Journal of Advanced Academics
jaa Volume 21 ✤ Number 2 ✤ Winter 2010 ✤ pp. 204–232

Associations of Students' Beliefs With Self-Regulated Problem Solving in College Algebra

Victor Cifarelli
University of North Carolina at Charlotte

Tracy Goodson-Espy
Appalachian State University

Jeong-Lim Chae
University of North Carolina at Charlotte

Many stakeholders in the mathematics community believe that the college course commonly entitled College Algebra (hereafter referred to as College Algebra) is not helping students become quantitatively literate citizens (Hastings, 2006). The Conference to Improve College Algebra called for a new perspective (Small, 2002), encouraging math departments to redefine the course and describe desired student outcomes. Mathematics should be viewed as sense-making and problem solving rather than as a collection of facts and procedures (De Corte, Verschaffel, & Op 'T Eynde, 2005). This perspective is compatible with the CRAFTY College Algebra Guidelines (Mathematical Association of America, 2007) that define competencies and recommend pedagogical practices. Redefining College Algebra requires consideration of how students view mathematics and how they view their role in summary

We examined the associations between the expressed mathematical beliefs of students and their self-regulated actions in solving mathematics problems. We conducted surveys and interviews that focused on students' self-regulated problem solving and identified students' self-reported beliefs about mathematics. Our findings suggest that even though students may possess rigid instrumental views about mathematics, they may still be able to achieve success by incorporating some general heuristics into their problem solving if they have first broadened their definition of mathematics to legitimize such activity. Instructional activities that allow students opportunities to share and defend their ideas for solving particular problems prior to actually solving them help students develop self-advocacy and contribute to a proactive sense of agency. Students need support to develop as self-regulated problem solvers. This can be achieved through coaching and one-on-one tutoring; however, it is difficult to achieve in classroom practice. For students to broaden their view of mathematics and what their role as a mathematical problem solver can be, they must be provided with ample problem-solving opportunities. Encouraging students to reflect on their problem solving helps promote the monitoring and assessment necessary for self-regulated learning to occur.

Cifarelli, V., Goodson-Espy, T., & Chae, J.-L. (2010). Associations of students' beliefs with self-regulated problem solving in College Algebra. *Journal of Advanced Academics, 21,* 204–232.

doing mathematics. A primary goal of reform-based mathematics instruction is for students to develop into problem solvers who can self-initiate, monitor, and sustain their actions while solving problems. It is questionable whether College Algebra students perceive their role in the course in these terms.

The goal of this study was to improve understanding of the links between students' mathematical beliefs and their problem-solving actions, with the purpose of developing more effective instructional strategies. Knowing how students' beliefs are associated with their problem-solving actions enables us to better anticipate the learning needs of students and develop appropriate instructional materials designed to address those needs. As the College Algebra course is revamped nationally, it is important to consider the role that mathematical beliefs play in students' problem-solving behaviors so that teachers can design instructional activities that promote positive mathematical beliefs in support of productive problem solving. The current study is the latest in a series of studies we conducted (Cifarelli, 1998; Cifarelli & Goodson-Espy, 2001) to build and validate a model of the mathematical problem-solving actions of college students. In our earlier studies, we surveyed the mathematical backgrounds of these students and found that students' prior difficulties in K–12 mathematics classes influenced their views of mathematics as practice (Cifarelli, 1998) and identified ways in which students' views impacted how they conceptualized mathematical problems (Cifarelli & Goodson-Espy, 2001). The current study builds on these results by examining the students' problem-solving skills, focusing on the ways they develop and self-regulate their actions as they solve problems. Unlike our previous studies, the current study examines the problem-solving actions of students along two levels of knowledge. First, we consider the role that students' mathematical beliefs play in their problem solving (De Corte et al., 2005). Second, we adopt a focus on the students' cognitive actions as they complete mathematical tasks in individual interviews. In doing so we highlight the problem-solving processes that students use. In adopting this focus, we provide an account of the associations between the students' mathematical beliefs and self-regulated problem solving. In addition to helping us learn more about the learning needs of these students, we believe that the results will help other mathematics educators design and implement more effective approaches for promoting self-regulated problem solving with this population.

Mathematical Beliefs

The study draws from the constructivism of Piaget (1970) and von Glasersfeld (1991) and the social cognitive psychology of Bandura (1986, 1997) and Zimmerman (1995, 2005), with the following implications. First, mathematics learning is viewed as the construction of knowledge that is problem-based (Wheatley, 2004). Thus, we must observe students while they are solving mathematics problems to explain how they develop knowledge from actions. Second, the student's mathematical beliefs are viewed as conceptions the student holds about mathematics and ideas of how he or she can act within a mathematical context. Op 'T Eynde, De Corte, and Verschaffel (2002) defined mathematical beliefs as describing what students see as true in mathematics, in the classroom, and within themselves. This perspective on beliefs suggests a useful framework. A third implication concerns the types of analyses conducted. Even though self-reports of mathematical beliefs and attitudes from surveys are valuable, another source of data includes observations of students solving mathematics problems. Students' mathematical beliefs and self-regulated activity are inferred from their interpretation of problems, the goals they explore, and the strategies they pursue to develop solutions. The analyses include comparing and contrasting case studies to develop explanations for the students' actions.

Self-Efficacy for Self-Regulated Learning

Defined by Zimmerman (2002), self-regulation is further described by Usher and Pajares (2008) as a metacognitive process where students examine and evaluate their thought processes and discover pathways to success. In addition to knowing self-regulatory strategies, students must believe that they can apply them effectively; this is called "self-efficacy for self-regulated learning" (Usher & Pajares, 2008, p. 444). Bandura and Zimmerman's notions provide two crucial components. First, Bandura's (1997) characterization of self-efficacy and Zimmerman's follow-up studies (Zimmerman, 2002, 2005) suggest the need to consider features of beliefs not addressed elsewhere. For example, the *generality* of self-efficacy beliefs refers to transferability. The *strength* of self-efficacy beliefs refers to the certainty with which one can perform a specific task (Zimmerman, 1995). Second, Zimmerman's (2005) view of self-regulation as a cyclical process of adapted action is consistent with this study's interpretation of structured action. While solving problems, students must self-initiate and plan their actions, develop goals for action, monitor and assess their actions, and adapt their actions in order to achieve goals. Hence, Zimmerman's cyclical model of self-regulatory processes (Zimmerman, 2005; Zimmerman & Cleary, 2006) provides a means to classify students' problem-solving actions.

Mathematics educators have conducted few studies concerning the role that self-beliefs play in self-regulated activity. Although they have acknowledged the importance of learners developing metacognitive control of their mathematical actions (Cai, 1994; Schoenfeld, 1985, 1992; Silver, 1982; Wong, Marton, Wong, & Lam, 2002), the prominent studies of self-regulated learning are centered in social cognitive psychology, drawing from the work of Bandura (1986, 1997, 2006) and Zimmerman (1995, 2005). Although some compatibility exists between perspectives, there are important differences. As mathematics education researchers emphasize metacognitive processes in self-regulated activity, social cognitive psychologists also cite the learner's self-beliefs as an important source of self-regulated learning (De Corte et al., 2005; Zimmerman, 1995, 2005). For example, Pajares and his colleagues examined the role of self-beliefs in mathematics achievement (Pajares, 1996a, 1996b, 1999; Pajares & Miller, 1994; Usher & Pajares, 2006, 2008). Results include the finding that self-efficacy is a predictor of mathematics performance (Pajares & Miller, 1994). In later studies, Pajares and colleagues found that a strong relationship exists between self-efficacy beliefs and the use of self-regulatory skills to solve mathematics problems (Pajares, 1996a, 1996b) and that measures of self-efficacy are maximally predictive when they are measured in a manner specific to the academic task at hand (Usher & Pajares, 2008). These findings suggest the need for multiple means to assess students' self-efficacy beliefs.

Instrumental Understanding Versus Relational Understanding

This study sought to describe students' expressed beliefs about mathematics. Yackel's (1984) Mathematical Beliefs Survey provided a framework for soliciting these beliefs. The survey was based on Skemp's (1976) seminal research that defined *relational understanding* of mathematics as the ability to use knowledge that is based on what has been previously learned through the use of multiple strategies and representations. Skemp defined *instrumental understanding* of mathematics as driven by procedures that must be memorized, as "rules without reasons" (p. 6). This orientation was useful for this study because it provided information concerning how students view mathematics and what they consider their role to be in mathematical activity. Students come to College Algebra with beliefs about what mathematics is, what mathematics classrooms are like, and what they see as their role in doing mathematics. Some students' beliefs seem to support their problem solving and mathematical achievement; others do not. This study sought to describe the types of beliefs that supported effective problem-solving actions.

Research Questions and Hypotheses

The study addressed the following research questions: What are some prominent mathematical beliefs of students enrolled in

College Algebra? How do student's self-efficacy beliefs relate to self-regulated problem solving activity? The study incorporated a set of working hypotheses, based on the literature on student learning in College Algebra, to frame the analysis:

- Hypothesis 1: Students holding procedural conceptions of mathematics will apply idiosyncratic interpretations of rules to solve problems and students believing in mathematics as a conceptual system will apply more complex and sophisticated problem-solving strategies.
- Hypothesis 2: Students exhibiting high self-efficacy beliefs will be more persistent in problem solving and will apply more complex and sophisticated strategies than students exhibiting fragile self-efficacy beliefs.

Methods

Triangulation Mixed Methods Design

The study adopted a triangulation mixed methods design with an emphasis on the qualitative process (Creswell & Plano Clark, 2007). In triangulation mixed methods design, the quantitative and qualitative procedures are conducted separately to maintain independence and both sets of findings are integrated into the results. The qualitative portion of the present study described the beliefs of students exposed during problem-solving interviews, and a survey instrument provided a quantitative measure of the students' beliefs. Qualitative data were examined through a within-case analysis and a cross-case analysis using the theme analysis method. We first provided a detailed description for each case and identified problem-solving and mathematical beliefs themes within and across cases. The themes that were identified within and across cases were finally compared to the mathematical beliefs results from the survey data.

Participants and Setting

The sample consisted of 139 students enrolled in College Algebra at a university in the southern U.S. The sample was comprised of 81 females and 58 males, including 97 Caucasians, 21 African Americans, 8 Asian Americans, 7 Latinos, 3 American Indians, and 3 international students. There were 26 freshmen, 53 sophomores, 41 juniors, and 19 seniors.

All students enrolled in four sections of the course during fall/spring semesters of 2000–2001 were surveyed. All students were invited to participate in a series of tutorial interviews with the class instructor, and 12 volunteered. The participants included 3 female African American students, 8 Caucasians (5 males and 3 females), and 1 male Latino student. Six of the students were majoring in business, 4 in political science or history, and 2 in nursing. Episodes from two case studies are presented to provide illustrations of the detailed type of analysis conducted and to provide insight into how the students' beliefs, as evidenced in their survey results and through their interview comments, were associated with their problem-solving decision-making.

The College Algebra Course

The class met twice every week for 90 minutes. Although traditional College Algebra lecture topics were covered, the format was less traditional. Classes combined lecture and student work sessions. The same instructor taught all sections and employed a conversational style, encouraging students to work problems and ask questions. The instructor possessed both a bachelors and master's degree in mathematics, as well as a doctorate in mathematics education, specializing in research on problem solving. Students worked with peers to solve problems, and the instructor and tutors provided assistance as needed. Approximately half of each class was dedicated to having students work/discuss problems. The course textbook was *College Algebra Concepts and Models* by Larson, Hostetler, and Hodgkins (2000).

Procedures

Data consisted of survey responses, videotaped protocols of interviews, written transcriptions, participants' written work, and the researcher's field notes. All tests, interview questions, and the survey were pretested during previous semesters.

The Mathematical Belief Systems Survey. Yackel (1984) developed the Mathematical Belief Systems Survey to determine college students' expressed beliefs about mathematics and to measure how likely they were to favor Skemp's (1976) instrumental versus relational understanding of mathematics. The two-part survey utilizes a 5-point Likert scale. Part 1 of the survey uses the instrumental versus relational understanding framework to ask questions that probe the students' beliefs about mathematics and that also ask the students to characterize their own problem-solving behaviors. Questions stated as positive relational statements were coded with *Strongly Agree* (*SA*; 5.0) reflecting a strongly relational view. Overall survey scores were labeled as follows: (1.0–2.0) *instrumental*, (2.1–3.0) *somewhat instrumental*, (3.1–4.0) *somewhat relational*, and (4.1–5.0) *relational*. Part 2 of the survey examines students' attitudes toward mathematics. Questions stated as positive attitudes toward mathematics were coded with *SA* (5.0) reflecting a positive attitude. Quillen (2004) used the survey in a doctoral study and found a Cronbach alpha of .89. A reliability analysis was also constructed for this study. The survey was given to all 139 participants at the beginning of the courses, and the Cronbach alpha was determined for Part 1 (.73) and for Part 2 (.87).

Interviews. The instructor conducted individual problem-solving interviews with 12 volunteers. Interviews occurred roughly bi-weekly all semester and lasted about 40 minutes each. The initial interview included questions to confirm the participant's survey responses. In addition, the interviewer asked about the participant's mathematical K–12 history. Subsequent interviews consisted of having participants solve problems. Some of the questions came from homework or practice test questions, and others were homework problems posed by the participant.

The interviews were conducted following Cobb and Steffe's (1983) principles of clinical interviews. The interviewer's questions ranged from questions that asked the participant to explain an action that was performed to questions intended to induce the participant to consider a new problem. For example, whenever an extended period of silence was accompanied by an absence of writing, the interviewer asked a question, such as "What are you thinking?" These kinds of questions cause only minor interruption of learners' actions and do not threaten the data's validity (Ericsson & Simon, 1993; Schwarz, 1999). Periods of self-reflection may indicate instances where learners are monitoring and assessing their actions to aid their understanding of the problem situation and thus can be seen as important indicators of self-regulating activity (Cobb & Steffe, 1983). Finally, upon completion of a task, the interviewer might pose a question to stimulate the participant to consider a new situation. Such questions gave the participant opportunities to consider new goals to further develop mathematical knowledge (Zimmerman, 2005).

Data collected for each participant included both video and written protocols. Videotaped protocols served three purposes. First, the video protocols were crucial for isolating and examining nonverbal processes that may play important roles in how students proceed in problem solving. Second, research suggests that students develop their knowledge in problem-solving situations within clusters of activity (Schoenfeld, 1992) that are situational and episodic in structure (Hall, Kibler, Wenger, & Truxaw, 1989), and that unfold via activity (Kieran & Pirie, 1991). These types of structuring activities appear related to the phases of self-regulation identified by Zimmerman (2005). Third, an interview is a social interaction in which the interviewer and the participant share a dialogue. Hence, viewing videotape gives the researcher an opportunity to analyze the dialogue as an observer and allows for reinterpretation and revision of the participant's activity, allowing for a continual communication between theory and data (Roth, 2005). Written protocols, comprising written transcripts of participants' responses and the written students' work, were also used in the analysis providing an additional perspective. Transcripts allowed the researchers a means to identify and make reference to examples of significant solution activity when they occurred.

In analyzing the interviews, the researchers acknowledged Zimmerman's (2005) cyclical model of self-regulation and partitioned participants' solution activity into episodes that evidenced critical aspects of their solution activity: (a) task analysis and development of goals, (b) ongoing self-monitoring and progress assessment, and (c) problem-solving results. Three experienced College Algebra researchers, one of whom taught the project courses, completed the data analysis. The analysis was conducted in the following phases.

Coding of interview protocols. Two researchers independently examined each video protocol to identify episodes where the participant faced problematic situations. In such situations, the participants viewed the methods they had used previously as not applicable and determined that new ideas were needed. Once video episodes were identified, the written transcripts were annotated to indicate the participant's statements. The researchers compared coding of the videos and written work to find patterns of agreement and to resolve areas of discrepancy. The third researcher was also brought into the analysis at this point to analyze and resolve points of disagreement.

Because the study examined how self-efficacy beliefs interact with self-regulated activity, it was important to partition the transcript protocols accordingly. Participants might indicate their beliefs about the problem by making explicit statements such as, "I think that the problem may involve these kinds of equations so I will need to generate and solve some equations from the information." In this example, the participant's hypothesis statement expresses confidence in being able to complete both the task as well as the goal that the student has in mind. In many cases, such explicit statements are absent. The researchers probed the participants' actions when they worked in silence, by asking questions such as, "Why did you perform these particular actions?" These questions induced the participants to comment and thus provided the researchers with material with which they could make inferences about the participant's beliefs, hypotheses, and potential solution actions. The researchers applied a coding scheme to refer to the videos that helped them identify instances of these processes within the participant's actions. In developing this scheme, the researchers considered the participants' self-efficacy beliefs and problem-solving actions in the following terms: the participant's initial idea about the problem situation and assessment of difficulty; the participant's subsequent formulation of a question to explore (the student's goal development); and the student's actions to develop answers to the question (the student's problem-solving). These definitions are consistent with contemporary definitions of problem posing and solving (Schoenfeld, 1992; Wheatley, 2004) and also with Zimmerman and Cleary's (2006) planning and monitoring phases of self-regulation. Table 1 summarizes the operational codes and contains examples of how the codes help identify instances of self-efficacy beliefs and self-regulated problem solving.

Preparation of a case study for each interviewee. Based on the analysis of the video and written protocols, the researchers prepared a case study for each interviewee. Each case study contained a summary of the participant's beliefs observed during the interviews and documentation of the problem-solving actions.

Results

The results are presented in terms of our original hypotheses. Because each is based on knowledge of the participants' beliefs and what they believed as they solved problems, we present results from both the survey analysis results for the interviewees and episodes from two participants, Zach and Bonnie, to comment directly on our hypotheses.

Survey Data Analysis

The Mathematical Beliefs Survey (Yackel, 1984) was developed in two parts. Part 1 of the survey explores students' beliefs of mathematics in terms of instrumental versus relational under-

Table 1

Operational Codes for Identifying Self-Efficacy Beliefs and Self-Regulated Action

Process	Indicators
Self-Efficacy • Does the student demonstrate confidence in developing goals for action? • Are the student's goals consistent with the problem's conditions? • Does the student demonstrate persistence in his or her goal-directed actions?	Evidence of self-efficacy in the student's explicit verbal statements or inferred actions, such as: • "I think that I can solve the problem by the same way that I solved those other problems." • "The problem is about linear equations, so I will solve this problem by linear equations." • "I tried that strategy, and it did not work. This other strategy might be useful so I will try it." • The student's actions appear purposeful and designed to answer the central question.
Self-Regulation • Are the student's goals well defined to answer the problem's central question? • Is there evidence that the student can monitor and assess the results of his or her actions and reformulate goals when needed?	Evidence of self-regulation in the student's explicit verbal statements or inferred actions, such as: • "I think that I can solve the problem by examining this set of conditions." • Student demonstrates planning in actions. • "This result only addresses a special case of the problem. I will now consider other cases in order to fully solve the problem." • Student hypothesizes about problem difficulty.

standing and asks students to characterize their own problem-solving behaviors. Part II of the survey examines students' attitudes toward mathematics. For Part 1 of the survey, 6 participants scored at the instrumental level and 6 participants scored at the somewhat instrumental level. These subgroups had mean averages of 67.8 and 76.0 on the final exam. On Part 2 of the survey, 5 participants scored as having negative attitude, 3 participants scored somewhat negative attitude, 2 scored somewhat positive attitude, and 2 exhibited a positive attitude score. The group of participants having either negative or somewhat negative attitude had a mean average of 70 on the final exam and the group of participants having either positive or somewhat positive attitude had a mean average of 77.5 on the final exam.

Interview Data

The subsequent interviews with participants were an attempt to document the relationships between students' self-efficacy beliefs and their self-regulated problem solving.

We noted that the interviewees as a group performed at a level consistent with other participants in the study. The interviewees had mean exam averages of 64.41, 71.81, and 72.09 as compared to 66.03, 73.14, and 72.19 for the noninterview participants. On the cumulative final exam, the interviewees had a mean average of 72.27, which was similar to the mean for noninterview participants (72.58).

In characterizing participants' self-regulation strategies to solve problems, it was useful to assess them on the basis of how their initial planning progressed to specific strategies to carry out. Certain participants tended to favor one method to solve problems throughout the interviews. The observations of participants' overall organization of their actions suggested three categories of strategies (personal interviews, January–April, 2001). Certain participants used a systematic trial-and-error (ST) strategy that was exemplified by careful monitoring and assessment of results. Four of the participants we interviewed demonstrated the ST strategy. A second category of participants appeared to be more haphazard in their investigations, relying on a random trial-and-error approach (RT) to initiate and assess their actions. A total of 3 participants demonstrated the RT strategy. A third group of participants did not exhibit RT, and relied less on their own self-generated explorations and more on what they could recall from other sources (e.g., teacher's solutions in class, detailed solutions in the textbook). We refer to this last strategy as imitative modeling (IM). Based on our observations of the participants, 4 were identified as demonstrating the IM strategy.

Data in Support of Hypotheses

Hypothesis 1. Hypothesis 1 compares the problem solving between participants identified as having instrumental beliefs about mathematics versus those identified as having relational beliefs about mathematics. Because the survey data identified the interviewees as instrumental and somewhat instrumental in their beliefs, we can only comment directly on the problem solving of these two groups. Of the 6 participants identified as instrumental in their beliefs, 5 of these consistently used an IM strategy to solve problems (personal interviews, January–April, 2001). Of the participants identified as somewhat instrumental, 5 of these consistently used the conceptual strategy of ST to solve problems (personal interviews, January–April, 2001). The finding that the participants who are somewhat instrumental in their beliefs used a more conceptual strategy to solve problems than participants who are instrumental in their beliefs supports Hypothesis 1.

Hypothesis 2. Hypothesis 2 predicts that participants exhibiting high self-efficacy beliefs will be more persistent in problem solving and will apply more complex strategies than participants exhibiting low self-efficacy beliefs. Our interview data provide some support for this hypothesis. Two of the participants were identified as having positive attitudes about mathematics (personal interviews, January–April, 2001). In addition to using more complex solution strategies, these participants regularly demonstrated persistence in their problem solving when difficulties arose. In contrast, the 5 participants identified as having negative attitudes about mathematics struggled whenever difficulties arose in the course of their problem solving (personal interviews, January–April, 2001).

Case Studies

Zach. Zach majored in business and was an above-average student in the class, receiving a grade of B-. Zach completed 4 years of high school mathematics including Algebra 1, geometry, Algebra 2, and advanced mathematics. Zach's survey (Part 1) score of 3.0 suggested that he was at the break between *somewhat instrumental* and *somewhat relational* in his mathematical beliefs, and his survey (Part 2) score of 4.5 suggested that he had positive attitudes about mathematics. Zach appeared highly motivated and engaged in solving problems throughout the interviews and was consistently generous in his self-reporting. We characterized Zach's self-efficacy beliefs for each problem that he attempted to solve using the following considerations. In a particular problem-solving situation a student, on the basis of an initial task analysis, may consider a strategy and have a sense of where the proposed action may lead and the likelihood of success. Encountering difficulty, the student may rethink original ideas and pursue another path. Such "online" decisions draw from the student's beliefs about mathematical activity and the student's role. The strength of one's self-efficacy beliefs is related to the student's will to remain persistent and focused when unexpected problems arise during mathematical activity. The following episode highlights some ways that Zach demonstrated these types of solution activity.

Although Zach's survey responses appeared to minimize the importance of using rules, Zach's actions in the interviews suggested otherwise. He relied on formal rules to solve most problems (personal interviews, January–April, 2001). However, he also had interesting ideas of what he believed mathematics learning and problem solving to be about. In the first interview, for instance, he stated that "What math is, is a learning process. You have to try different ways to get the answer" (personal interviews, January 24, 2001). Based on our sense of his beliefs about mathematics learning, we hypothesized that Zach would demonstrate self-regulation in his problem solving in at least two ways: Zach would be likely to demonstrate variety in his solutions even when he found himself faced with unexpected results, and Zach would demonstrate more persistence than other students in his problem-solving actions.

Example of self-regulating activity: Rationalizing the denominator of a fraction. In the second interview, Zach was asked to solve a problem involving rationalizing the denominator of a fraction (personal interview, February 14, 2001). Zach's solution is summarized in Figure 1.

Zach recalled, generally, how to rationalize the denominator yet could not remember specifics. Zack had a sense of how to solve the problem but did not appear to understand how equivalent fractions were used. His work was consistent with his survey comments (see Figure 1) because he systematically varied the signs of the second fraction until he found the correct answer. When explaining, he commented that in mathematics one has to be prepared to try a variety of approaches to solve problems. Zach appeared to bring a high level of confidence to most problems he attempted (personal interviews, January–April, 2001). His view that most problems could be solved by trying different approaches suggested that he held a relational view of mathematics. These impressions were confirmed by survey data (Part 1 survey score 3.0—*somewhat instrumental/relational*; Part 2 score 4.5—*positive attitude*). He consistently used self-questioning in the course of his actions to monitor and assess the usefulness of results (personal interviews, January–April, 2001). In most cases the immediate result was either the confirmation of a current conjecture or development of a new supposition to explore.

Our hypotheses that Zach would demonstrate variety in his solutions and be persistent when unexpected problems arose were confirmed throughout the interviews. Zach had a good "big picture" view of how to solve particular problems, although his misunderstanding of formal concepts sometimes interfered when he carried out actions. For example, in trying to rationalize the denominator of the initial problem, his switching involved systematically varying the signs in the second fraction, without understanding that the second fraction must have a numerical value of one (personal interview, February 14, 2001). This sug-

Protocol Statements	Zach's Written Work
Zach: From my understanding of how to do these, we have to get square root from the denominator cause you are not supposed to have square roots there. Are you supposed to multiply by $(5-\sqrt{3})$ on the top and $(5+\sqrt{3})$ on the bottom, or do I keep them the same? Or, do I switch them? Let's see.	$\frac{2x}{5-\sqrt{3}} \bullet \frac{5-\sqrt{3}}{5+\sqrt{3}} =$
Zach: Okay. These will cancel out, this will still give you $\sqrt{3}$ in denominator, get $\frac{2x}{5+\sqrt{3}}$, you're back where you started.	$\frac{2x}{\cancel{5-\sqrt{3}}} \bullet \frac{\cancel{5-\sqrt{3}}}{5+\sqrt{3}} = \frac{2x}{5+\sqrt{3}}$
Zach: Let's try a plus sign up here. You don't want to cross multiply these, multiply straight across. Is that right? Let's see. That will not work, still do not get rid of square root sign.	$\frac{2x}{5-\sqrt{3}} \bullet \frac{5+\sqrt{3}}{5-\sqrt{3}}$ $(5-\sqrt{3}) \bullet (5-\sqrt{3}) =$ $5(5-\sqrt{3})-\sqrt{3}(5-\sqrt{3})$ $25-5\sqrt{3}-5\sqrt{3}+3$
Zach: I could try both same sign on this side. Let's change it to plus and try. Yeah, it is correct. The reason I got it right is I went through some things when same sign didn't work. I switched them, I still had $\sqrt{3}$ in denominator. And the like signs allowed me to get rid of it. I tried different things and that is how I got it right.	$\frac{2x}{5-\sqrt{3}} \bullet \frac{5+\sqrt{3}}{5+\sqrt{3}} =$ $\frac{2x(5+\sqrt{3})}{25-3} = \frac{2x(5+\sqrt{3})}{22}$

Figure 1. **Rationalizing the denominator of a fraction: Zach's solution.**

gested that he did not have a strong understanding of equivalent fractions. Nevertheless, he remained persistent to check all possibilities, assessing his progress with each calculation, and he did find the correct solution. Zach applied what we coded as an ST.

Three other interviewees also applied this as their prime problem-solving methodology.

Bonnie. Bonnie was a reentry student, returning to school some 20 years after she had graduated. In contrast to Zach, Bonnie struggled with almost every kind of problem that she attempted. She placed in the lower third of the class on all exams, receiving grades ranging from D to C+. However, she did surprisingly well on the final exam and received a C- in the class.

In the initial interview, Bonnie emphasized the importance of the teacher's actions: "If I can just follow all the steps, I will be okay." Her solution activity during the interviews relied on her memory of how the instructor solved similar problems. Hence, we hypothesized that she looked to imitate others in developing solutions and would have particular difficulty solving multistep problems.

Bonnie seemed to have some competence performing basic skills such as factoring trinomials and solving first and second degree equations, yet lacked confidence for more complex problems. For example, in reporting her actions, Bonnie consistently phrased her comments in question form (e.g., "This is an example of a first degree equation isn't it?"), looking to the interviewer for confirmation that her actions were correct—she seldom proceeded into new problems without seeking such approval.

As a study aid early in the semester, Bonnie used the algebra videotapes in the University Learning Center. These tapes were prepared by the mathematics department and cover most of the needed algebraic skills and also provide helpful study suggestions. According to the Center's records, she was the first student to check them out in 20 years. Similar to her focus on mimicking the instructor's actions, Bonnie came to rely on the tapes as a model for her actions, convinced that her chances for success in the class depended on her mastering the tapes.

Despite lacking in self-confidence, Bonnie did demonstrate successful problem solving during the interviews. We present her solution here as a contrast to Zach.

Example of self-regulating activity: Graphing a quadratic equation. In the fourth interview, Bonnie was asked to graph

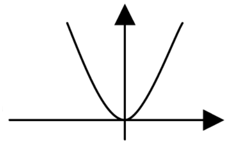
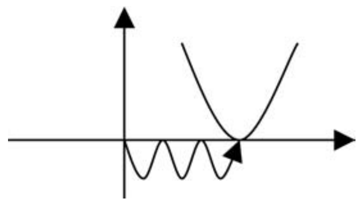
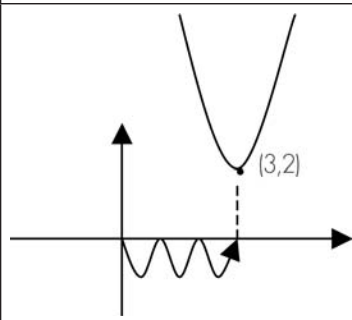
Protocol Statements	Bonnie's Written Work
Bonnie: There is nothing in parentheses—I would do that first. So, I am going to insert parentheses first.	$f(x) = x^2 - 3x + 2$ $f(x) = (x^2 - 3x) + 2$
Bonnie: Okay, we need to start with just this parabola piece, $f(x) = x^2$.	$f(x) = x^2$ 
Bonnie: The $-3x$ means that it will move to the right by three places.	
Bonnie: The +2 means it moves up two, to the point at (3,2). And since it's x^2, means it is going to be one of these types of graphs.	

Figure 2. **Graphing a quadratic function: Bonnie's solution.**

the following quadratic function, $f(x) = x^2 - 3x + 2$ (personal interview, February 28, 2001). Bonnie's solution is summarized in Figure 2.

Bonnie referred to how the instructor from the videotapes solved problems by applying horizontal and vertical shifts to the quadratic $f(x) = x^2$ without making a table of points. From her statements it was clear that she had some sense of how to solve the problem by using this transformational approach. However, her insertion of parentheses in the first step led her to the inaccurate conclusion that the graph needed to be shifted horizontally +3 units. She was clearly discouraged upon learning that her answer was incorrect, replying, "Well, I haven't gone past the first tape yet." Despite this setback, she was able to follow the interviewer as he coached her through the problem in a step-by-step fashion. With tutorial assistance she was able to generate the correct vertex and intercepts, as well as sketch a correct graph.

In general, Bonnie explained her actions by referring back to the tapes and the need to be able to replicate the actions she had seen on the videos. This reliance on trying to replicate the actions of others suggested to us that Bonnie brought a low level of confidence to most problems she attempted (personal interviews, January–April, 2001). Her view that most problems could be solved by recalling and imitating the actions of the instructor suggested that she held an instrumental view of mathematics. These impressions were confirmed by survey data (Part 1 survey score of 1.5—*instrumental*; and Part 2 score of 2.0—*somewhat negative attitude*). Unlike Zach, her use of self-questioning in the course of her actions was not to monitor and assess the usefulness of results, but rather to prompt her memory of how the instructor may have solved the problem in class (personal interviews, January–April, 2001). In most cases, the immediate result was a replication of the steps that the instructor used to solve the problem.

Our hypotheses that Bonnie would demonstrate limited variety in her solutions and not be persistent when unexpected problems arose were confirmed throughout the interviews (personal interviews, January–April, 2001). Bonnie seldom demonstrated a "big picture" view of how to solve particular problems and often applied idiosyncratic rules when she carried out actions. For example, in trying to graph the quadratic function, she saw the need to insert parentheses to group the terms. Although she recalled aspects of how to use transformations to graph the parabola, her application of the concept was incorrect (personal interview, February 28, 2001). This suggested that she did not have a strong understanding of transformational graphing techniques. Bonnie applied what we coded as an IM. Three other interviewees also applied this as their prime problem-solving methodology.

Discussion

What are some prominent mathematical beliefs of students enrolled in College Algebra? The survey results findings are consistent with other studies concerning students' mathematical beliefs (Kloosterman, 2002; Op 'T Eynde et al., 2002; Yackel & Rasmussen, 2002). Holding an instrumental view of mathematics was found to be the most dominant factor, a finding compatible with previous results concerning the mathematical beliefs of students enrolled in College Algebra. Many students view learning mathematics as a process of memorizing procedures (Underwood-Gregg & Yackel, 2000) and believe that they need to learn a new method for each class of problems (Carlson, 1997).

The current results indicate how students with an instrumental understanding of mathematics can still find ways to become successful. For example, despite Zach's reliance on rules, his belief that doing mathematics has an exploratory side and that trying different ways to approach problems (ST) is an important strategy to use across problems enabled him to be successful throughout the interviews. We observed this ST strategy in 4 of the participants (personal interviews, January–April, 2001). This finding is important for the following reason. The research literature on relational and instrumental learning contains few examples of instrumental learners demonstrating success in solving problems. The findings here suggest that even though students may possess rigid instrumental views about mathematics, they may still be able to achieve success by incorporating some general heuristics into their problem solving, if they have first broadened their definition of mathematics to legitimize such activity.

How Do Student's Self-Efficacy Beliefs Relate to Self-Regulated Problem-Solving Activity?

The results of the interviews were somewhat mixed with regard to answering this question. Because most studies of mathematical self-efficacy ask students to rate their confidence to solve particular tasks (Pajares, 1996a), the results here provide an illustration of some ways that students actually demonstrate self-efficacy while solving problems.

We observed participants who consistently expressed confidence in their capabilities to perform specific tasks and were able to carry through that confidence in the face of unexpected results (personal interviews, January–April, 2001). These were students who, like Zach, were able to employ ST approaches to solve problems. Even when results did not lead directly to success, they were able to assess the usefulness of results and reformulate the problem as needed, thus demonstrating the phases of self-regulation hypothesized by Zimmerman and Cleary (2006). However, we also observed participants like Bonnie who were not as structured in their actions even though they sometimes brought high self-efficacy beliefs to the particular task being solved. In particular, the 8 participants characterized as RT or IM in their actions were not always able to self-regulate their actions when unexpected situations arose (personal interviews, January–April, 2001). For these students, although sometimes very confident about their prospects for solving particular problems, a sense of self-efficacy would not always carry them through to make progress toward the solution. In particular, 5 students consistently used the strategy of "recall-and-imitate actions of outside sources" in order to solve problems (IM). The strategy was effective when the students solved single step problems. However, in solving multistep problems, the limited usefulness of the strategy became apparent. Their subsequent self-regulated activity was often characterized by inappropriate strategies that seldom resulted in progress towards the solution.

The results must be viewed with care. First, the College Algebra population includes students at all levels of academic experience and background, ranging from first-year students encountering the material for the first time, to older reentry students who have not taken mathematics for some time. Because we could not control for these characteristics, care should be exercised in comparing the results to all other College Algebra classes. Second, despite the inclusion of both surveys and interviews in the analysis, these techniques at best give a glimpse of the complex processes that underlie students' beliefs and self-regulated learning. More studies are needed that employ both traditional and nontraditional instruments to examine problem-solving processes.

Teaching and Learning Implications

Encourage Proactive Agency in Problem Solving

In assessing the self-regulating problem solving of participants, we noted their ongoing assessment of results and problem reformulation as processes that enabled them to persist in their actions. Participants such as Zach viewed themselves as in control and aggressively switched course whenever unexpected problems arose. In contrast, other participants appeared to view themselves as reactive problem solvers like pawns on a chessboard. Instructional activities that allow students opportunities to share and defend their ideas for solving particular problems prior to actual solving help develop self-advocacy in students and contribute to a proactive sense of agency. Students need support to develop as self-regulated problem solvers. This can be achieved through coaching and one-on-one tutoring but is difficult to achieve in classroom practice. Such coaching can also be done via well-designed computer-aided instruction. Computer-aided College Algebra projects at the University of Alabama and Louisiana State University have reported success using this approach.

Create Problem-Solving and Problem-Posing Opportunities for Students to Generalize Their Self-Efficacy Beliefs

We agree with Pajares' (2006) assertion that self-efficacy beliefs can be generalized. In order for generalization to occur, we suggest two types of activities. First, students must be presented with rich problem-solving and problem-posing opportunities, including experiences with open-ended problems, such as those described by Becker and Shimada (2005), and classic problem-posing tasks as described by Brown and Walter (2004) and English (2003). Second, instructional activities that allow students opportunities to experience success in solving problems and then reflect on their successes with a view to solving new problems appear to be useful activities that will help students to generalize their self-efficacy beliefs across problem situations.

Promote Reflection and Discussion in Classroom Discourse

In order for students to broaden their view of mathematics and what their role as a mathematical problem solver can be, they must be provided with ample problem-solving opportunities. Encouraging students to reflect on their problem solving helps promote the monitoring and assessment necessary for self-regulated learning to occur. We must carefully listen to students and observe what they do rather than conduct classroom activities based on our expectations of what we *think* they will say and do. Due to large class sizes, it is difficult for teachers to engage in the lengthy discussions represented in our interviews. However, more one-on-one communication can be facilitated using mathematical journaling that invites students to write their thoughts about the decisions they make and the difficulties they face while solving problems. This information, in turn, provides teachers with opportunities to respond to the beliefs and questions of individual students.

References

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory.* Englewood Cliffs, NJ: Prentice-Hall.

Bandura, A. (1997). *Self-efficacy: The exercise of control*. New York, NY: Freeman.

Bandura, A. (2006). Guide for constructing self-efficacy scales. In F. Pajares & T. Urdan (Eds.), *Adolescence and education, Vol. 5: Self-efficacy beliefs of adolescents* (pp. 307–337). Greenwich, CT: Information Age.

Becker, J., & Shimada, S. (2005). *The open-ended approach: A new proposal for teaching mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Brown, S., & Walter, M. (2004). *The art of problem posing* (3rd ed.). Mahwah, NJ: Lawrence Erlbaum.

Cai, J. (1994). A protocol-analytic study of metacognition in mathematical problem solving. *Mathematics Education Research Journal, 6,* 166–183.

Carlson, M. (1997). Obstacles for college algebra students in understanding functions: What do high-performing students really know? *The AMATYC Review, 19,* 48–59.

Cifarelli, V. V. (1998, March). *Student portraits: A study of the mathematical beliefs and conceptions of college algebra students*. Paper presented at the Western Region North Carolina Council of Teachers of Mathematics Conference, Charlotte, NC.

Cifarelli, V. V., & Goodson-Espy, T. (2001). The role of mathematical beliefs in the problem solving actions of college algebra students. In M. van den Heuval-Panhuizen (Ed.), *Proceedings of the 25th meeting of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 265–272). Utrecht, The Netherlands: The Freudenthal Institute.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14,* 83–94.

Creswell, J. W., & Plano Clark, V. L. (2007). *Designing and conducting mixed methods research*. Thousand Oaks, CA: Sage.

De Corte, E., Verschaffel, L., & Op 'T Eynde, P. (2005). Self-regulation: A characteristic and a goal of mathematics education. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of self-regulation* (pp. 687–726). Orlando, FL: Academic Press.

English, L. (2003). Engaging students in problem posing in an inquiry-oriented mathematics classroom. In F. Lester & R. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten-grade 6* (pp. 187–198). Reston, VA: National Council of Teachers of Mathematics.

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis: Verbal reports as data* (Rev. ed.). Cambridge, MA: The MIT Press.

Hall, R., Kibler, D., Wenger, E., & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition & Instruction, 6,* 223–283.

Hastings, N. B. (2006). *A fresh start for collegiate mathematics: Rethinking the courses below calculus.* Washington, DC: MAA Notes Series.

Kieran, T. E., & Pirie, S. E. B. (1991). Recursion and the mathematical experience. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 78–101). New York, NY: Springer-Verlag.

Kloosterman, P. (2002). Beliefs about mathematics and mathematics learning in the secondary school: Measurement and implications for motivation. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 247–269). Dordrecht, The Netherlands: Kluwer Academic.

Larson, R., Hostetler, R. P., & Hodgkins, A. V. (2000). *College algebra: Concepts and models* (3rd ed.). Wilmington, MA: Houghton Mifflin.

Mathematical Association of America. (2007). *CRAFTY guidelines for college algebra*. Washington, DC: Author.

Op 'T Eynde, P., De Corte, E., & Verschaffel, L. (2002). Framing students' mathematics related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 13–37). Dordrecht, The Netherlands: Kluwer Academic.

Pajares, F. (1996a). Role of self-efficacy beliefs in the mathematical problem solving of gifted students. *Contemporary Educational Psychology, 21,* 325–344.

Pajares, F. (1996b). Self-efficacy beliefs in academic settings. *Review of Educational Research, 66,* 533–578.

Pajares, F. (1999). Current directions in self-efficacy research. In M. Maehr & P. R. Pintrich (Eds.), *Advances in motivation and achievement* (pp. 1–49). Greenwich, CT: JAI.

Pajares, F. (2006). Self-efficacy during childhood and adolescence. In F. Pajares & T. Urban (Eds.), *Adolescence and education, Vol. 5: Self-efficacy beliefs of adolescents* (pp. 339–367). Greenwich, CT: Information Age.

Pajares, F., & Miller, D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem solving: A path analysis. *Journal of Educational Psychology, 86*, 193–203.

Piaget, J. (1970). *Genetic epistemology* (3rd ed.). New York, NY: Columbia University Press.

Quillen, M. A. (2004). *Relationships among prospective elementary teachers' beliefs about mathematics, mathematics content knowledge, and previous mathematics course experiences* (Unpublished master's thesis). Virginia Polytechnic Institute and State University, Blacksburg, VA.

Roth, W. M. (2005). *Doing qualitative research: Praxis of method*. Dordrecht, The Netherlands: Sense.

Schoenfeld, A. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem-solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan.

Schwarz, N. (1999). Self-reports: How the questions shape the answers. *American Psychologist, 54,* 93–105.

Silver, E. A. (1982, September). *Thinking about problem solving: Toward an understanding of metacognitive aspects of mathematical problem solving.* Paper presented at the Conference on Thinking, Suva, Fiji.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77,* 20–26.

Small, D. (2002). An urgent call to improve traditional college algebra programs. *MAA Online.* Retrieved from http://www.maa.org/t_and_l/urgent_call.html

Underwood-Gregg, D., & Yackel, E. B. (2000). Supporting students' conceptualization of algebraic expressions and operations using composite units. In M. L. Fernandez (Ed.), *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 147–152). Columbus, OH: ERIC Clearinghouse for Science, Math, and Environmental Education.

Usher, E. L., & Pajares, F. (2006). Sources of academic and self-regulatory efficacy beliefs of entering middle school students. *Contemporary Educational Psychology, 31*, 125–141.

Usher, E. L., & Pajares, F. (2008). Self-efficacy for self regulated learning: A validation study. *Educational and Psychological Measurement, 68,* 443–463.

von Glasersfeld, E. (1991). *Radical constructivism in mathematics education.* Dordrecht, The Netherlands: Kluwer Academic.

Wheatley, G. H. (2004, February). *Mathematics learning from a philosophic and psychological perspective.* Paper presented at the annual conference of the Research Council on Mathematics Learning, Oklahoma City, OK.

Wong, N., Marton, F., Wong, K., & Lam, C. (2002). The lived space of mathematics learning. *Journal of Mathematical Behavior, 21,* 25–50.

Yackel, E. B. (1984). *Mathematical beliefs survey.* West Lafayette, IN: Purdue University.

Yackel, E. B., & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 313–330). Dordrecht, The Netherlands: Kluwer Academic.

Zimmerman, B. J. (1995). Self-regulation involves more than metacognition: A social cognitive perspective. *Educational Psychologist, 29,* 217–221.

Zimmerman, B. J. (2002). Becoming a self-regulated learner: An overview. *Theory Into Practice, 41,* 64–70.

Zimmerman, B. J. (2005). Attaining self-regulation: A social cognitive perspective. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of self-regulation* (pp. 13–39). Orlando, FL: Academic Press.

Zimmerman, B. J., & Cleary, T. J. (2006). Adolescents' development of personal agency: The role of self-efficacy beliefs and self-regulatory skill. In F. Pajares & T. Urdan (Eds.), *Adolescence and education, Vol. 5: Self-efficacy beliefs of adolescent*s (pp. 45–69). Greenwich, CT: Information Age.