More about how to teach FRACTAL GEOMETRY *with music*

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Padula (2005) described how mathematics teachers can channel the passion of students interested in electronic music by teaching the mathematics that has been used to create it. She hypothesised that the study and enjoyment of music may help the study of mathematics since both mathematics and music are symbol systems and pattern (and its recognition) are important to both.

Some excellent music has been composed on and for the computer by composers Gyorgy Ligeti and Charles Wuorinen, (Quaglia, 2000), and Larry Sitsky (Nisbet, 1991); however much of it would not engage the average teenager. Apart from such serious works, there has been some good dance 'techno' music produced, but it did not, and still does not, receive the support of the art-music establishment (Crotty, 2007). It has, however, been enthusiastically received by many teenagers and young adults. 'Techno' computer-based music is enjoyed by many students and is known to most. 'Techno' is based on fractal geometry.

Fractal geometry

Mandelbrot's (1983) formulation of fractal geometry meant that scientists could measure not just triangles, rectangles and circles (Euclidean geometry) but unusual shapes they had previously called: grainy, hydralike, in between, pimply, pocky, ramified, seaweedy, strange, tangled, tortuous, wiggly, wispy, wrinkled and the like. He noticed that the number of distinct scales of length of natural patterns is for all practical purposes infinite. In nature these patterns can be observed in the branches of a tree, coastlines, snowflakes, mountain ranges and the florets of a cauliflower. Informally, Mandelbrot (1998) declares fractal geometry to be the systematic study of certain very irregular shapes in either mathematics or nature, wherein each part is very much like a reduced size image of the whole.

Infinite perimeter in a finite space

Mandelbrot used the example of the coastline of Britain being an infinite length, confined within a finite space (Farrell, 1998). In measurements of coastlines, smaller units mean greater accuracy, but just as fractions can always be infinitely smaller, so can our units of measurement. Formerly we relied on calculus to help us to understand perimeter and area by creating rectangles of smaller and smaller dimensions but this meant relying on simple models. With fractal geometry we can more closely approximate nature. When fractals are demonstrated in nature, they have what is called *statistical self-similarity*. This means that the statistics of the pattern repeat themselves on different levels.

The Mandelbrot set is based on the quadratic equation $f(z) = z^2 + c$ where both z and c are complex numbers, pairs of real numbers (Sabine, 2004). The employment of Mandelbrot's actual equation $Z \leftrightarrow z^2 + c$ mapped to the sound-making and colour-pixel elements of computers enabled composers to use computers to create fractal music. When fractals are generated by a computer they have self-similarity (Padula, 2005; Diaz-Jerez, 1999).

Moon (1987) described fractal geometry more formally as a geometric property of a set of points in an n-dimensional space having a quality of self-similarity at different length scales and having a noninteger fractal dimension less than n. 'Fractal dimension' means a quantitative property of a set of points in an n dimensional space which measures the extent to which the points fill a subspace as the number of points becomes very large. Although the mathematics is simple, you need a computer to do the large number of calculations.

Fractal music

Diaz-Jerez (1999) described how students can listen to and create modern musical compositions on computers using Mandelbrot's non-linear iterative fractal equation $Z \leftrightarrow z^2 + c$ (the answer is fed back into the equation). The first iteration of Mandelbrot's equation produces a pair of coordinates. Then on the second, third, fourth, etc. iterations the equation generates related coordinates. If you process say, 1 000 000 iterations you will produce 1 000 000 pairs of very closely related coordinates that belong to a complex pattern (Sabine, 2004), e.g., the Mandelbrot and Julia sets. These coordinates, when mapped to frequency, duration and amplitude parameters on a simple x–y graph are employed to create music. (Oliver (2002) described how students can plot Julia orbits using an Excel spreadsheet.)

Different mathematical topics can be used to generate fractal music. 'Earthworm algebra' is just one and is described by Diaz-Jerez (2000) in the following manner:

> Take any whole number A, a constant multiplier B and a number which will be the maximum number of digits allowed C. Next, multiply A and B. Take the result and multiply it again by B. Repeat this process until the number of digits in the result get past C. Now sever the result by truncating it to its rightmost C digits. Multiply again by B and sever again to C digits, and so forth. It turns out that all earthworms (mathematically speaking) eventually enter a cycle, an infinite loop of repeating values.

The following example is given with 2 as the first number, 3 as the multiplier and 2 for the maximum number of digits:

A	B	C (maximum digits)
2	3	2
6	18	
18	54	
54	62 (severed)	
62	86 (severed)	
86	58 (severed)	
58	74 (severed)	
74	22 (severed)	
22	66	
66	98 (severed)	
98	94 (severed)	
94	82 (severed)	
82	46 (severed)	
46	38 (severed)	
38	14 (severed)	
14	42	
42	26 (severed)	
26	78	
78	34 (severed)	
34	2 (severed)	
2	6	
6	18	

As shown above and Diaz-Jerez (2000) states, this particular combination makes a cycle of twenty different values. Some combinations generate 'worms' thousands of values long. The sequences seem to be almost random, but if you listen to the music they generate you'll notice hidden patterns, patterns which illustrate the importance of pattern to mathematics and music.

Students may conduct general searches for the intersection of, or connections between, mathematics and music, or, fractal-music composition on the Internet. O'Haver (2002) has links to many interesting sites. One is: *The Sound of Mathematics* (Cummerow, 1998) where students may hear music based on topics such as:

- combinatorics (www.geocities.com/Vienna/9349/combpent.mid);
- Pascal's triangle (www.geocities.com/Vienna/9349/pascal.mid); and
- trigonometry (www.geocities.com/Vienna/9349/functions.html#trig).

Conclusion

Mathematics is abstract. "This abstraction poses a challenge to the teacher and student alike, and both will need to draw on knowledge of the world and link this to mathematical knowledge and its application in various situations" (VCAA, 2005). One of the ways this can be done, and done pleasurably, is through the study of music — its production, appreciation and composition. In this way students can not only see, and perhaps produce, say Pascal's triangle, they can hear it. The advantages of using all of the students' senses in this way seems pedagogically obvious, since students learn in often individual and multi-sensory ways. They can also learn about the importance of pattern to mathematics by listening for patterns in mathematically structured music, such as fractal music.

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From Helen Prochazka's Scrapbook

Mathematics on YouTube

In mathematics, the Klein four group or Vierergruppe, often symbolized by the letter V, is the group $Z_2 \times Z_2$, the direct product of two copies of the cyclic group of order 2. It was named Vierergruppe by Felix Klein in 1884. All elements of the Klein group, except the identity, have order 2. But since 2006 there has been a musical Vierergruppe in the world of higher mathematics, an acapella choir whose members are mathematicians from the Northwestern University (USA) who are known as Klein Four Group.

Their very skilful and amusing rendition of their song "Finite Simple Group (of Order 2)" became a hit on YouTube, bringing higher mathematics to the masses. Since that success, the Klein Four Group have put together an album of their choral work — "a fourteen song journey of mathematically tinged music."