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# Mathematics *Is* Motivating

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Mathematics is motivating; at least, it should be. I argue that mathematical activity is an inherently attractive enterprise for human beings because as intellectual organisms, we are naturally enticed by the intellectual stimulation of mathematizing, and, as social beings, we are drawn to the socializing aspects of mathematical activity. These two aspects make mathematics a motivating activity. Unfortunately, the subject that students often encounter in school mathematics classes does not resemble authentic mathematical activity. School mathematics is characterized by the memorization and regurgitation of rote procedures in isolation from peers. It comes as no surprise that many students have little motivation to continue mathematics study because it lacks intellectual and social appeal. I suggest several practical changes in school mathematics instruction that are drawn from the literature. These changes will lead to instruction that more readily engages students with the subject because they are rooted in the intellectually and socially appealing aspects of mathematics.

Mathematics *is* motivating. Or at least, it should be. Mathematical activity is an inherently attractive enterprise for human beings because of its intellectual and social aspects. This may be difficult to believe, especially when "so many people find mathematics impossibly hard" (Devlin, 2000, p. 1) and many openly admit strong dislike for the subject (Paulos, 1988). Certainly, critics might argue, a few gifted individuals might have a special inclination toward mathematical study. But, can mathematics be appealing for everyone? I claim that it can be; mathematics has the potential to be interesting for everyone because it is an intellectual and social endeavor. In the following sections, I detail what is meant by authentic mathematical activity, describing both its intellectual and social aspects. Comparing authentic mathematical activity to typical school mathematical activity, I suggest ways that teachers can draw on the intellectual and social aspects of mathematical activity to motivate and engage students in the study of mathematics.

## Mathematical Activity

Because "mathematics is a woefully misunderstood subject" (Devlin, 2000, p. 3), for the purposes of this article I define *authentic mathematical activity*, or *mathematical activity*, to be *what mathematicians do when they do mathematics*. I use examples from the life of Stanislaw Ulam, a Polish mathematician, to describe authentic mathematical activity.

Not as well-known as Euler, von Neumann or Einstein, Ulam was a rather ordinary mathematician. Ulam (1976) humbly stated, "after all these years, I still do not feel much like an accomplished professional mathematician" (p. 27). Much of Ulam's descriptions of his work focused on the people he met, was inspired by, and worked with. Many of the experiences he reported illustrate intellectual and social aspects of mathematics.

## Intellectual Aspect of Mathematics

Mathematical activity is the most intellectual endeavor of all the sciences.

> Mathematics is a creation of the brain... Mathematicians …work …without any of the equipment or props needed by other scientists…. mathematicians can work without chalk or pencil and paper, and they can continue to think while walking, eating, even talking. This may explain why so many mathematicians appear turned inward [which is] quite pronounced and quantitatively different from the behavior of scientists in other fields…. I have spent… on the average two to three hours a day thinking…. Sometimes… I would think about the same problem with incredible intensity for several hours without using paper and pencil. (Ulam, 1976, p. 53)

Since mathematics is "a thinking, flexible subject" (Boaler, 1999, p. 264), mathematical activity is

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characterized by a variety of mental methods, such as cognitive wrestle and creativity.

*Cognitive Wrestle*

Mathematicians wrestle with cognitively demanding problems that have no clear solution path. Such wrestle is invigorating, and "most mathematicians begin to worry when there are no more difficulties or obstacles" (Ulam, 1976, p. 54). Mathematicians rarely make progress at steady rates; rather, they struggle with little apparent progress, and then major strides are made suddenly. The mathematician Banach "worked in periods of great intensity separated by stretches of apparent inactivity" (Ulam, p. 33). Ulam described the importance of "'subconscious brewing' (or pondering) [which] produces better results than forced, systematic thinking" (p. 54), being "a discontinuous process. Nothing, nothing, at first, and suddenly one gets …it" (p. 70). Part of this cognitive wrestle involves exploring alternative possibilities through mental experimentation.

> I always preferred to try to imagine new possibilities rather than merely to follow specific lines of reasoning or make concrete calculations. …Forcing oneself to persist in a logical exploration becomes a habit after which it ceases to be forcing since it comes automatically. (Ulam, 1976, p. 54)

*Creativity*

Mathematicians are creative by generating mathematical content; they do this by posing and solving problems. During this process they create the very fabric of mathematics, weaving a thread that connects to the work of others. For example, when only 25, Ulam (1976) "established some results in measure theory which soon became well known [by solving] certain set theoretical problems attacked earlier by Hausdorff, Banach, Kuratowski, and others" (p. 55). In turn, the Russian Besicovitch solved a problem posed many years earlier by Ulam. These events also illustrate that problem posing is another part of creative mathematical generation.

Devlin (2000) claimed that all people, everywhere, have "a mind for mathematics" (p. 1), that every human being with a functioning brain has "an innate facility for mathematical thought" (p. xvi). The intellectual aspects of mathematics, such as cognitive wrestle and creative generation, are fundamental attributes of our species: This helps explain why all human cultures mathematize the world. The predisposition for patterned thinking is even seen in newborns (Dehaene, 1997).

Researchers know that "large parts of the brain are active when a person is doing mathematics" (Devlin, 2000, p. 12). Because we are intellectual beings, the intellectual appeal of mathematics makes it naturally enjoyable; people's brains like doing mathematics. Mathematics is by far children's favorite subject in school, at least well into the fourth grade (National Council of Teachers of Mathematics [NCTM], 2000).

## Social Aspect of Mathematics

Besides its intellectual character, doing mathematics is also highly social. Every mathematician works within a mathematical community. Mathematicians are social on both a local and global scale. Ulam, in particular, described communication and collaboration.

*Communicating*

Mathematicians are engaged in constant communication, in part to help them learn more mathematics. This may entail reading about mathematics, listening to lectures, or discussing mathematical ideas with knowledgeable others. Ulam was influenced by mathematicians' books at an early age, such as Sierpinski's *Theory of Sets*, Steinhaus's *What is and What is Not Mathematics*, and Poincaré's *La Science et l'Hypothèse*, *La Science et la Méthode, La Valeur de la Science*, and *Dernières Pensées* (1976, p. 21). Kuratowski was an early teacher of Ulam who made a formidable impression, and, in part, was responsible for starting him on a career in mathematics. Teachers and mentors play a significant role in mathematicians' development. Many famous mathematicians bring to mind their mentors: Euler as a student of Bernoulli; Ramanujan as a student of Hardy; and Dedekind and Riemann as students of Gauss.

Mathematicians recognize the benefit that flows from sharing and networking (Davis & Simmt, 2003). Ulam (1976) described how he would engage in mathematical discussions with friends and colleagues. During a break while attending an International Mathematical Congress, he got lost in the nearby woods, and bumped "into Paul Alexandroff and Emmy Noether [who were] walking together [among the trees] and discussing mathematics" (p. 46).

The view of an isolated mathematician working long hours alone in the office with little interaction is almost everywhere false (Wiles' work on the Fermat theorem being a notable exception).

> Much of the … historical development of mathematics has taken place in specific centers [or] a group in which mathematical activity flourished.

> Such a group possesses more than just a community of interests; it has a definite mood and character in both the choice of interests and the method of thought. Epistemologically this may appear strange, since mathematical achievement, whether a new definition or an involved proof of a problem, may seem to be an entirely individual effort, almost like a musical composition. However, the choice of certain areas of interest is frequently the result of a community of interests. Such choices are often influenced by the interplay of questions and answers, which evolves much more naturally from the interplay of several minds. (p. 38)

The work of a mathematician incorporates his or her surrounding influences. For example, Ulam (1976) also wrote that "most of my mathematical work was really started in *conversations* with Mazur and Banach" (p. 33, emphasis added). Even gatherings in a local café provided opportunities for sharing and discussing mathematics. A large notebook was permanently kept in the café and brought out by a waiter upon demand; it was the central repository of the group's ideas. Ulam later translated the notebook and "distributed it to many mathematical friends in the United States and abroad" (pp. 49–51).

*Collaborating*

Beyond communicating about mathematics, active collaboration is also a natural part of mathematicians' sociality. Ulam (1976) worked with many distinguished mathematicians during his career; "Collaboration [with Mazur and Banach] was on a scale and with an intensity I have never seen surpassed, equaled, or approximated anywhere—except perhaps at Los Alamos during the war years" (p. 34). Ulam said that upon his arrival in the United States, he and Borsuk "started collaborating from the first… my first publication in the United States ….was a joint paper with Borsuk" (p. 41). Later, he said, "a whole series of papers which we [Steinhaus and I] wrote jointly came from …collaboration" (p. 43). Ulam also worked with John von Neumann.

The joint work of mathematicians results in mutually accepted definitions, terms, strategies, methods, and algorithms. From parking lots to offices, academic lunchrooms to conference halls, mathematicians scribble and get stuck, share questions and solution attempts, backtrack and refine, reattempt and debate; they question, raise counterexamples, reason, argue, collectively justify, and develop communal metaphors (Polya, 1945). As such, many believe mathematics, as a domain, transcends any individual perspective (Boaler, 1999; Davis & Simmt, 2003; Devlin, 2000). It is not a static knowledge domain—an external thing to be internalized by a learner—but rather a socially created, culturally dependent, fallible domain (Ernest, 1990).

Mathematics exists due to the collective actions of many people over thousands of years. It belongs to no one and yet is accessible to all; it is a constant, communal, and humanistic creation (Romberg, 1994). Great discoveries by many individuals and groups have woven the tapestry of current mathematical thought; people like the Pythagoreans, the Arab algebraists, Cardano's band, and Newton and Liebniz have all contributed their threads. As Leopold Kronecker, a nineteenth century mathematician, remarked, "God made the integers, all else is the work of man" (quoted in Devlin, 2000, p. 15). All mathematics—fundamental axioms, appropriate terminology, conventional representations, mathematically valid propositions—is socially driven, "a cultural product" (Ernest, 1990). From this perspective, mathematics is much more than numbers or computations; it emerges through correspondence, questions, and group deliberations.

## Inherent Mathematical Appeal

I am not alone in believing that mathematics can be interesting for everyone. The authors of the NCTM *Standards* (2000) opine that mathematics is a meaningful, richly rewarding subject that all can learn and enjoy. Additionally, when given the opportunity to engage in meaningful mathematical tasks that maintain their cognitive integrity, students not only tolerate mathematical work, but report satisfaction and enjoyment (e.g., Boaler, 1999). These findings are not exclusive of any particular personality or culture. In addition, I have seen ample evidence to suggest that students, either high or low achieving, when allowed to engage in mathematics, are drawn to the activity (Ricks, 2007). There is something intrinsically motivating in the subject.

School mathematics can share this attraction if students are able to engage in authentic mathematical activity. Unfortunately, "most people do not know what mathematics is" (Devlin, 2000, p. xvii), perhaps because they have not experienced authentic mathematical activity and are thus dissuaded from further mathematics study. School mathematics is characterized by the memorization and regurgitation of rote procedures in isolation from peers (Burton, 2004; Stigler & Hiebert, 1999). Therefore, it comes as no surprise that, devoid of its intellectual and social appeal, mathematics is not motivating for many students and that many do not continue formal mathematics study past high school. A corrective possibility is to harness the intellectual and social potential of mathematics activity; allowing students to engage in mathematical activity in their own classrooms affords simple, straightforward options to improving mathematics instruction by returning to root motivational aspects of the subject.

## The Lack of Intellectual and Social Appeal in School Mathematics

School mathematics is characterized by learning definitions and practicing procedures (Stigler & Hiebert, 1999), activities that lead to intellectual boredom. The essential attributes of mathematical activity—cognitive struggle and creative generation—are absent. "The questions people [mathematicians] worried about and the struggles they went through trying to answer them almost never appear [in school mathematics]; instead we see the results of the struggles, neatly packaged into pieces of boxed text" (Cuoco, 2001, p. 169). School mathematics is, quite bluntly, an intellectual wasteland, a pseudo-mathematics. Richards (1991) describes the intellectually stagnating *initiation-reply-evaluation* sequence as the common form of classroom interaction. No wonder students are confused; no wonder they avoid further mathematics study!

*Intellectual Lack in School Mathematics*

Mathematics teachers often view their job as showing "a few standard facts and algorithms" to students, and, later, "supervis[ing] some drill and practice" (Romberg, 1994, p. 314). Students are expected only to memorize the various rules and procedures the teacher demonstrates (Boaler, 1999; Davis, 1994): Independent thought is not an expectation. The intellectual possibilities for "relatively sophisticated levels of mathematical reasoning, well beyond what is typically thought of as appropriate for primary school mathematics" (Yackel, 2000, p. 20) are rarely met. The current level of intellectual engagement in learning school mathematics pales in comparison to what could happen if children were allowed to think things through for themselves (Davis, 1994).

School mathematics is also not viewed as a creative endeavor. The curriculum is set, the teacher and textbook are the authorities in classrooms. There is no room for questioning, no room for exploration, no room for experimentation. "In many schools, mathematics is perceived as an established body of knowledge that is passed on from one generation to the next. Instead of seeing [theorems, formulas, and methods of mathematics] as the *products* of doing mathematics, these artifacts are seen *as* the mathematics" (Cuoco, 2001, p. 169). Said Burton (2004):

> It has long been my opinion that the mathematics experienced by students in formal education, and the ways in which it is encountered, offer explanations for [the] decline in interest. Public interest books about mathematics are readily bought so it cannot be that people have no wish to engage with mathematics…. once students make a choice to study mathematics, many of them report experiences that are not conducive to holding them in the discipline. The same pattern holds whether they are studying mathematics at school, as a pre-university choice, at university as undergraduates or even at doctoral level. (p. 4, emphasis added)

Contrasting this intellectually diluted school mathematics with the work of mathematicians is enlightening: "as a result of such limited experiences, many students are prejudiced against the broader, more interesting aspects of mathematics" (Romberg, 1994, p. 290).

*Social Lack in School Mathematics*

Similarly, school mathematics deprives students of the natural socializing appeal of mathematical activity. Students are expected to sit quietly and listen to the teacher with little to no interaction with others (Davis, 1994). The necessary mathematical interactions needed for full mathematical activity are absent. This severely curtails children's abilities to make "judgments about what is acceptable mathematically, for example, with respect to mathematical difference, mathematical sophistication, mathematical inefficiency, mathematical elegance, and mathematical explanation and justification," and it deprives them of autonomous "mathematical power" (Yackel, 2000, p. 21).

In school mathematics, students usually do problem sets alone, do homework alone, and take quizzes and tests in isolation. When do they have a chance to engage in social mathematical work? School mathematics perpetuates beliefs that heterogeneous class makeup is an "obstacle to effective teaching" and that "the tutoring situation is best, academically, because instruction can be tailored specifically for each student" (Stigler & Hiebert, 1999, p. 9). When they do work in groups, students usually do only superficial computational exercises, even though the group could enable individual students to overcome personal barriers in the problem-solving process. Rarely is mathematical understanding created by the group as a whole.

School mathematics neglects the social aspects that make mathematics so appealing—the ability to participate in larger mathematizing collectives working toward shared meanings and common objectives. Classroom dialogue is characterized by univocal "number talk" rather than socially intertwined, mutually specified, dialogic functioning (Davis & Simmt, 2003; Richards, 1990, Wertsch & Toma, 1995). Such absences of mathematical activity in school mathematics classrooms led one researcher studying U.S. mathematics lessons to bemoan, "I have trouble finding the mathematics [in these lessons]" (quoted in Stigler & Hiebert, 1999, p. 26).

The great ironic tragedy is that most students who claim to have little motivation to study mathematics have never really experienced authentic mathematics. To deal with a lack of motivation, non-mathematical strategies are often employed to maintain students' attention in mathematics classes. However, such strategies do not work. Some examples are: (i) interrupting mathematics instruction to talk about a more interesting subject, (ii) using candy or prizes to excite students, (iii) presenting the lesson in the context of competitive games, or (iv) letting students work together on projects where the focus often shifts from mathematical ideas to creating attractive displays (Stigler & Hiebert, 1999).

## Making School Mathematics Authentic

Mathematics learning does not require games, dramatic teacher presentations, external motivators, or even connections to real world activities, all common suggestions to motivate students in traditional mathematics classrooms. It requires instead a return to the core components of mathematics. The reason motivation is an issue at all is that current school mathematics is neither intellectual nor social; by focusing on "habitual, unreflective, arithmetic problems" (Richards, 1991, p. 16), *school mathematics strips from the subject the very constituents that provide for meaningful mathematical experiences*. Slight, subtle changes in the way mathematics is taught can significantly increase students' motivation to learn mathematics.

For example, teachers can engage students in (1) cognitively challenging (Stein, Smith, Henningsen, & Silver, 2000) and (2) socially oriented activities in mathematics classrooms (Stein & Brown, 1990). Students can then be involved in the genesis of mathematical ideas in a group setting. These two components make mathematics a motivating activity:

> When a class follows an inquiry tradition of instruction, many of the 'tasks' that children engage in are tasks that they set for themselves as they attempt to reason about the dynamic interactions that occurs in small group interactions and in whole class discussions. In a real sense, by choosing what they reflect on, students individualize instruction for themselves in ways that only they can do. (Yackel, 2000, p. 20)

We can see how this understanding of what makes mathematics motivating is reflected in current trends in mathematics education. For example, the common recommendation to structure lessons around *central challenging tasks* (Stigler & Hiebert, 1999) would support the intellectual and social requirements of mathematical activity. A teacher's ability to recognize, modify, or develop a central activity that is cognitively demanding, while, at the same time, maintaining the intellectual integrity of the task as students struggle, allows the mathematics to retain its intellectual vitality. Students' mathematical experience would be less likely to degrade into mimicry, repetition, and boredom. Jointly working on a central task also provides for more robust whole-class discussions; the class shares the common foundations necessary to truly collaborate on mathematical work. The class can begin to emerge as a mathematical community through developing a common vocabulary and engaging in collective sense making. Ball & Bass (2003) write:

> Making mathematics reasonable is more than individual sense making. Making sense refers to making mathematical ideas sensible [and] comprises a set of practices and norms that are collective, not merely individual or idiosyncratic… That an idea makes sense to me is not the same as reasoning toward understandings that are shared by others with whom I discuss and critically examine that idea toward a shared conviction. (p. 29)

A second trend in mathematics education—*relinquishing 'mathematical authority'* (Cobb, Yackel, & Wood, 1992; Smith, 1996)—also respects the intellectual and social dimensions of mathematical activity. By purposely removing herself or himself as the source of mathematical truth, the teacher enables students to collectively develop mathematical knowledge.

In fact, most current trends in mathematics education respect and enable the intellectual or social dimension of mathematics. For instance, *establishing sociomathematical norms*—establishing an environment for shared ways of mathematical sense-making and making explicit appropriate means of questioning, justifying, and reasoning—enables the social aspect of professional mathematicians' work in the classwork (Cobb, 1994; Cobb, Wood, & Yackel, 1990;Yackel, 2000; Yackel, Cobb, & Wood, 1999). The trend toward *whole-class discussions*, where the teacher orchestrates a respectful space for students to discuss and question each others' thinking, (Cobb, Wood, & Yackel, 1990; Yackel, Cobb, & Wood, 1999); *joint mathematizing* or encouraging collaboration, where students combine their mathematical efforts (Grossman, et al., 2001; Ricks, 2007); and *equalizing participation of students* so particular students do not dominate class discussions (Noddings, 1989; Webb, 1995) are recommendations that attempt to catalyze the types of social interactions characteristic of mathematicians' work. *Eliciting students' mathematical diversity,* the teacher selection of different manners of student approaches and solutions to mathematical problems (Bennie, Olivier, & Linchevski, 1999; Borba, 1992; Linchevski, Kutscher, & Olivier, 1999; Linchevski, Kutscher, Olivier, & Bennie, 2000a, 2000b; Smith, 1992), helps students experience the range of creativity that is a hallmark of mathematical problem solving. *Emphasizing dialogic functioning* (Wertsch & Toma, 1995), when students think about each others' thinking, also contributes to the intellectual work needed to make sense of others' ways and means of operating mathematically.

The ultimate goal of mathematics instruction should be for students to become "lifelong mathematics learners" (Cuoco, 2001, p. 169). What might this look like in a classroom? Although this article is not the place for delving into the specifics of intellectual and social rejuvenation of mathematics classes, I do offer three categories of examples at the levels of task, lesson, and overall curriculum to whet the reader's appetite (e.g., similar to Usiskin, 1998).

*Mathematical Tasks*

The most obvious place for change is at the level of mathematical activities. Stein et al. (2000) detailed a task framework for measuring a task's cognitive demand, ordered from lowest to highest: memorization, procedures without connections, procedures with connections, and doing mathematics. For them, *doing mathematics* is a cognitively demanding activity "requiring complex and non-algorithmic thinking" where no rehearsed approach is used (p. 16), as opposed to *memorization*, defined as the recall or reproduction of previously learned material, or *procedures*, defined as emphasizing algorithms to produce correct answers with little explanation of thinking. They consider *doing mathematics* tasks to be the most beneficial student activities, and their study details several classroom case studies of teachers iteratively attempting to setup and implement *doing mathematics* tasks appropriately.

Definitions can be developed through investigating, rather than having definitions presented by the teacher at the beginning of lessons as though they were axiomatic. For example, developing a definition for *trapezoid* could lead to intriguing intellectual and social possibilities. As there is no accepted standard definition for *trapezoid* (Wolfram, 2010). Are trapezoids quadrilaterals with *at least* one pair or *only* one pair of parallel sides? There are tantalizing ramifications of this choice, such as implications for trapezoidal classification as a subset or superset of parallelograms, how the trapezoid and parallelogram area formulas relate, etc. Students can explore how the taxonomy of other shapes change with similar definition modification. The teacher can mediate a class discussion about which definition is better and why, and the class can then adopt this specific definition in their future work.

*Mathematical Lessons*

Teachers can also structure their lessons to accentuate the intellectual and social aspects of mathematics. Instead of a lesson structured around teacher presentation, demonstration, and modeling of pre-packaged mathematical procedures (Cuoco, 2001), the teacher can pose challenging tasks, and then allow individual and/or group work followed by whole class discussions (Yackel, 2000). There are many examples of such lessons available for teachers, e.g., videos from the Annenberg collection (WGBH Boston, 1995). Another excellent example is the released 1995 Trends in Mathematics and Science Study (TIMSS) videos (NCES, 2003), including one of an eighth grade Japanese geometry lesson that revolved around a single task of dividing land equally. This lesson's unified structure is particularly powerful when juxtaposed against the piecemeal problem review and teacher lecture of an eighth grade U.S. geometry lesson in another TIMSS video.

Lessons from Deborah Ball's 1989 third grade classroom offer further examples of rich intellectual and social mathematics lessons (Ball, 1991). In the lesson known as *Shea's Numbers*, a student claims that the number six is both an even and an odd number. Rather than telling Shea that he is mistaken about the definitions (a typical teacher's response), Ball instead allows him to fully explain his reasoning. Doing so allows others to value his argument, and to recognize that certain even numbers ($2, 6, 10, \ldots, 2 + 4n$) have an odd number of twos; these the class calls *Shea numbers*.

*Mathematical Curriculum*

Mathematics teachers can also structure their units, courses, and curriculum to more accurately mimic the work of mathematicians. Examples of this abound, such as Moses' (2001) Algebra Project where students learn algebra from real-life experiences, the Moore Method (Corry, 2007) of building mathematical structure from a small set of teacher-provided axioms, and Anderson's (1990) mathematics courses "emphasizing that ordinary people create mathematical ideas and 'do' mathematics" (p. 354). Deborah Ball's (1991) year-long third-grade class offers another glimpse at such curriculum innovation because she allows students to reason through their thinking in whole class discussions.

## Conclusion

Current school mathematics strips from the subject the very constituents that provide for meaningful mathematical experiences. A solution to the crisis may be far easier than some think and this solution would not require more rigorous standards, more standardized testing, more funding for smaller classes, or more content training—only a return to the fundamental aspects that make mathematics so intriguing.

The primary way to engage students in mathematics classrooms is to allow them to experience mathematical activity. Mathematical activity is a motivating activity because it connects the ubiquitous human capabilities of intellectualizing and socializing (Devlin, 2000). More specifically, mathematical activity welds together the intellectual and social dimensions of human beings as they collaboratively wrestle with and jointly create mathematical terrain in a process of social mathematizing.

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