How to multiply by adding

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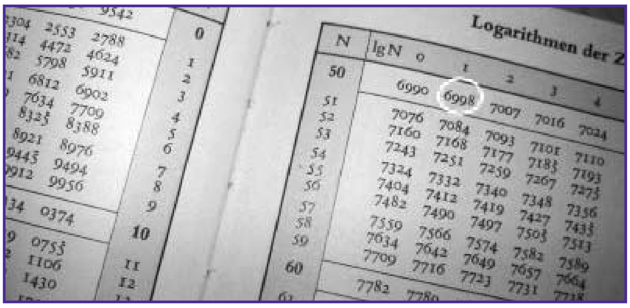


The powers that be

These days, multiplying two numbers together is a breeze. You just enter the two numbers into your calculator, press a button, and there is the answer! It never used to be this easy. Generations of students struggled with tables of logarithms, and thought it was a miracle when the slide rule first appeared. I remember in my university days, carrying a slide rule on one's belt was a badge of honour!

It is easy to make a simple slide rule.

1. Carefully copy two identical scales as show below on two pieces of cardboard *A* and *B*. The markings are spaced at 1 cm intervals, and the numbers are just powers of 2. You can continue the scales to the right as far as you like.

A	1	2	4	8	16	32	64	128	256	512

B	1	2	4	8	16	32	64	128	256	512

Now these pieces of card can be used to multiply numbers together. For example, to multiply 16 by 32, we place the cards like this.

A	1	2	4	8	16	32	64	128	256	512				
				B	1	2	4	8	16	32	64	128	256	512

We now read off $16 \times 32 = 512$.
(Check this.)
Use the cards to find 8×128, 32×64, 64×16.

This simple law is based on the index law:

$$2^m \times 2^n = 2^{m+n}.$$

In our example, $16 = 2^4$ occurs 4 cm from the left-hand end of the scale; $32 = 2^5$ occurs 5 cm from the end. The product,

$$512 = 2^4 \times 2^5 = 2^{4+5} = 2^9$$

is 9 cm from the end. This number is located by simply adding the 4 cm and 5 cm lengths together. We have thus converted our multiplication problem into an easier addition problem.

Of course one of the inadequacies of the slide rule is its accuracy. It only ever gives *approximate* solutions.

The same calculation can be done from a table. We say that the number $2m$ has *logarithm* m, and write $\log_2 (2^m) = m$.

For example, $\log_2 16 = 4$.

The subscript 2 shows that we are considering powers of 2; it is called the *base* of the logarithm.

2. Below is a table of some powers of 2 and their logarithms to base 2. Write out in words the steps you would take to evaluate 16 × 32 using this table.
Extend the table, and use it to find 8 × 128, 32 × 64, 64 × 16.

Number	Logarithm
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9

Any positive number other than 1 can be taken as the base of a system of logarithms. *Common logarithms* have base 10. Mathematically, the most useful base is the number *e*. Although this may not seem to be at all natural to you, logarithms to the base *e* are called natural logarithms, sometimes written as *ln* rather than $\log_e$.

Extensions

3. (a) Construct a slide rule using powers of 3. Use it to evaluate 27 × 81, 9 × 243, 81 × 279.
   (b) Construct a simple table of logarithms to base 3. What is $\log_3$ 243? What is $\log_3$ 2187? Use your table to evaluate the products in (a).

4. (a) Why is 1 not suitable as a logarithmic base?
   (b) Can you suggest why common logarithms have base 10? What number has common logarithm 1? 2? 3?

5. Try to obtain a manufactured slide rule.
   (a) See if you can multiply simple numbers using the *A* and *B* scales.
   (b) On the *A* scale, measure the actual distance between
   (i) the 1-mark and the 10-mark
   (ii) the 10-mark and the 100-mark.
   What do you find? Why is this so?

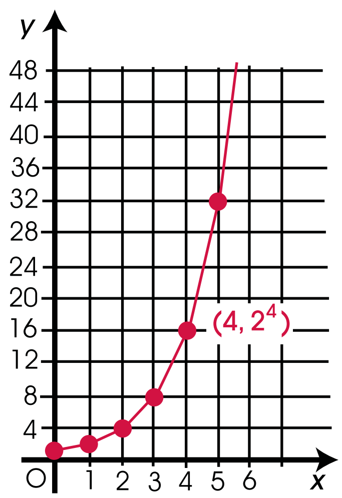
6. Read the chapter 'How to forget the multiplication table' in the old book Mathematician's Delight by W. W. Sawyer. You will find that there is a relation between logs and posts!

Getting the picture

So far we have talked about *powers* (or *exponents*, as they are called), but there has been no sign of our mystery number *e*. Let's remedy that! As an introduction, we take the powers of 2, and plot them on a graph. Here are the powers:

n	1	2	3	4	5	6	7	8
2^n	2	4	8	16	32	64	128	256

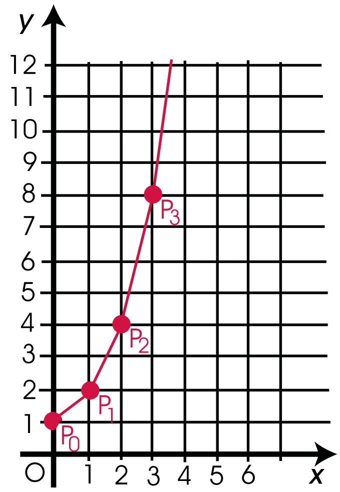
Here is how they appear on a graph. We see that the powers increase very rapidly. In the graph we have marked the points (1, 2), (2, 4), (3, 8), (4, 16), (5, 32), as well as the point (0, 1) corresponding to 20 = 1. If we draw a smooth curve through these points, then it seems reasonable that the 'in between' powers of 2 such as $2^{3/2}$ and $2^{7/8}$ will lie on this curve. The curve has equation $y = 2^x$ ($x \geq 0$).



7. (a) Reproduce this curve on a large piece of squared paper.
   (b) Using a different colour, plot the points (n, 3^n) for n = 0, 1, 2, 3, … and join them by a smooth curve.
   (c) Use a third colour to draw a curve through the points (n, 4^n) for n = 0, 1, 2, 3, …

(d) Can you see that these curves are related by a 'horizontal stretching'? Remembering that $e \approx 2.7$, sketch in the approximate position of $y = e^x$ for $x \geq 0$.

All these curves are closely related. They are called *exponential curves* because they map the powers of numbers. Suppose we now reproduce the above graph, but with a larger y-scale.



8. (a) Use the figure at right to complete the following table.

Point	y-coordinate of point	Segment	Slope of segment
P_0	1	$P_0 P_1$	1
P_1		$P_1 P_2$	
P_2			
P_3			
P_4			

(b) Construct a similar table for the points $(n, 3^n)$. Make an observation about your results in each case.

This admittedly rather rough argument suggests that for exponential curves, "slope is proportional to height," or

(slope at point P) = C (y-coordinate of P), where C is constant for a given curve.

With a little calculus, it can be shown that for points on the curve $y = 2^x$, $C \approx 0.69$; for points on the curve $y = 3^x$, $C \approx 1.1$; and for points on the curve $y = e^x$, $C \approx 1$ exactly. This simple fact is what makes the number e special. We have yet to explore some of the consequences of this property.

Bibliography

Scott, P. R. (1974). *Discovering the mysterious numbers.* Cheshire.

Sawyer, W. W. (1944). *Mathematician's delight.* Penguin.

From Helen Prochazka's

Scrapbook

G. H. Hardy was unorthodox, radical and regarded by his peers as the 'purest of pure mathematicians'. The New Yorker described A Mathematician's Apology , first published in 1940, as 'one of the most eloquent descriptions in our language of the pleasure and power of mathematical invention'.

Hardy uses the word 'apology' in the sense of a formal justification of mathematics, not as a plea for forgiveness. He was feeling the approach of old age and wanted to explain his mathematical philosophy to the next generation of mathematicians. On page 81 he wrote:

The Greeks were the first mathematicians who are still 'real' to us to-day. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. The Greeks first spoke a language which modern mathematicians can understand: as Littlewood said to me once, they are not clever schoolboys or 'scholarship candidates', but 'Fellows of another college'. So Greek mathematics is 'permanent', more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. 'Immortality' may be a silly word, but probably a mathematician has the best chance of whatever it may mean.