also help since graphs naturally lend themselves to holistic interpretation. However, engaging students in this type of tasks should be done with caution. Requiring mastery might lead to rote learning or misconceptions especially for students who are not ready for this leap. The idea of function as an object, "as a static 'thing', when introduced too early is doomed to remain beyond the comprehension of many students" (Sfard, 1992, p. 77).

Summary

The study identified four growth points in students' understanding of function in equation form. These are:

GP 1: Equations are formulas or procedures for generating values
GP 2: Equations are representations of relationships
GP 3: Equations represent properties of relationships
GP 4: Functions are objects that can be manipulated or transformed

Each growth point describes a particular stage of understanding function in equation form in terms of the abstraction and generality involved in the students' thinking. The growth points are described in terms of big picture ideas to focus on conceptual understanding. This framework of growth points has the potential to provide teachers with a structure for assessing students and designing classroom instruction that would facilitate students' attainment of object conception of function.

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Author

Erlina R. Ronda, UP NISMED, Quirino Avenue, University of the Philippines, Diliman, Quezon City, 1101 PHILIPPINES. Email: <erronda@up.edu.ph>

Appendix

Set 1

Imagine water flowing through a pipe into a container. The following equations show how the water level or height of the water (w) in the container was related to the number of minutes (t) when the pipe was opened for 10 minutes.

$w = t + 8$ for the first four minutes ($t = 0$ to 4)

$w = 3 \times t$ for the remaining six minutes ($t = 4$ to 10)

where,

w refers to the water level (height) in centimetres

t refers to the number of minutes

Please use the above information to answer the following questions.

From the given information, do you think the height of the water in the container is increasing at the same rate throughout the 10 minutes? Circle the letter corresponding to your answer.

Yes, the water level increases at the same rate throughout the 10 minutes.
No, the water level is <u>not</u> increasing at the same rate throughout the 10 minutes.

Please show or explain how you obtained your answer.

From the given information, do you think the container already contains water before the pipe was opened? Circle the letter corresponding to your answer.

Yes, the container already contains water before the pipe was opened.
No, the container does not contain water before the pipe was opened.

Please state or show how you obtained your answer.

Which equation shows the fastest change in y when x takes values from 1 to 10? Please show/ explain how you worked out your answer.

a. $x + y = 100$ b. $y = 6x - 3$ c. $4y = 8x$
d. $y = 75 + 5x$

Solution or Explanation:

The relation of s with p is shown in the equation $s = 5p + 3$. The relation of p

with n is shown in the equation $2p = 6n$. From this information, please write the equation that will show the relation of s with n.

Please show your solution.

Examine the two tables shown below. The set of values in the table on the left shows specific values of $y = 2x^2 + 3$. Please write the equation whose values are shown in the table on the right.

Please show or explain how you obtained your answer.

x	y
-1	5
0	3
1	5
2	11
3	21

x	y
-1	3
0	1
1	3
2	9
3	19

$y = 2x^2 + 3$

____

Solution or Explanation:

The relationship between x and y in Table 1 is $y = 2x + 1$. In Table 2, the values of x and y in Table 1 were swapped or interchanged. Please write the equation which shows the relationship between x and y in Table 2?

Show how you obtained your answer.

Table 1

x	y
0	1
1	3
2	5
3	7
4	9

Table 2

x	y
1	0
3	1
5	2
7	3
9	4

$y = 2x + 1$

____

Solution or Explanation:

Set 2 (Given the next day)

The relation of s with p is shown in the equation $s = 5p + 3$. The relation of p

with
is shown in the equation $2p = 6n$. If n = 5, what is s? Please show your solution.

4.1 Examine the two equations shown below. The specific values of $y = x^2 + 3x + 3$ is shown in the table on the left. Fill in the table on the right with values of $y = x^2 + 3x$. Please explain/show how you obtained your answer.

$y = x^2 + 3x + 3$

x	y
0	3
1	7
2	13
3	21
4	31

y = $x^2 + 3x$

x	y
0	
1	
2	
3	
4	

Explanation or solution: