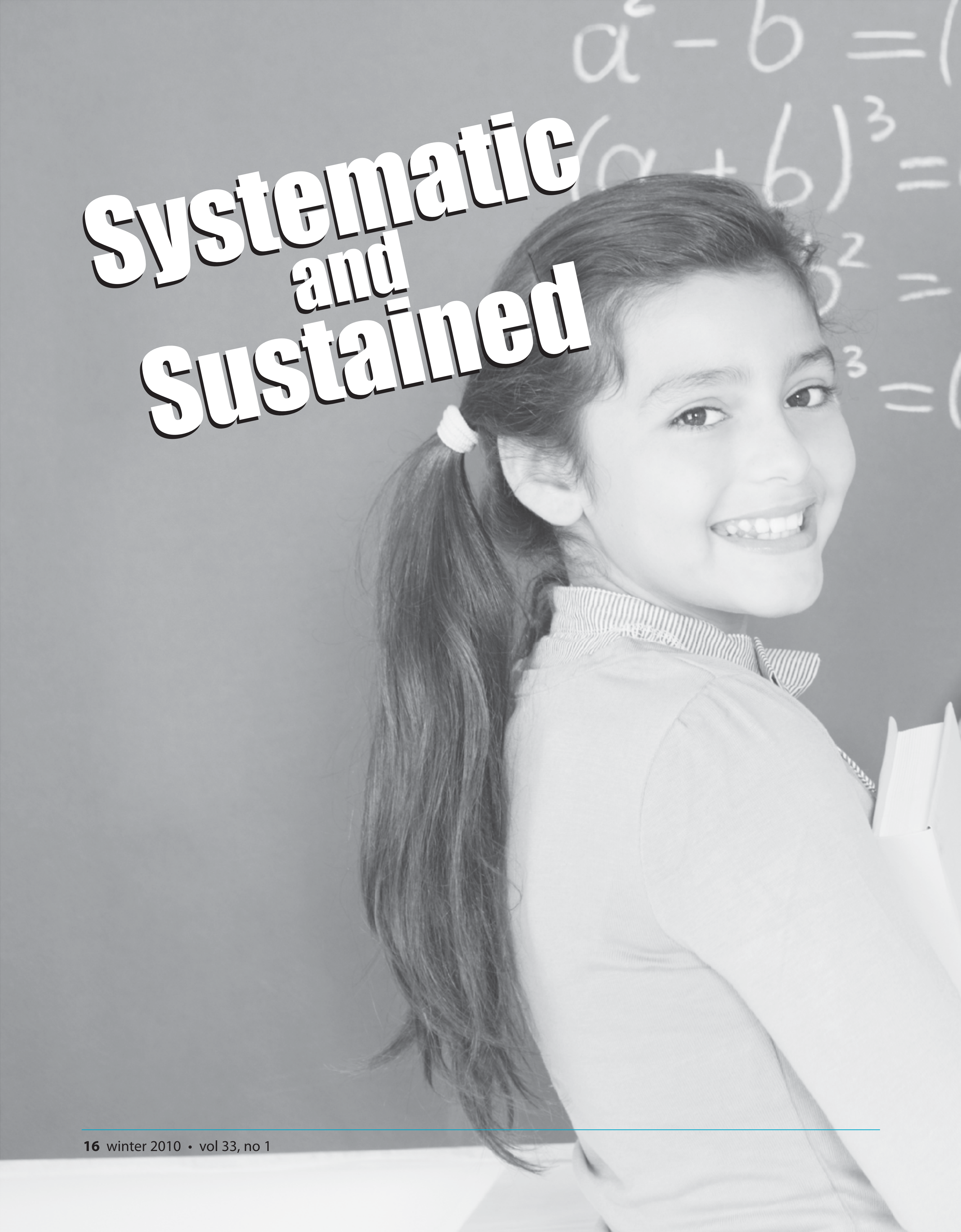
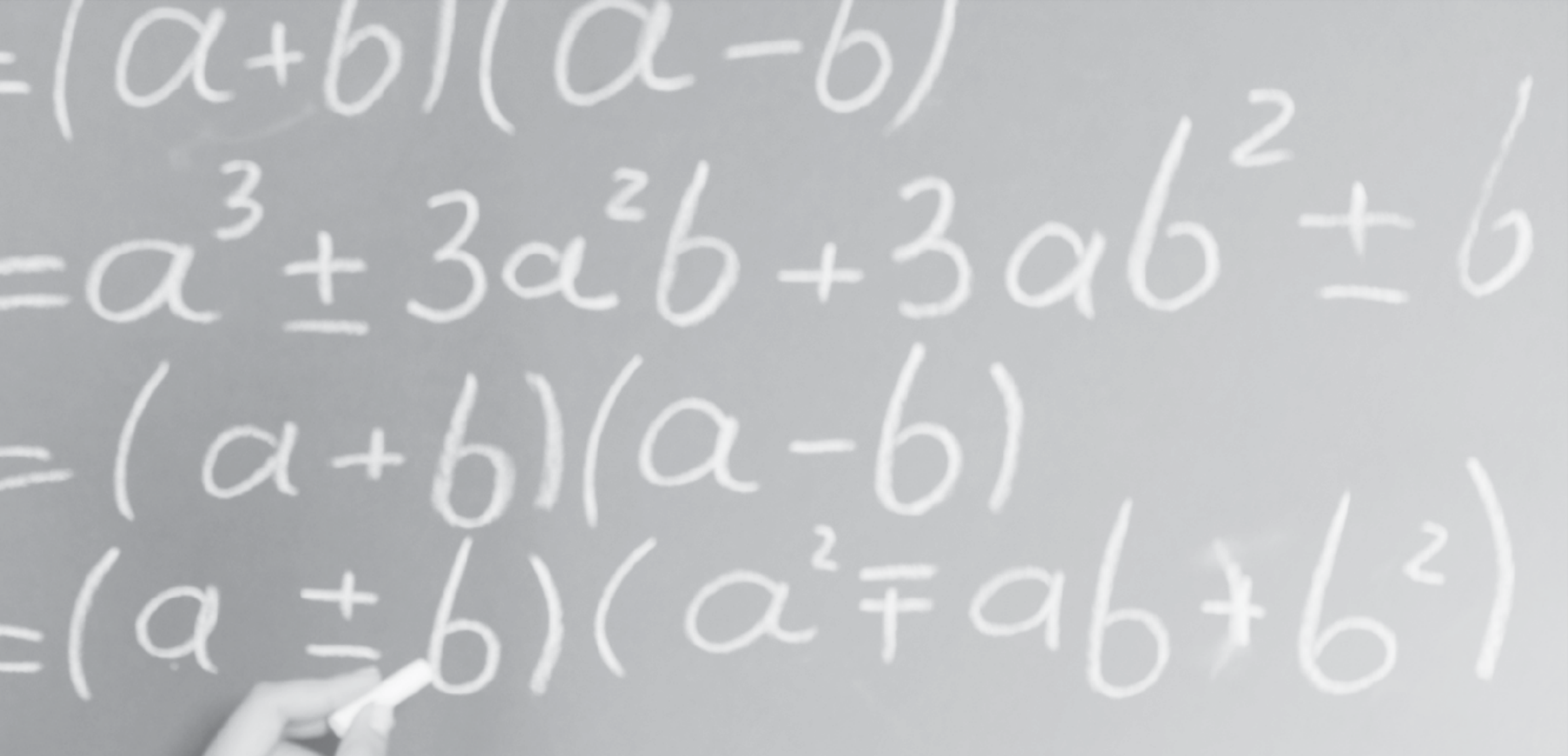
Systematic and Sustained





Powerful Approaches for Enhancing Deep Mathematical Thinking

by Thomas R. Tretter

Teachers of gifted students, as do many teachers, face the daunting task of crafting effective instruction for their students in a context of too much content to cover in too short of a time span (Schmidt, McKnight, & Raizen, 1997). With the advent of the No Child Left Behind Act (NCLB; 2001) and its focus on high-stakes testing, systems of accountability, which include high-stakes testing programs, have been instituted by almost every state in the United States (Moon, Brighton, & Callahan, 2003). This educational focus on accountability and high-stakes testing appears to have influenced a number of instructional practices, resulting in unintended consequences (Jones, Jones, & Hargrove, 2003; Moon et al., 2003). Among other changes, teachers report mandated use of curriculum pacing guides and the imposition of rigid timelines in the use of those curriculum guides, resulting in a narrowing of the curriculum (Scot, Callahan, & Urquhart, 2009). This shift in educational focus ensures that all students reach a minimum competency, but moves away from challenging and stretching our strongest students to cognitively grow to the best of their ability (Scot et al., 2009). In classroom situations of mixed-ability students, the gifted students in such settings are likely to not have their potential tapped to the fullest possible extent, simply because of the logistical challenges a teacher in those situations must manage.

Developing appropriate curriculum to challenge gifted students can be difficult, whether in a mixed-ability classroom or in a

1. ABC + ACB = CBA where A, B, and C are each different, unique digits. What digit does each letter represent?
2. Seven sisters each have a brother. Counting Mr. and Mrs. Hope, how many are there all together?
3. You have some money. You divide it in half and then spend $10, you then divide it in half again and again spend $10. You are left with no money. How much money did you start with?

Figure 1. Set of three typical mathematical brainteasers for late elementary school.

self-contained gifted classroom. Four potential ways that a curriculum can be modified for gifted students have been suggested: acceleration, enrichment, sophistication, and novelty (Gallagher & Gallagher, 1994). Of these four, enhanced sophistication of the curriculum may be both the most effective for gifted students while simultaneously being the most difficult for teachers to achieve (Coleman, 2001). Burns, Purcell, and Hertberg's (2006) list of characteristics of curricula and instruction effective for gifted students also included aspects closely related to enhanced sophistication. Their list included characteristics such as a high ceiling for content expectations, cognitive engagement with a variety of teaching strategies, and authentic and open-ended extension activities linked to content goals. This article proposes several specific strategies that take two typical approaches for gifted students—enrichment and novelty—and transforms those experiences into a highly sophisticated approach that may more strongly enhance students' abilities for deep mathematical thinking.

Two features of the proposed sophisticated curricular experiences are that students engage with systematic and sustained experiences. A focus on systematic curricular experiences will enhance sophistication by intentionally building and strengthening students' cognitive frameworks. This systematization must be undertaken with a cognitively sophisticated endpoint as the target, for example, by focusing students' attention to overarching themes and big concepts that can be used to structure knowledge in a particular field. Because of the cognitive complexity of truly sophisticated thinking within a content area, the development of this sophistication requires multiple experiences over time, and hence the curricular experiences must be sustained over time in order to have maximum impact.

The curriculum approaches described below are likely to benefit both gifted students and those not identified as gifted, but because of the intentional focus on sophistication, gifted students may be the biggest beneficiaries of receiving this instruction. Although the particular strategies outlined below are proposed as a way to enhance deep mathematical thinking, they are likely to be beneficial for students' thinking in other related areas, such as science, as well.

This article employs a strategy that will be one of those recommended for use with students to enhance the characteristics of "systematic and sustained." A few particular examples are selected for presentation in some depth, from which larger pedagogical implications and impacts are then later discussed. As with any particular examples, these are chosen simply because they offer an opportunity to illustrate some of the more fundamental concepts being raised and not because they are inherently more useful or valuable than any number of other possible choices that might serve equally well.

Mathematical Brain Teasers

A common approach for enriching a mathematics curriculum is to periodically pose mathematical puzzles for students to solve; sometimes these are called brainteasers. Without a thoughtful and intentional approach to utilizing the potential of these activities to enhance student thinking, this particular approach may or may not serve as a highly effective cognitive strategy. To illustrate an instructional approach for enhancing the sophistication of this pedagogical technique, a few typical brainteasers are posed in Figure 1. These particular examples will serve to illustrate larger principles of intentional curriculum design that will be discussed. The three examples were chosen because they represent a reasonably broad range of types of mathematical brainteasers within a few examples, and these particular brainteasers were actually assigned to a fifth-grade classroom. In some cases, the same brainteaser could be effective for a wider range of students, as will be highlighted in comments below.

In many cases, a teacher might have students try brainteasers such as these, then report answers and possibly solution approaches, which sometimes would be the end of the experience. Although some students may benefit from this exercise of finding out the right answer and checking their work, there is a potential to craft this process into a more powerful experience for students. After discussing particular solution approaches for the problems in Figure 1, an opportunity to incorporate this particular mathematics enrichment exercise into a long-term sustained approach is discussed. Each particular brainteaser will first be addressed separately.

Pedagogical Uses of ABC + ACB = CBA

Limited Guess and Check. The best approach to a solution for this type of problem depends on the pedagogical goal. If the goal is for students to solve the problem any way they can, an effective solution approach is to use the structure of the problem and some logical reasoning to limit the options, and then guess and check because there are only a limited number of possibilities to work through. An example of this type of thinking might unfold as follows. If A + A = C (the hundreds place), then A must be below 5 because C is a single digit. If A is 1–4 (less than 5, but not zero because then it wouldn't be needed in the first two numbers), then C must be 2–9 (if a 1 is carried, then you can get an odd digit for C). Then students can pick a value for C to try out the possibilities, computing a value of A based on what they chose for C, and then trying out a value for B, until arriving at an answer that works. This particular approach does have students wrestling with logic and eliminating options to the point where they have to only guess and check a relative few options. Although these skills are valuable, there could be other more powerful conceptual goals for students to wrestle with in solving this problem.

If the goal is to enhance students' critical thinking about place value and the underlying rationale involved with addition and carrying values over to the next higher place value, then a more complicated approach will both accomplish that and arrive at the right solution in a deductive fashion. Although the creators of these brainteasers may not have had a particular pedagogical goal in mind when crafting the problem, a teacher can approach this particular assignment from different perspectives. Because a core feature of elementary mathematics includes understanding place value and carrying to next higher place, a focus on these core concepts would be pedagogically appropriate for late elementary students. The downside is that the logic of this analytical second approach (described below) is probably a bit of a stretch for most fifth graders, although with guidance I think most could understand it, especially if they are mathematically gifted. However, I suspect that even after being walked through the analytical approach, some students wouldn't necessarily be able to solve a similar problem the same way, whereas the "guess and check" approach might be replicable by most students.

1. *Hundreds place value*: Because A + A = C, then C is bigger than A.
2. *Units place value*: Because C + B = A, then C + B would need to carry a 1 to the next place value because C is already bigger than A (Step 1 above), and adding B to it makes it bigger. Thus, the sum of C + B would have to be a total of 1_ where the second (currently blank) digit is the value of A, and the 1 would be carried to the next place value.
3. *Tens place value*: We have B + C = B, but we now know that we will be carrying 1 from the units place value, so we really have B + C + 1 = B. The only way this can be is if C + 1 = 10, because then you would get something that would carry the 1 to the hundreds place value and leave the digit "B" as the tens place value. An example or two might help students understand this:

   B + 10 = B could mean (just to pick a random example) 3 + 10 = 3 because of the 13, the 3 stays in tens place and the 1 is carried to hundreds place. With a couple of examples, students will see that this is true for any digit B (7 + 10 = 7; 2 + 10 = 2; 8 + 10 = 8) because the 1 is always carried.

4. Because C + 1 = 10 is now known, that means that **C = 9** must be true.
5. Now you start going back and using the known value of C to compute the others, starting with A. From Step 1 above, A + A + 1 = C (the plus 1 is because in Step 3 we figured out that we would be carrying a 1 to the hundreds place), so A + A + 1 = 9 means that **A = 4**.
6. From Step 2 above, we have C + B = A, which with the numbers we now know is 9 + B = 4. Because this involves carrying 1 (see Step 2), then 9 + B must be 14 and hence **B = 5**.
7. Having figured out **A = 4, B = 5, C = 9**, it is a good idea to check if that really works (it does) as a general problem-solving strategy.

Figure 2. An analytic approach to solving ABC + ACB = CBA.

Analytical Approach. Figure 2 details the logic of an analytical approach to this same problem. All of the mathematical ideas contained in this approach should be familiar to late elementary students, so that even if they couldn't have generated this sequence of logic, it is likely that they can follow it.

After students understand the particular approach to solving this problem, it is pedagogically helpful to assist students in generalizing from the specific case to general problem-solving strategies (a topic further addressed later in this arti-

Table 1
Two-Column Chart to Organize Solution Working Backward

Problem in Words	Dollar Values Computed From the Words
no money	\$0
spend \$10	\$10 (so that you are left with 0 after spending \$10)
cut in half	\$20 (so that when cut in half, you will have \$10)
spend \$10	\$30 (so that you are left with \$20 after spending \$10)
cut in half	\$60 (so that when cut in half, you will have \$30)

cle). This is a part of a systematic use of brainteasers to enhance deep mathematical understandings. Particular suggestions to guide students to organize a systematic approach are presented later in the article. In this case, the general approach involves the following considerations. Use each place value to generate an equation or piece of knowledge, one by one (leads to Steps 1, 2, and 3 in Figure 2). A key feature here is the concept that with addition, sometimes you have to carry the value of 1 to the next higher place value, which is how you can get a resulting digit in a given place value that is lower than either of the two added together (e.g., 9 + 5 results in a digit of 4 because of carrying the value of 1 from the 14). With these multiple pieces of information from each place value equation, you will be able to determine the value of one of the digits, which you then use to go back to the original three pieces of knowledge and substitute known values to compute the other unknown digits.

Nonmathematical Brain Teaser

The second brainteaser in Figure 1, asking for the total number of people, is not a mathematical brainteaser at all, but rather a semantic one. Although the element of asking for a number may lead some to think of it as a mathematical exercise, a closer analysis of the thinking necessary to solve the puzzle reveals that the solution hinges on realizing the meaning of the word "brother" in the context of this family. Thus, while these sorts of puzzles can be fun and engage students, I would not recommend that anything of this nature be chosen to advance instructional goals of deepening mathematical thinking of students. If a teacher approaches the use of brainteasers as a vehicle to engage kids with systematic and sustained mathematical thinking, then he or she would bring an evaluative lens to bear on puzzles of this nature that would lead to an intentional choice to bypass this puzzle.

Working Backward

Organize Information. The basic strategy for solving the third brainteaser presented in Figure 1 is to work backward. In this case, one option is to create a two-column chart with the text from the problem explaining what is happening in one column, and the resulting dollar values in the other (see Table 1). This serves as a useful graphical way to organize the information that is likely to lend clarity to the solution process. A teacher could help students start this solution by having them write the text backward; thus the end point of "no money" is the first entry in Table 1, first column. After students write all of the text first (i.e., the left column in this table is completely filled out before writing anything in the right column), then students begin figuring out the dollar amounts starting at the top with \$0 because that is what "no money" means.

The other dollar amounts are always figured out by thinking backward. For example, the words, "spend \$10," means that you have to add \$10 to your \$0 so that, when you spend \$10, you are left with the \$0 at the end. Likewise, when the problem says "divide in half," in the computation of the dollar amount you have to double the current amount so that, when cut in half, you end up with the dollar amount in the row above.

Algebraic Approach. There is an algebraic way of doing the same thing, although this approach may not be appropriate for late elementary students. However, for older students in middle school or early high school, whenever they are learning algebra, this solving of the problem in two ways can lead to insights about the power and meaning of algebraic symbols. An exercise like this puzzle, when solved in both manners presented here, can help students bridge the divide to algebra.

To illustrate the algebraic approach, let "x" be the original sum of money. Then have students write an equation, step-by-step, by incorporating the words into the equation. To illustrate how the final equation below is built up step-by-step, start with x dollars, then "divide in 2" to get the first part on the left side ($\frac{x}{2}$). Next the words say to "spend \$10," which mathematically is represented by subtracting 10. Put parentheses around all of that to now have ($\frac{x}{2}$ – 10). Then divide all of that by 2, and finally subtract 10 ("spend \$10) from that whole amount. The result, on the other side of the equals sign, is 0 because this series of steps results in "no money." The final algebraic equation the students generated would be $(\frac{x}{2} - 10) \div 2 - 10 = 0$.

Starting with the equation above, using basic algebra will solve for *x*. I would highlight for students who did the chart system above how this is exactly the same thing—they are "working backward" to "undo the words," except that this time the words have first been translated into the language of algebra before the undoing begins. In this case, "undoing," means to solve the equation. First add 10 to 0 to move that to the other side of the equals sign. Then multiply by 2 to move the lowest "2" to the other side. The algebraic solution proceeds in this manner, each step undoing something to move toward isolating the variable *x*. With guidance, most students who have studied some beginning algebraic manipulations can see how this algebraic solution is the exact same thing as the chart solution above, where the "working backward" strategy is enacted by the "undoing" of the algebraic equation until the unknown *x* is isolated.

Most elementary and middle school students will find the working backward chart the most intuitive, and this is a powerful thinking strategy for solving problems. For students who are studying algebra, incorporating the second, algebraic way to solve the problem can strengthen their ability to create appropriate variables and use the text of a problem to algebraically express the situation. This is followed by a reinforcement of a fundamental algebraic strategy for solving equations—they step-by-step undo the operations until the chosen variable is isolated. Depending on the students, it may be pedagogically useful to have them interact with both solution approaches (chart and algebraic) with a goal of reinforcing the meaningfulness of both the algebraic equation generation and the general algebraic solution approach of isolating a variable to solve the equation.

Systematic and Sustained, Part 1—Synthesize Across Cases

Approaching the particular selected mathematical brainteasers as described above may have some additive educative value over a simple teacher-led confirmation of right answers, but mathematical stimuli of this nature can be employed in a more systematic and sustained way to strengthen and deepen students' mathematical thinking. Clearly, the particular solutions to these particular problems aren't of inherent interest—they are meaningless in and of themselves—but the underlying idea is that students should take away some deeper understanding of mathematics from engaging with these (and future) problems of this sort. Keeping a focus on pedagogical techniques to enrich the sophistication of curriculum for gifted students (and for all students), a teacher may choose to utilize problems similar to those in Figure 1 as a springboard to initiate longer term, more generalizable student learning.

I propose that an effective impact from these kinds of assignments is for students to begin to develop a typology of both problem types and problem-solving strategies. In particular, one pedagogical strategy could be to require students to keep a problem-solving journal (with numbered pages) where they were to record the following information for each of these brainteaser sorts of problems (see Table 2):

(a) Give the strategy employed a name.
(b) Describe the strategy.
(c) Explicitly outline the characteristics of problems that tend to be successfully addressed by this particular strategy.
(d) Summarize the particular mathematical knowledge needed to solve this problem.

With this approach, the goal is to have students begin to generalize from the specific problems and evaluate the underlying structure of a problem, rather than focus solely on the particulars of the specific problems they did that day or week. Younger students' terminology may be somewhat different from the examples shown in Table 2 (e.g., "finite possibilities"); it is the concepts that matter, not the terms used.

After a while, students will have a journal with a fairly large number of entries. As particular sets of brainteasers would be due for class discussion, the teacher can review the characteristics outlined in Table 2 for each problem, so that in case students got stuck or had incorrect ideas, they had the opportunity to correct them in their journal. Because of the typically time-intensive manner of following up on the solutions to the brainteasers, some practical implications are the necessity to keep the number of problems to a manageable minimum, and to very intentionally and carefully select those problems that will be most pedagogically useful.

Several times a year (e.g., once every quarter or semester), the teacher could have students review their journal entries and synthesize across them to create a master list of problem-solving strategies on one page. Simultaneously, students would summarize the description and the characteristics of each strategy. However, because the particular content knowledge is problem-specific, there usually is not any value in including that category in the synthesis activ-

Table 2
Journal Entries for Sample Problems in Figure 1

Categories	
	Problem: ABC + ACB = CBA
Name	Guess and check (Note: This would likely be the strategy employed by most younger students, rather than the analytical approach, but this is a valid and useful strategy.)
Description	This strategy involves choosing particular values for unknowns, substituting them in the problem, and checking if it is correct.
Characteristics	Must be only a relatively small number of finite possibilities.
Knowledge	• Understand place value • Understand significance of carrying a 1 to the next higher place value • Deductive reasoning skills (for reducing the number of possibilities to try)
	Alternative Algebraic Entry for ABC + ACB = CBA
Name	Generate system of algebraic equations (Note: This would be appropriate for students who are studying algebra. As noted in text, it may be pedagogically useful to have students interact with both this algebraic approach and the guess and check approach.)
Description	This strategy involves writing a series of algebraic relationships, obtaining one relationship among variables for each of the three place value situations. Once one of these equations leads to a numerical value for one of the variables, then use this known value to substitute back into the other equations until all variable values are known.
Characteristics	Must be able to get as many independent equations (or relationships among variables) as there are unknown variables.
Knowledge	• Understand place value • Understand how to translate known relationships into algebraic representations • Understand algebraic significance of carrying a 1 to the next higher place value
	Problem: "How much money did you start with?"
Name	Work backward
Description	Starting with a known final situation, work backward through the information given to determine the starting situation.
Characteristics	The problem must be structured so that it gives a final situation, gives details of the process to get there, and is asking something about the initial situation.
Knowledge	• Translate words such as "spend $10" into mathematical operations • Organize sequence of steps (e.g., in a chart) to be clearly tracked • Understand how to undo a mathematical operation by doing its opposite • Translate words into algebraic expressions (if using algebra method) • Solve algebraic equations (if using algebra method)

Strategy

Description

Characteristics

Page numbers of examples

Figure 3. Example of a synthesis chart for problem-solving strategies.

ity. Finally, it is useful to have students reference which pages on their journal to find each particular strategy. Many students find a synthesis chart (see Figure 3) useful as an organizing tool.

After completing and reviewing their synthesis charts, students begin to appreciate that certain strategies are more flexible and powerful than others (e.g., by looking at how often they were used in the page number column). They also begin to recognize and synthesize essential characteristics of problems that are critical for crafting potential solution approaches. Essentially, students develop the ability to answer the all-important question, "What aspects of this particular problem are crucial for figuring out how to go about solving it?" Although all of this takes valuable classroom time (reviewing in detail each problem as it was assigned, and then taking time to synthesize and create the master list), this learning is of most importance, and without it, the time spent on brainteasers may or may not lead to enhanced mathematical thinking.

By selectively choosing problems and including a strong emphasis on the mathematical knowledge underpinning each particular problem, teachers and others can appreciate that the time spent on these activities isn't just supplementary to the mathematical content to be covered, but that it directly strengthens the mathematical content knowledge that is the focus of the curriculum at that time. From this perspective, inclusion of a focus on problem solving (or brainteasers) isn't an extra to an already crowded curriculum, but is in fact a central part of that curriculum. Applying mathematical content knowledge in unique ways is a core component of a strong mathematics program for students of all ages.

Different Approach to Systematic and Sustained

A second example highlighting ideas how to engage students' mathematical thinking in systematic and sustained ways is offered to illustrate that such a pedagogical focus isn't necessarily limited to traditional mathematics problem-solving scenarios. As was done with the mathematical brainteasers, an example will be developed in some detail to engage the reader with details of how a sophisticated approach may be employed to challenge, motivate, and deepen the thinking of gifted students. As before, this detailed example is not intended to serve as a comprehensive list of suggestions, but rather is intended to serve as a solid foundation from which readers can generalize to other implementation examples in which they are interested.

This second example is based on a mathematical analysis of the two-player game of Sprouts. Figure 4 is a description and illustration of this game that is reprinted from a previous issue of *Gifted Child Today* (Tretter, 2003).

For two players. Rules:

1. Begin with a given number of dots, keeping the number relatively small to limit the duration of the game.
2. On your turn, connect dots with an arc. An arc can connect any two dots subject to the restrictions below.
3. An arc can connect a dot to itself.
4. A new dot is placed at the midpoint of each new arc.
5. No dot can have more than three arcs coming from it.
6. No arc can cross another arc.
7. The winner is the player who makes the last valid move.

Sample game with three starting dots (new moves each turn are shown dashed).

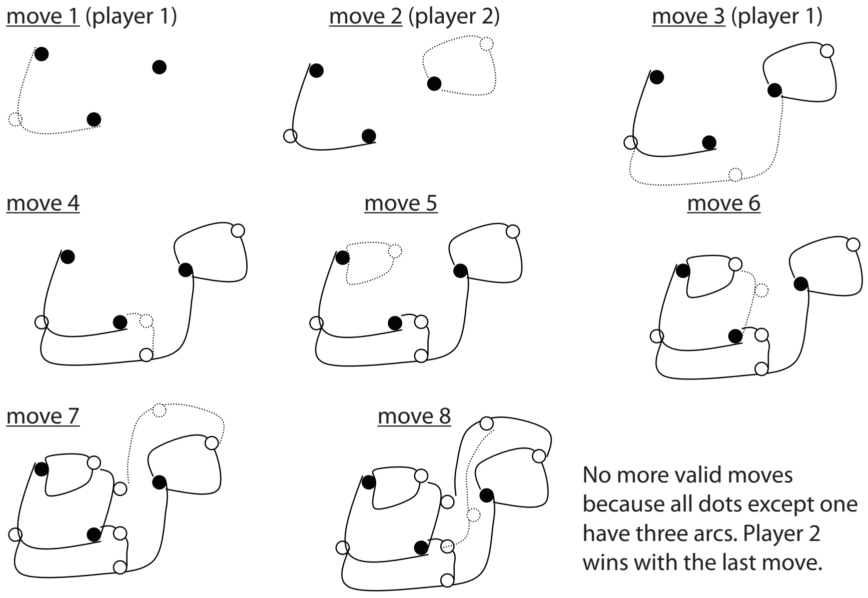


Figure 4. The game of Sprouts.

Note. Adapted from "Gifted Students Speak: Mathematics Problem-Solving Insights," by T. R. Tretter, 2003, *Gifted Child Today*, *26*(3), p. 28. Copyright 2003 by Prufrock Press Inc. Adapted with permission.

Thorough Analysis of Original Game

Play the Game. When using the game of Sprouts to focus and enhance mathematical thinking, the second step is to have students thoroughly analyze the game strategy. The first step, of course, is to have students play enough games to comfortably understand the rules, and likely by this point they have already started developing strategies to win (this seems almost instinctual). For all of the mathematical strategies and suggestions I include below, I recommend that the teacher allow students to discover as much of this as possible on their own. The mathematical summaries provided below are intended to serve as guides for leading student thinking, rather than as information to be delivered directly to students. This text will present one technique for thinking about the mathematical structure of the game, but it is likely that students will come up with their own unique notation, organization, or other strategies that on the surface may look different, but if on target will likely be mathematically similar to the ideas presented below. Encouraging individual sense making is helpful for students to experience the greatest cognitive growth from these experiences.

One pedagogical strategy that tends to work well is to alternate pair playing time with whole-class strategy discussion time; this format permits each student to develop his or her own intuition and ideas, and then benefit from sharing with the rest of the class so that, collectively, the students are exposed to a range of mathematical thinking about this situation. Depending on the ages or abilities of the students, a teacher may need to provide more or less guidance from time to time. In general, I find that two major cycles of pair playing time and whole-group discussion time tend to be necessary.

The first cycle is relatively short, with each pair playing roughly 8–10 games with 3 dots as the starting situation each time. At this point, it is often useful for the whole class to regroup to explore its thinking about what students are finding. Depending on past experiences of students, many may find it difficult to get mathematical traction to start with the analysis process, which is why this step is useful after a short time. Helping students begin to think mathematically about the critical features of the game often can facilitate the process of students bringing mathematical language and logic to subsequent analyses. The second cycle is designed for longer term, sustained engagement where pairs of students develop their mathematical perspectives. A teacher may find it most useful to work with pairs or small groups from time to time rather than the whole group at this point, but as always the pedagogical details depend to a large degree on the dynamics of each particular classroom.

Helping students begin to think mathematically about the critical features of the game often can facilitate the process of students bringing mathematical language and logic to subsequent analyses.

Begin Mathematical Analysis of Game. After numerous games, I have students begin their analysis by making a table with columns for number of starting dots, number of moves to win, and who wins (Player 1 or 2). They should then search through numerous examples to discern patterns. What they will eventually discover is that each starting dot provides a total of three available "arc connections" due to Rule 5 (no more than three arcs per dot). Also, each move uses up two connections (one at the starting dot and one at the ending dot, which can be the same dot). Each move also generates one additional arc connection from the new dot placed in the middle because that new dot already has two arcs connected at its placement, leaving one additional arc connection available.

Hence, each move uses up two arc connections and creates one more, for a net loss of one arc connection per move. Thus, with three starting dots, 3 x 3 = 9 arc connections are available, and with each move resulting in a net loss of one connection, students first think there must be nine moves to win. However, after the last dot is placed only one arc connection is available, and this is unusable because you need two connections to make a move (starting and ending points for the arc), hence there is only a total of eight moves in a 3-dot game to win (as shown in the sample game in Figure 4).

If your students are comfortable with algebraic notation, they can let n signify the number of starting dots. Then the total number of moves to end the game will be $3n - 1$ (the minus 1 because the last connection is not usable). Given the number of starting dots, students can figure out the maximum number of possible moves and thus the likely winner of the game.

Identify Key Strategy From Analysis. Students will quickly point out that this likely winner from the analysis above isn't always the winner, however. Because of Rule 6, no arc can cross another; students can trap a dot inside a closed figure so that it is isolated from dots outside the closure. This effectively renders another arc connection useless because you can't cross an arc to connect to it, so this strategy will allow a different player to win than that given by $3n - 1$. Of course, if the other player can isolate a second dot, the advantage switches again—hence the central strategy of the game. With a little practice, students quickly figure out that given n starting dots and knowing their player position (1 or 2), they can readily identify if their strategy is to isolate an even number of dots or an odd number, with their opponent trying to do the opposite.

There are also a number of patterns that emerge in the number of moves to win as a function of *n*, such as triangular numbers, combinations, and portions of Pascal's triangle. Depending on your students, you may or may not have them investigate for these underlying patterns. See Tretter (2003) for multiple examples of mathematical situations that also ultimately came back to some of these same foundational mathematical ideas, providing opportunities to deepen and connect student mathematical thinking across a robust mathematical landscape.

Systematic and Sustained, Part 2—Alter the Rules

Having mastered the basic game strategy through such mathematical analysis (and many played games, of course), students then alter the rules and reanalyze their new game. I recommend complete freedom for them to make their alterations. Table 3 lists some common rule alterations that students have invented and the end result of their analysis.

Students will probably come up with more variations than those listed in Table 3. Have students keep a log of the rule variations, strategy implications, and a final decision about whether or not that results in an interesting game and why. This sort of longitudinal approach to interacting with this one specific context can be integrated into the curriculum over time, enabling students to devote more of their cognitive abilities to thinking about the underlying mathematical structure because the particular game context will be very familiar after the initial sessions are completed.

Eventually, this leads to a discussion of what makes for a fun game. Generally, a good game should involve a combination of luck and skill. If it is all skill, then the more skilled player will always win, which isn't interesting to play. If it is all luck, then there is no thinking required, which makes for a boring game. For example, the slot machines in Vegas would be very boring as a game—anyone or anything that can put in money and pull a lever has an equal chance of winning. There is no mental engagement in the process as a game; I suppose the possibility of a monetary payoff must be the motivator, rather than the thrill of the game. Blackjack, on the other hand, involves both luck and some skill, even if the probability odds are against you. If luck isn't really a factor (such as in Sprouts), then the skill part of the game should be complicated enough that even more skilled players can be occasionally outwitted by those less skilled. Sprouts tends to fall in that category because it is sometimes difficult to visualize ahead of time which dots will become isolated with which moves, and so players can find themselves on the wrong side of the dot-isolation strategy even if they are very cognizant of what they are trying to do.

Systematic and Sustained Beyond Mathematics

The particular examples highlighted above were set in the context of deepening students' mathematical thinking. However, a systematic and sustained approach is likely to be equally effective in any number of content contexts. For example, explorations of cultural contexts may mesh with social studies curricula. Students could be guided to systematically explore historical and cultural trends across time that contextualize and help explain contemporary circumstances. A systematic and sustained approach could be applied in a science curriculum, guiding students to explore why some scientific theories are so powerful for explaining a disparate set of natural phenomena, and how development of these theories has in turn led to new scientific insights. Enhancing the sophistication of instruction for gifted students through a focus on systematic and sustained experiences is a pedagogical approach that is portable across many different content contexts.

Students who experience curricula that have been designed with a focus on sophisticated learning are likely to deepen their understanding of a particular content area, and also are likely to be cognitively situated for effective future learning. Systematic and sustained experiences can be designed to scaffold over time the cognitive complexity expected of students, providing students opportunities to grow from a multitude of cognitive starting points. The possibilities for structural connections students could make will particularly enhance the opportunities for gifted students. As gifted students gain experiences with enhanced sophistication in curriculum, there may be a positive feedback loop reinforcing and strengthening future learning cycles. Eventually, students may reach a point of sustaining sophisticated learning independently from the guidance of a teacher—a wonderful way to teach oneself out of being needed. **GCT**

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Table 3
Rule Change Possibilities for Sprouts and Their Analysis Results

Rule Change	Result of Game Analysis
Allow four arcs per dot	This results in each move using up two connections (start and end) but creating two connections (the middle dot that has two arcs automatically and then allows for two more with the new rule). Thus, this game never ends because the number of connection points never changes. It's not an interesting game, but it is interesting to let the students figure this out and figure out why.
Allow five (or greater) arcs per dot	This results in an increasing number of connection points, and so again the game never ends. It's not interesting as a game.
Put two dots in the middle of each new arc instead of one	Results in creating two connections and using two connections for each move, again causing the game to never end. It's not interesting as a game. (Three or more dots in the middle increases the number of connections available per move, and so is also not interesting as a game that never ends.)
Require that the starting dots be collinear	No difference because the arrangement of the dots is irrelevant. This sometimes results in dramatic contortions of the arcs drawn to fit around and between the dots. Students sometimes try to draw an arc so close to another that a new arc can't "slip between them." Use this to initiate a discussion of the zero-width of a mathematically idealized line (or arc), implying that there is always room for a zero-width arc to get through even if the real-world pencil used to represent it draws an arc that does have width. If students want to make it a rule that the real-world representation has to fit between, by all means they can do so. That often leads to arguments about what fits and what doesn't, so I turn it back to them to refine their rule to be well-defined because it was their rule and not mine. If they can't settle on a clear definition, then frequently they have to toss out the rule.
Allow arcs to cross	This effectively means it is no longer possible to isolate a dot, and so the winner is completely determined by the number of starting dots and the player position because there will always be $3n - 1$ moves to win. This is not interesting as a game because the outcome is predetermined.
Not allow a dot to connect to itself	I haven't actually had students do this one, so don't know for sure the outcome. It might be fun to have students try it. My first intuition is that it would be harder to isolate a dot because one can't create a circle back to itself (although not impossible, because you could have a pair of dots each connect to the other, forming a circle that way), which could impact the dynamics of the game.
Give starting dots different characteristics, such as some allow three arcs to connect whereas others allow only two	This is infinitely variable because you can always add more starting dots with different criteria. Students may wish to see if there is some systematic pattern that can be analyzed, such as equal numbers of two-arc and three-arc starting dots, twice as many of one as the other, and the like.
Make this a three-player game	This is tougher because your goal of isolating or not isolating a dot will shift depending on how many turns you've had. For example, if you're Player 3, you have move #3, #6, #9, and so on, and an even or odd move wants different numbers of dots isolated to win. Then students have to decide if there is one winner (last move) or they can make the last move the loser so that there is one loser per game. They could also play four-person games with two teams of two who are trying to work together but are not allowed to communicate, so that they have to figure out their partner's strategy and work with that (attempting to isolate certain dots or keep others from being isolated). This works best with larger number of starting dots.

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