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Extending Ourselves: Making Sense of Students' Sense Making

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This article discusses three solutions to the *Tower of Hanoi* problem offered by students in a mathematics content course for prospective elementary school teachers. The course uses standards-based pedagogy and teaching via problem solving. Within this work, we consider the growth supported by collaboration at both the students' level and the teachers' level. In each case, we offer new understandings of algebraic representations and number systems.

Change is a curious thing. Sometimes change happens abruptly, like the change brought on by a sudden, unexpected occurrence—the addition of a new family member, winning the lottery, and other kinds of surprise. Sometimes change is slow, happening over a long period of time. Sometimes it's so slow that the change is hardly noticeable.

Education is often like that. Students spend years in school, often believing that not much is changing until they look back to see how far they have come. Education can be like that for teachers, too. As teachers, we consider ourselves constructivists, actively trying to build a model of our students' thinking and base our curricular decisions on that model (Steffe & D'Ambrosio, 1995). We anticipate learning about our students' thinking; in fact, that is our goal. We also teach via problem solving (Schroeder & Lester, 1989; Lester, & Mau, 1993; Lester et al. 1994), giving our students problems to solve and then helping them formalize and use mathematical symbols to record their thinking. We anticipate that this will help students' comfort zones with mathematics. What we describe in this paper is something we had not anticipated—making new meaning of mathematics for ourselves.

We teach mathematics content to prospective elementary and middle school teachers prior to their admission into the School of Education. Our students come to us with a variety of backgrounds. Their beliefs and attitudes about mathematics are varied—some never achieved success in mathematics and others never really struggled. Many of the students come with a mass of jumbled mathematics, recalling formulas tangled with other formulas and frequently using them at inappropriate times. Most of the students, however, learned mathematics sitting silently, working independently of their classmates. Many of them have developed a perspective of the nature of mathematics that we want to change. As part of the growth process we hope to create in our students, we require them to keep a reflective journal. At times, the journal prompts ask students to consider extension questions about the mathematics they are learning and at other times the prompts ask them to consider their learning processes. As one of our students explained in an early journal entry,

> After the first day of class, I knew that my math thought process for this class was going to alter what I have learned previously. I've been so used to memorizing formulas and different ways to solve problems, i.e.: y = mx + b or A = 1/2 (bh)
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> My brain has become a one-track mind. Now I'm having to do things differently. . . . We had to figure out some kind of formula. *This is a reversal from what I've been taught in the past. . . . I'm figuring out how to do something on my own without someone telling me "This is exactly how you do it."* (emphasis added)

As teachers, we have multiple goals for our students: We want them to alter their perception of the nature of mathematical thought, to develop enough self-confidence to persevere, to build their problem-solving strategies, to develop self-monitoring techniques, and to learn to justify their thinking. To begin this task of change, we present them with problems and encourage them to work in groups to solve the problems. Our entire curriculum is built around small group problem solving. There is no lecture and no explanation from us as teachers. Rather, we pose the problems and then facilitate a whole class discussion where students determine the correctness of their work. One particular problem has pushed our thinking, eliciting change in us, and has pushed the students to engage in work they never believed they could do.

The Problem

One of our favorite problems is the *Tower of Hanoi Problem*.

> You are given five disks. Stack these disks so that they increase in size with the largest disk on the bottom. Imagine that these disks are stacked on a peg and that there are two other empty pegs on the table. The goal is to transfer all of the five disks to one of the other pegs using the fewest possible moves. You are to follow the conditions below:
>
> 1. Move only one disk at a time.
> 2. No disk may be placed on top of one smaller than itself.

Students are given Cuisenaire rods of different sizes to model the problem physically. They typically begin to solve this problem by moving disks and informally trying to count the moves. As we move around the room discussing students' solutions with them, it soon becomes clear to them that there is a need to keep track of moves and to make records of their work with the disks. We encourage them to watch each other closely and be certain that they have solved the problem in the minimal number of moves. Students begin to make real progress toward the solution when they identify a systematic way of moving the disks.

As is the case in all problems throughout this learning experience, students are encouraged to organize their data in ways that lend themselves to the emergence of useful patterns. Students' different counting strategies yield different forms of tabulation of data. In the following sections we describe the different counting strategies used by our students and the implications for the patterns that emerged.

At this point, it might be tempting to believe that we, as teachers, provided much of the information students needed to solve the problem. It may also tempting to believe that we told them exactly how to record their information. Although we did converse with them, we were exceptionally careful not to impose our thinking about the problem on them. As they recorded information, we might suggest that another arrangement would prove more useful. We might suggest that they organize around one idea rather than trying to accommodate several ideas and recording schemes all at once. However, we were unwilling to tell them exactly what information to record or how to organize it.

The Three Solutions

We use this problem during the first two weeks of class in order to help students begin the process of working in a group and begin to recognize the importance of organized record keeping. Typically, students begin by stabbing in the dark, grabbing at any solution strategy that strikes them. They tend to begin with the total number of disks suggested in the problem statement rather than finding a simpler problem and looking for a pattern. They record numbers rather randomly on the page. As we move around the room, we often have to ask what various numbers on the page represent because students failed to label their data. The process of organization often takes 45 minutes or more. As we encouraged students to start with a smaller number and to make organized lists (sometimes in tabular format), solutions began to emerge. Here we present *their* most polished solutions, formalized in standard mathematical notation. It is important to note that we provided this notation; students were rarely sophisticated enough to use mathematical symbols in conventional ways. It is also important to note that solutions remained on the board as successive groups presented their findings, providing us with the opportunity to compare solutions.

Solution One

A few groups counted the total number of moves for a certain number of disks. Affirmation of their data as the minimal number of moves came from two sources: a realization of a systematic way of counting moves and comparing findings with other groups' results. Their solution had the following entries.

Number of Disks	Number of Moves
1	1
2	3
3	7
4	15
5	31

and so forth.

They decided that for n disks, they should compute the nth power of 2 and subtract 1, presenting the generalization as $2^n - 1$. We considered this adequate notation causing us to accept this without revision.

Solution Two

Students presenting this solution moved one disk and recorded the number of moves, moved two disks and recorded the number of moves, moved three disks and recorded the number of moves. As they recorded the moves for four, five, and six disks, they began to see a recursive pattern. For example, when they moved four disks, they counted the number of moves for the first three disks, moved the fourth disk, and then commented that they already knew the number of moves necessary to move the three disks to the final peg. Their solution had the following entries.

Number of Disks	Number of Moves
1	1
2	1 + 1 + 1 = 3
3	3 + 1 + 3 = 7
4	7 + 1 + 7 = 15

In preparation for a whole-class discussion, we encouraged students to find a generalization of their pattern. Students in this group recorded and presented their pattern as "the number of moves = 2n+1".

During the whole-class discussion, we were concerned that students comparing *Solutions One and Two* might think that *n* represented the same number in each solution. In an effort to assess their understanding and compare *Solution Two* to *Solution One*, we asked about *n*—was it the same n as in *Solution One*? Students quickly told us no, that their *n* referred to the total number of moves from the previous number of disks. So for 4 disks, n=7, where 7 referred to the total number of moves for 3 disks.

Formal notation for recursive patterns was unfamiliar to our students. This was the moment where we, as teachers, inducted students into a new use of formal mathematical symbols. In helping students translate their verbal explanation into mathematical symbols, we suggested the following formal notation. If one considers a_n as the number of moves for n disks, then the number of moves for n+1 disks could be expressed as

$$a_{n+1} = a_n + 1 + a_n$$

or

$$a_{n+1} = 2a_n + 1$$

The language of recursive form versus closed form emerged from the comparison of *Solutions One* and *Two*. Throughout the semester, students periodically revisited this language, trying to determine for themselves whether or not their generalizations for other patterns were recursive or closed.

Solution Three

These students chose a different approach. They counted the number of times each individual disk moved. They recorded the following work, where disk 1 is the largest disk.

Disk	Number of moves	Total Moves
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63

and so forth.

In order to calculate the number of moves for four disks, they added 1 + 2 + 4 + 8. To calculate the number of moves for five disks, they added 1 + 2 + 4 + 8 + 16. They partially refined their understanding of this "doubling pattern" in the number of moves for each disk to a recognition of this pattern as successive powers of 2. However, they struggled to link 1 to 2^0. When this connection was resolved, then they presented the total number of moves as $2^0 + 2^1 + 2^2 + ... + 2^{n-1}$, for n disks.

In effect, they found the total number of moves to be the summation of powers of 2, starting with 2^0 and summing to 2^{n-1}, where n = the number of disks. We made the decision not to use a summation symbol with these students at this point. Our sense was that students had reached their limit of new formal notation and language.

In all three solutions, the same table of values emerges, yet the solutions are essentially very different. The difference lies in the students' work. Students who added powers of two did so because of the way they were counting the number of moves for each individual disk. Other students either used a recursive counting strategy or found the pattern in analysing the numbers directly from the table. Each solution is a direct result of students' actions and counting strategies.

Beginnings of Growth: Collaboration

When most groups had a formal solution, we called the students back to a whole-class discussion. Groups were asked to volunteer their solutions. As each of the solutions was presented to the rest of the class, some students had difficulty following the explanations. Students whose groups had approached the problems in very different ways had to be shown the actions of other groups and how other groups recorded those actions in order to understand and be able to decide whether the solution was correct. Additionally, students often had to question other groups about the group's thinking and the mathematical notation in order to make sense of the solution. In effect, collaboration across groups began.

For us, collaboration is joint meaning making and results in joint ownership of ideas. In an inquiry setting this means that we have collaborative conversations that help us develop new understandings through the understandings of others. Collaborative conversations help us to extend ourselves (Harste, 1994). These new understandings are the focus of our discussion in this section. We think of these new understandings as occurring in various layers of interactions.

The first layer of interaction occurs within the small group setting. Students engage in small group interactions where they articulate their thinking to each other. Many students experience internal tensions as they make sense of others' thinking in light of their own. The commitment of the group results in producing a group consensus to the solution of the problem. This commitment pushes the individuals in the group to persevere in overcoming the tensions among group members' diverse ways of thinking. When a student begins to see the other group members' ways of thinking, the student begins to extend personal understanding.

We can begin to understand the tensions students felt by listening to Callie's words as she writes in her journal about the experience of solving this problem.

> I have learned how important it is to take down in writing your observations about patterns quickly to help you find the answers to all different kinds of math problems. I have also learned better how to work in a group utilizing the benefits of each individual at the same time to get answers more quickly so that we don't stumble over each other as much. At first, however, our group could work on this a lot more. *I'm the one that often needs to be patient and shut up because I often confuse some of them trying to explain what patterns I see. It never seems to come out in words or concepts that they understand without me having to explain it for 20 minutes until I get it into something they understand*.
>
> I am beginning to understand also the kinds of thoughts you need to have stewing in the back of you mind when looking for a mathematical pattern. You shouldn't, I don't think, be looking specifically for any given equation you've learned before, but look for similar patterns of functions (or arithmetic processes) to give better clues of what the questions are asking.
>
> . . . all through my math classes I have been struggling with why math does what it does and always hated it because *I wasn't learning anything but how to do what I was told—make myself a math machine with no brain but to memorize and repeat*. For once in my life, I have finally gotten a glimpse of insight into how to learn math and why it's so interesting. (emphasis added)

The second layer occurs as groups share their solution techniques with other groups. In this particular problem (*Towers of Hanoi*), as students listen to others share their mathematical solutions a tension similar to the individual tension occurs. In order to understand Group A's solution, members of other groups must be able to build a representation of Group A's process. That is, they need to be able to see the actions Group A used to formulate their solution. Frequently, this required the presenting group to use overhead Cuisenaire rods and literally to move the rods and develop their data chart so that listeners could understand their solution process. The tensions among groups become the same as the tensions originally among individuals within a group. These are the moments of disequilibrium that can result in learning as students construct meaning (Mason, 1996).

Yet a third layer of interaction occurs as the teacher (or in this case, teachers) tries to follow the students' explanations and make sense of the students' thinking. Teachers, like students, have solutions in their minds, and it is often difficult for them to see another's solution. As teachers begin to understand students' solutions that differ from their own, teachers begin to extend themselves. That is, they begin to understand mathematics in deeper ways (Confrey, 1993).

Layers of Collaboration and the Development of Mathematical Connections

For us as teachers, a fourth layer of interaction and collaboration occurred. We co-taught this course which met two hours each day and three days each week. We considered it important for both of us to be in the room at all times, regardless of which of us was taking the lead for the whole-class discussion. This luxury of watching each other teach became the impetus for our own professional growth and joint inquiry.

Joint planning and analysis occurred daily—before, during, and after class. While students worked in small groups, we observed and discussed their thinking, strategies, and difficulties. We purposefully chose the order of presentations based on students' work, envisioning the sequence that would support the richest discourse among students. While one of us facilitated the large group discussion, the other took field notes that we later used in combination with students' written work and journals to build our interpretations of students' understanding. Throughout the entire semester, our joint inquiry focused on building a model of our students' understanding of mathematical concepts in order to shape our instruction. The example of the *Tower of Hanoi* is a snapshot of this inquiry that illustrates how unexpected student solutions can provide new mathematical meaning for teachers who engage in this inquiry.

As we began to understand students' thinking, we found our own tension in our desire to equate the algebraic solutions. We were clear that the students' actions were adequately and accurately represented in the symbols chosen; it was less obvious that we could easily demonstrate the algebraic equivalence of the three representations. Although we were able to use the formal mathematical demonstration commonly found in advanced algebra and calculus texts to demonstrate the equivalence of *Solutions Two* and *Three*, demonstrating the algebraic equivalence of *Solution One* to either *Solution Two* or *Three* was less straightforward. As we struggled to relieve our tensions, to find the symbol manipulation that we typically use as our mathematical currency, we found connections in mathematics that had not previously occurred to us. It was only when we began to look at the pattern of the numbers represented by *Solution Three* and *One* that we saw the numerical equivalence regardless of the value of n (see End Notes for the proof). Our realization of equivalence came from our understanding of place value, regardless of the base.

In effect, we built new mathematical meanings as we struggled to teach from a constructivist perspective (Confrey, 1993). We found ourselves thinking about place value in new ways. It was only as we struggled to put words to *Solution Three* that we came to think of the summation as representing a base two number in expanded notation.

We also found ourselves looking for alternate base (other than 10) representations in problems throughout the curriculum for the remainder of the semester. We began to recognize missed opportunities in other problems. Not only had we failed to make the mathematical connections in our own minds, we had failed to make those connections explicit for our students.

We began to think about mathematical symbols in other ways. Symbols could no longer be viewed as a set of instructions telling students what to do. As the students worked and we thought about their solutions, it became clear that the symbols resulted from the actions, not the actions from the symbols. Students' very personal ways of counting led to very personal solutions, which accounted for the mismatch of one group's counting scheme with another group's scheme. These mismatches led to tensions in understanding, and those tensions became springboards for learning. As we realized the power and importance of these springboards, we began to strive to recognize opportunities and to capitalize on them as fully as possible. These differences and the students' reactions to them allowed us to discuss thinking, emotional reactions, and appropriate responses to problems. Attending to all of the human responses to mathematics allowed us to view the students' solutions and their inability to understand others' solutions in a new way—that physical action is the source of mathematical representation and that different *actions* yield different algebraic solutions.

Our message through this example is that teachers who listen to students and who struggle to understand students *on their own terms* have the opportunity to learn new mathematics. Our desire to be constructivist teachers (Steffe & D'Ambrosio, 1995) and to respect students' thinking became the springboard for our doing mathematics, equating the logic of the students' solutions and finding symbolism to represent their thinking. We realized how students' rather straightforward thinking provides the backdrop for rich mathematical learning when we take the time and effort to model their actions and representations through formal mathematics. For us, this experience reinforced the need to use students' understandings as the catalysts for furthering their mathematical knowledge. Only when students' *thinking* is the focus will we truly be able to say that we are teaching mathematics. Only when our students, future teachers of elementary school children, are able to represent their own thinking will they be able to teach children to represent their own thinking. This change may only occur when we extend ourselves and our current understandings, when we take a close look at students' solutions and begin to make new sense of mathematics and mathematical thinking.

End Note – Our Proof for Algebraic Equivalence

It is reasonably easy to demonstrate that *Solution Two* ($a_{n+1} = 2a_n + 1$) and *Solution Three* ($2^0 + 2^1 + 2^2 + 2^3 + ... + 2^{n-1}$) are algebraically equivalent. In order to do this, start with the recursive form and successively substitute the algebraic form for a_n in each iteration.

Number of Disks	Algebraic Expression
1	$a_1 = 1$ (number of moves for one disk)
2	$a_2 = 2a_1 + 1$
3	$a_3 = 2a_2 + 1$
	$a_3 = 2(2a_1 + 1) + 1$
	$a_3 = 4a_1 + 2 + 1$
4	$a_4 = 2a_3 + 1$
	$a_4 = 2(4a_1 + 2 + 1) + 1$
	$a_4 = 8a_1 + 4 + 2 + 1$
5	$a_5 = 2a_4 + 1$
	$a_5 = 2(8a_1 + 4 + 2 + 1) + 1$
	$a_5 = 16a_1 + 8 + 4 + 2 + 1$

and so forth. Note that $a_1 = 1$, so a simple substitution yields a summation of powers of 2.

Now we need to demonstrate that the closed form, *Solution One*, is algebraically equivalent to the summation form, *Solution Three*.

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1$$

Although this can be demonstrated using geometric series, an opportunity to build additional connections exists. If we list successive terms in the summation on the left, we get $1 + 2 + 4 + 8 + 16 + \ldots + 2^{n-1}$. Writing this sum as a number in base two, we have $111111...1_{two}$ where there are n digits of 1. If we look at $2^n - 1$ and write it as a base two number, we have $1000000...0_{two} - 1$ where the first number has n zeros, that is, n+1 digits. Subtracting 1 from that number leaves us with $111111...1_{two}$, where the number has n digits of 1. That is,

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1$$

References

Confrey, J. (1993). Learning to see children's mathematics: Critical challenges to constructivist reform. In K. Tobin (Ed.), *The practice of constructivism in science education* (pp. 299-321). Washington, DC: American Association for the Advancement of Science.

Harste, J. (1994). Literacy as curricular conversations about knowledge, inquiry, and morality. In R. B. Ruddell, M. R. Ruddell, & H. Singer (Eds.), *Theoretical models and processes of reading* (pp. 1220-1242). Newark, DE: International Reading Association.

Lester, F. K., & Mau, S. T. (1993). Teaching mathematics via problem solving: A course for prospective elementary teachers. *For the Learning of Mathematics, 13*(2), 8-11.

Lester, F. K., Masingila, J. O., Mau, S. T., Lambdin, D. V., dos Santos, V. M. P., & Raymond, A. M. (1994). Learning how to teach via problem solving. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics: 1994 yearbook* (pp. 152-166). Reston, VA: National Council of Teachers of Mathematics.

Mason, J. (1996). Tensions. In R. B. Corwin, J. Storeygard, & S. L. Price (Eds.), *In talking mathematics: Supporting children's voices* (pp. 103-111). Portsmouth, NH: Heinemann.

Schroeder, T. L., & Lester, F. K., Jr. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton & A. P. Shulte (Eds.), *New directions for elementary school mathematics* (pp. 31-42). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P., & D'Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education, 26*, 146-159.

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