Features of GENERALISING TASKS

Help or hurdle to expressing generality?

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Introduction

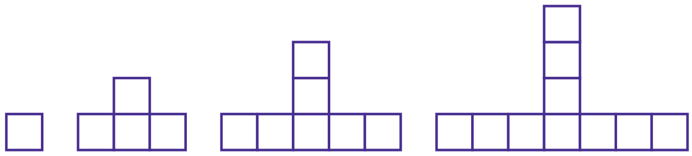
Suppose you intend to use the two problems shown in Figures 1 and 2 in a lesson on pattern generalisation and need to decide on their sequencing. On what criteria have you based your decisions? What criteria have you used to base your decision on? For teachers attempting to do this, there are some ways to help them approach it and make informed choices. One way to gauge the complexity of generalising problems is to look at their features, and this is basically what this article aims to explore and discuss further. This article sets out with two objectives: (1) to offer teachers a framework for considering the difficulty level of generalising problems in terms of task features, and (2) to raise issues for discussion on the possible influence the task features have on students' generalisation and reasoning.

Pattern generalising problems

Pattern generalising problems illustrated above are a common feature in school mathematics in many countries. By and large, such problems can be classified into two categories: numerical and figural. Numerical generalising problems list the pattern as a sequence of numbers whereas figural generalising problems set the pattern in a pictorial context. Figures 1 and 2 are two examples of figural generalising problems, and Figure 3 provides an example of the numerical type.

Numerical generalising problems like Figure 3 can sometimes be problematic due to a lack of further specific assumptions and explicit contexts that describe how the terms should continue. Without these, the given terms in the sequence are no longer a sufficient condition for students to predict the pattern, let alone formulate a rule to represent it! Consequently, the sequence is open to interpretation. For instance, the sequence in Figure 3 could develop in many ways such as 1, 4, 7, 10, 13, 1, 4, 7, 10, 13, … or 1, 4, 7, 10, 13, 13, 10, 7, 4, 1, 1, 4, 7, 10, 13, … However, we realise that, in most cases, many students are able to make a tacit assumption of following a recursive pattern of adding 3 each time to get the next term, possibly due to a didactical contract (Brousseau, 1997) to which they are conditioned. On the other hand, providing a description of how the sequence will continue not only takes away the challenge of spotting the pattern, but also makes the problem a clear giveaway. Such is a shortcoming of numerical generalising problems, of which teachers ought to be aware. Figural generalising problems are not necessarily free of this shortcoming, however. Rivera and Becker (2007) claimed that such problems can also be interpreted in other valid ways if the assumption and context are not clearly stated. But we think that these problems are less ambiguous than numerical tasks because, at least, the figures provide some kind of context about how the pattern grows, thus explaining the quantitative facet of the figural pattern.

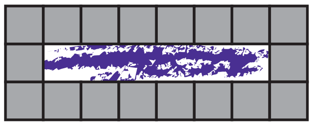
Ken builds a sequence of shapes with square tiles. He starts with one tile, and subsequently adds one tile each to the left, right and top of the preceding shape.



Can you help Ken find the number of square tiles needed to build shape N?

Figure 1. Inverted-T task.

Ken surrounds a rectangular flowerbed of dimensions 6 units by 1 unit with a layer of grey square tiles.



Can you help Ken find a rule for determining the number of grey square tiles needed to surround a row of flowerbed of any length?

Figure 2. The flowerbed task.

The first five terms of a sequence are given as follows:

1, 4, 7, 10, 13, …

Can you write down a rule for finding the nth term in this sequence if you were told what n is? Show how you obtained your answer.

Figure 3. A numerical generalising problem.

The idea behind these generalising problems is that students are expected to identify, recognise, extend and articulate the pattern. These skills play a pivotal role in a successful transition from arithmetic to algebra. Of crucial importance in this transition are two core aspects of algebraic thinking: the emphasis on relationships among quantities like the inputs and outputs (Radford, 2008), and the idea of expressing an explicit rule using letters to represent numerical values of the outputs (Kaput, 2008). With the potential of helping students develop algebraic thinking, it is then not surprising why growth patterns are often taken as a route to algebra from arithmetic. However, there seems to be a didactic cut in the transition from arithmetic to algebraic thinking (Filloy & Rojano, 1989). Research has consistently shown that the transition from recognising a pattern to expressing generality is by no means an easy feat. Most students have no problem recognising a pattern, yet the articulation and representation of the explicit rule in words or in algebraic notation remains challenging (English & Warren, 1995).

Some recent studies have shed light on the impact certain task features have on students' performance in growth patterns, with evidence pointing mainly to one particular task feature: the way in which the pattern is displayed. In one of these studies, Lannin, Barker and Townsend (2006) discovered that students' selection of strategy was influenced by the visual cue they saw in the change between consecutive diagrams. A similar finding was also reported in a study by Becker and Rivera (2006), who pointed out that two-dimensional diagrams presented in a sequence help students to visualise the basic core of the pattern that remains invariant and the part that is constantly growing, thereby allowing them to establish the explicit rule for the *n*th term. Contrary to these findings, some students in Warren's (2000) study were unable to spatially visualise a sequence of two-dimensional diagrams in a classic matchstick problem involving a row of squares.

In another study by Hoyles and Küchemann (2001) where high-attaining students were asked to find the number of grey tiles needed to surround a row of 60 white tiles when given a single diagram and a description of how it was constructed, 42% of the students correctly answered the question. What is surprising is that this question does not seem to be difficult, yet, taking into account the student abilities, the success rate was considered rather low. It, therefore, appears reasonable to think that the task is not easy for them. Although the students were allowed to generate their own sequence of diagrams to help them perceive the underlying structure inherent in the task, many still appeared to experience difficulties in handling this task. So, are students' difficulties attributable to the way the pattern is shown to them?

Apart from knowing that the way in which students deal with generalising problems is due in part to the depiction of pattern in the task, little else is known about what and how other task features might affect students' abilities to express generality. The next section presents the key features of generalising problems as well as justifications as to why teachers need to consider these features during task selection.

Features of generalising problems

The task features presented in this section are inferred from a broad range of generalising problems found in the literature. A striking feature that distinguishes the three aforementioned problems is what we call the *format of pattern display*. In Figure 3, the first few numbers of the pattern are listed sequentially using numerical symbols whereas the same set of numbers is disguised as diagrammatic figures in Figure 1, also in a sequential manner. The format is again different between Figures 1 and 2 in that the latter presents a single diagram of the flowerbed in contrast to a sequence of diagrams in the former. This task feature is thus concerned with whether the pattern is listed as a sequence of numbers, equations or diagrammatic figures, or simply as a single diagram. Teachers may wish to note that some tasks may depict the pattern both in diagrams and in numerical symbols.

Another feature that differentiates the three tasks is what we term as the *reference to the generator*. In some generalising problems, particularly the figural type, the independent variable can be connected to a certain component of the diagrams. Here, the independent variable will function as a generator — a term borrowed from Bednarz and Janvier (1996). It enables students to perceive the structure of the pattern and thereby to derive the rules. The generator is normally the ordinal number indicating the position of the diagram in the pattern. In Figure 1, the number of square tiles in the vertical arm of each inverted-T corresponds with the shape number: there are three tiles in the vertical arm of Shape 3, and four tiles in that of Shape 4. When this connection between the generator and the diagram is explicit, then it is said that the reference to the generator is embedded within the diagrams.

Creating an awareness of this relationship is of critical importance to generalisation. Not only can it help students to understand how the pattern grows with the generator, it can also draw their attention to the difference among the components within each diagram. For instance, when the generator is associated with the number of tiles in the vertical arm, the number of tiles in each of the two horizontal arms can then be compared with it. This is where teachers can lead students to notice that each horizontal arm always has one tile fewer than the vertical arm regardless of the shape number. Based on this interpretation, students can then be guided to write down the rule as $n + 2(n - 1)$.

In the case of Figure 3, the link between the generator and the terms is rather inconspicuous. For illustration, take the third term of the sequence, that is, the number 7: the "threeness" of the generator is not manifested very clearly in the number 7. As a result, the generator-term relationship appears to be concealed, making it harder for students to recognise the inherent pattern. When this relationship is not explicit, then the reference to the generator is said to be external to the diagrams. To summarise, this task feature looks at whether the link between the independent variable and the pattern is clearly evident.

Some teachers might think that the generator in a numerical generalising problem can never be linked to the terms, and we hope to dispel this thinking with an example in Figure 4. This example illustrates that a link could be forged between the line number and the number of consecutive odd integers in each equation, as well as the square number on the right side of the equation. For instance, the number of consecutive odd integers in 1 + 3 + 5 + 7 is 4, which is the line number, and this line number is also the base number of the square number 16. Clearly, this is a numerical problem with the reference to the generator embedded within the equation.

Study the mathematical statements below carefully.

Line 1:	1	=	1
Line 2:	1 + 3	=	4
Line 3:	1 + 3 + 5	=	9
Line 4:	1 + 3 + 5 + 7	=	16

Figure 4. Numerical generalising problem involving identities.

We shall continue to introduce three more task features through two new generalising problems. Both the tasks in Figures 1 and 5 are known as problem isomorphs. They are technically identical in the sense that they share the same linear rule conforming to the type involving one variable. But as the problems show, what distinguishes the tasks is the choice of *visual representation* of the given diagrams. In Figure 1, the diagrams are shown in two dimensions whereas those in Figure 5 are three-dimensional. With evidence in the literature pointing to students' difficulties in conceiving three-dimensional arrays (Battista & Clements, 1996), teachers ought to realise that the diagrams of cube arrays in Figure 5 might pose a challenge to students, depending on their spatial visualisation abilities. Students may end up with a wrong rule if their failure to understand the arrays impedes the counting of the number of cubes used to build the towers.

Figures 5 and 6 differ in one other task feature, apart from the visual representation of diagrams. As mentioned previously, the task in Figure 5 involves a linear function. However, the function in Figure 6 is quadratic. In other words, the distinguishing task feature is the type of functions involved. Teachers might wish to take note that the latter task might be more complex for students to deal with because finding the quadratic rule by the common recursive method is not as straightforward as one expects it to be. When considering the function type, teachers ought to look at the simplified form of the explicit rule. We like to stress this point because students might give a feasible solution that resembles a function of higher degree. For instance, the explicit rule[1] for the task in Figure 5 can be expressed as:

number of cubes
in the Nth tower
$= N^2 - (N + 2)(N - 1)^2$,

which can be easily mistaken as a cubic function.

The last task feature to be discussed herein is the *number of independent variables* present in the equation of the function. In Figure 2, the rule for the flowerbed task is a linear function in one variable:

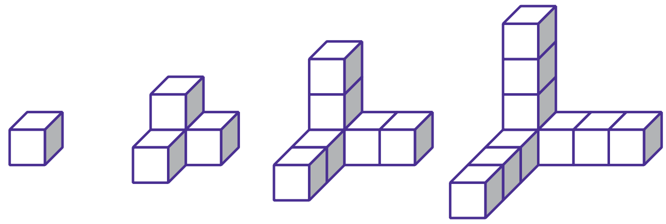
$$\text{number of grey tiles} = 2l + 6,$$

where l is the length of the flowerbed. Imagine if students are then asked to find the number of 1 unit square tiles needed to surround a flowerbed of any length and width, instead of just a row of any length. How does the rule for the modified flowerbed problem compare with the original rule? Symbolically, the new rule can be expressed as:

$$\text{number of square units} = 2l + 2w + 4,$$

where l is the length of the flowerbed and is its width. This rule is a linear function in two variables. Although both rules share the same function type, the difference in the number of variables involved can alter the task

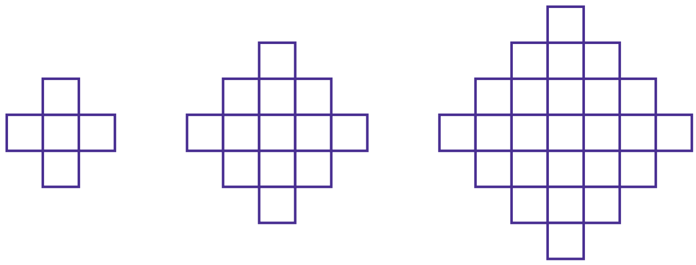
Ken builds a sequence of towers with cubes. He starts with one cube, and subsequently adds one cube each to the front, side and top of the preceding tower.



Can you help Ken find a rule for determining the number of cubes needed to build Tower N? Show how you obtained your answer.

Figure 5. The tower task.

Ken builds a sequence of shapes with tiles.



Can you help Ken find a rule for determining the number of tiles needed to build shape N. Show how you obtained your answer.

Figure 6. The expanding crosses task

1. To obtain $N^2 - (N + 2)(N - 1)^2$, begin with a $(N \times N \times N)$ cube. First take away a $(N \times (N - 1) \times (N - 1))$ cuboid, leaving only two perpendicular surfaces of the original cube. Now taking away a $(N - 1)^2$ cuboid from each of these surfaces will result in Tower N. So the number of cubes in Tower N is given by $(N \times N \times N) - (N \times (N - 1) \times (N - 1)) - 2(N - 1)^2$ which simplifies to $N^2 - (N + 2)(N - 1)^2$.

difficulty and affect students' performance. Thus students may find the modified task elusive, depending on their abilities to represent quantities with algebraic symbols and to operate on them. It is, therefore, important for teachers to be aware of the possible difficulties students might experience when formulating an explicit rule for the modified flowerbed problem. As is emphasised in the preceding paragraph, we need to consider only the least possible number of independent variables needed in the explicit rule given that students may use more than one symbol to represent the same variable.

In summary, this section draws attention to five key features of generalising problems, including justifications for considering them. Table 1 offers an overview of the task features presented herein.

Table 1. Key features of generalising problems.

Features	Pattern Generalising Problems	
	Numerical	Figural
Format of pattern display	• sequence of numbers • sequence of identities	• sequence of diagrams • single diagram
Type of function	• linear • non-linear (e.g., quadratic)	• linear • non-linear (e.g., quadratic)
Number of variables involved	• one • two or more	• one • two or more
Reference to the generator	• external to number • embedded within identities	• external to diagram • embedded within diagram
Visual representation of diagram		• 2D • 3D

Concluding remarks

Pattern generalising problems offer a very rich context for exploring relationships among quantities, expressing generality and representing the same relationship in different ways. Selecting appropriate tasks for students to work on in class is by no means a straightforward process, but there are ways to handle it. To offer support for teachers, the present article addresses this issue by introducing a framework for considering the tasks' complexity through an account of task features that might affect the way students handle pattern generalising problems. We identified and elaborated five task features. Although studies cited above produce evidence of students' difficulty in recognising a pattern and representing it symbolically, it is still unclear to what extent task features attribute to this difficulty. Further research is certainly necessary to look into the role that task features have on the way students engage in generalising problems. Some questions that still need to be addressed include the following: Which task features are instrumental in student success in dealing with generalising problems? How do various task features influence students' use of strategies? What visual cues do successful students see that unsuccessful students don't notice? How do students reason algebraically with various task features? What types of thinking processes do students show when justifying their generalisations for the different task features? The answers to these questions might hold some important implications for teaching and curriculum design.

Finally, getting students to do generalising problems correctly is a matter of utmost importance. What is also equally important is to get them to do the right generalising problems at the right time in class. This is, in essence, the message we hope this article has put across and illuminated. We hope that the framework will enable teachers to know not only which task features to vary to make the generalising problems more appropriate for their students' learning needs, but also which type of tasks is simple and which is more complicated for students to handle with a view to designing more effective lessons. Since generalising problems that present diagrams sequentially help students to recognise the pattern as well as to construct the correct explicit rule, it seems a good strategy for teachers to use these kinds of tasks as introductory activities to allow students to figure out the underlying pattern structure. In contrast, if the pictorial context is found to complicate students' perception of the inherent pattern, then teachers need to provide more guidance to help their students cope with these problems.

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