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Solving Algebra and Other Story Problems with Simple Diagrams: a Method Demonstrated in Grade 4–6 Texts Used in Singapore

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Out of the 38 nations studied in the 1999 *Trends in International Mathematics and Science Study* (TIMSS), children in Singapore scored highest in mathematics (National Center for Education Statistics, NCES, 2003). Why do Singapore's children do so well in mathematics? The reasons are undoubtedly complex and involve social aspects. However, the mathematics texts used in Singapore present some interesting, accessible problem-solving methods, which help children solve problems in ways that are sensible and intuitive. Could the texts used in Singapore be a significant factor in children's mathematics achievement? There are some reasons to believe so. In this article, I give reasons for studying the way mathematics is presented in the elementary mathematics texts used in Singapore; show some of the mathematics problems presented in these texts and the simple diagrams that accompany these problems as sense-making aids; and present data from TIMSS indicating that children in Singapore are proficient problem solvers who far outperform U.S. children in problem-solving.

Why Study the Methods of Singapore's Mathematics Texts?

What is special about the elementary mathematics texts used in Singapore? These texts look very different from major elementary school mathematics texts used in the U.S. The presentation of mathematics in Singapore's elementary texts is direct and brief. Words are used sparingly, but even so, problems sometimes have complex sentence structures. The page layout is clean and uncluttered. Perhaps the most striking feature is the heavy use of pictures and diagrams to present material succinctly—although pictures are never used for embellishment. Simple pictures and diagrams accompany many problems, and the same types of pictures and diagrams are used repeatedly, as supports for different types of problems, and across grade levels. These simple pictures and diagrams are not mere procedural aids designed to help children produce speedy solutions without understanding. Rather, the pictures and diagrams appear to be designed to help children make sense of problems and to use solution strategies that can be justified on solid conceptual grounds. Because of this pictorial, sense-making approach, the elementary texts used in Singapore can include problems that are quite complex and advanced. Children can reasonably be expected to solve these problems given the problem-solving and sense-making tools they have been exposed to.

Thus the strong performance of Singapore's children in mathematics may be due in part to the way mathematics is presented in their textbooks, including the way simple pictures and diagrams are used to communicate mathematical ideas and to provide sense-making aids for solving problems. If so, then teachers, mathematics educators, and instructional designers in the U.S. will benefit from studying the presentation of mathematics in Singapore's textbooks, so that they can help children in the U.S. improve their understanding of mathematics and their ability to solve problems.

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Using Strip Diagrams to Solve Story Problems

One of the most interesting aspects of the elementary school mathematics texts and workbooks used in Singapore (Curriculum Planning and Development Division, Ministry of Education, Singapore, 1999, hereafter referred to as *Primary Mathematics* and *Primary Mathematics Workbook*) is the repeated use of a few simple types of diagrams to aid in solving problems. Starting in volume 3A, which is used in the first half of 3rd grade, simple "strip diagrams" accompany a variety of story problems. Consider the following 3rd grade subtraction story problem:

Mary made 686 biscuits. She sold some of them. If 298 were left over, how many biscuits did she sell? (*Primary Mathematics* volume 3A, page 20, problem 4)

The problem is accompanied by a strip diagram like the one shown in Figure 1.

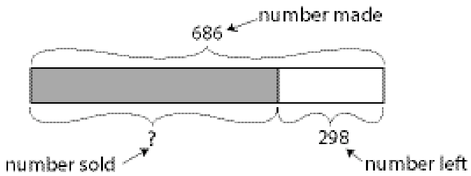

Figure 1: How Many Biscuits Were Sold?

On the next page in volume 3A is the following problem:

Meilin saved $184. She saved $63 more than Betty. How much did Betty save? (*Primary Mathematics* volume 3A, page 21, problem 7)

This problem is accompanied by a strip diagram like the one in Figure 2.

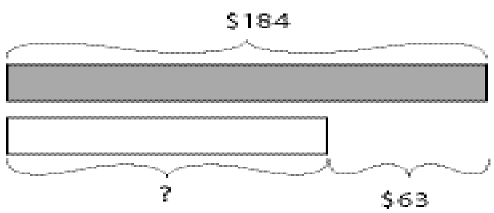

Figure 2: How Much Did Betty Save?

These two problems are examples of some of the more difficult types of subtraction story problems for children. The first problem is difficult because we must take an unknown number of biscuits away from the initial number of biscuits. This problem is of the type *change-take-from, unknown change* (see Fuson, 2003, for a discussion of the classification of addition and subtraction story problems). The second problem is difficult because it includes the phrase "$63 more than," which may prompt children to add $63 rather than subtract it. This problem is of type *compare, inconsistent* (see Fuson, 2003). The term *inconsistent* is used because the phrase "more than" is inconsistent with the required subtraction. Other linguistically difficult problems, including those that involve a multiplicative comparison with a phrase such as "N times as many as", are common in *Primary Mathematics* and are often supported with a strip diagram. Consider the following 3rd grade problem, which is supported with a diagram like the one in Figure 3:

A farmer has 7 ducks. He has 5 times as many chickens as ducks….How many more chickens than ducks does he have? (*Primary Mathematics* volume 3A, page 46, problem 4)

(Note: The first part of the problem asks how many chickens there are in all, hence the question mark about all the chickens in Figure 3 below.)

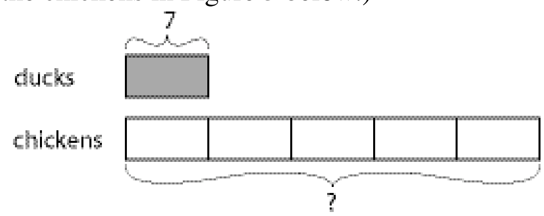

Figure 3: How Many More Chickens Than Ducks?

Although the strip diagrams will not always help children carry out the required calculations (for example, we don't see how to carry out the subtraction $184 – $63 from Figure 2), they are clearly designed to help children decide which operations to use. Instead of relying on superficial and unreliable clues like key words, the simple visual diagram can help children understand why the appropriate operations make sense. The diagram prompts children to choose the appropriate operations on solid conceptual grounds.

From volume 3A onward, strip diagrams regularly accompany some of the addition, subtraction, multiplication, division, fraction, and decimal story problems. Other problems that could be solved with the aid of a strip diagram do not have an accompanying diagram and do not mention drawing a diagram. Fraction problems, such as the following 4th grade problem, are naturally modeled with strip diagrams such as the accompanying diagram in Figure 4:

David spent 2/5 of his money on a storybook. The storybook cost $20. How much money did he have at first? (*Primary Mathematics* volume 4A, page 62, problem 11)

Without a diagram, the problem becomes much more difficult to solve. We could formulate it with the equation $(2/5)x = 20$ where x stands for David's original amount of money, which we can solve by dividing 20 by 2/5. Notice that the diagram can help us see why we should divide fractions by multiplying by the reciprocal of the divisor. When we solve the problem with the aid of the diagram, we first divide $20 by 2, and then we multiply the result by 5. In other words, we multiply $20 by 5/2, the reciprocal of 2/5.

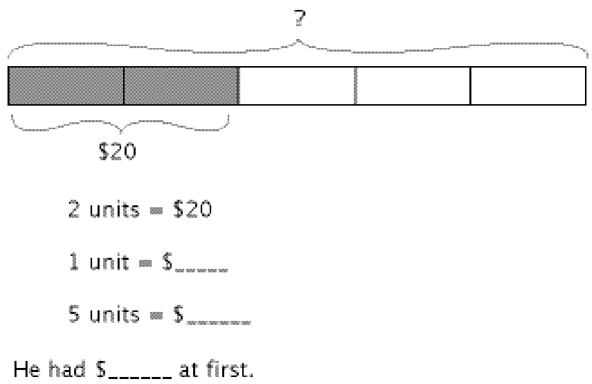

Figure 4: How Much Money Did David Have?

The problems presented previously are arithmetic problems, even though we could also formulate and solve these problems algebraically with equations. But starting with volume 4A, which is used in the first half of 4th grade, algebra story problems begin to appear. Consider the following problems:

> 1. 300 children are divided into two groups. There are 50 more children in the first group than in the second group. How many children are there in the second group? (*Primary Mathematics* volume 4A, page 40, problem 8)
>
> 2. The difference between two numbers is 2184. If the bigger number is 3 times the smaller number, find the sum of the two numbers. (*Primary Mathematics* volume 4A, page 40, problem 9)
>
> 3. 3000 exercise books are arranged into 3 piles. The fist pile has 10 more books than the second pile. The number of books in the second pile is twice the number of books in the third pile. How many books are there in the third pile? (*Primary Mathematics* volume 4A, page 41, problem 10)

These problems are readily formulated and solved algebraically with equations, but since the text has not introduced equations with variables, the children are presumably expected to draw diagrams to help them solve these problems. Notice that from an algebraic point of view, the second problem is most naturally formulated with two linear equations in two unknowns, and yet 4th graders can solve this problem.

The 5th grade *Primary Mathematics* texts and workbooks include many algebra story problems which are to be solved with the aid of strip diagrams. Some do not have accompanying diagrams, but others do, and some include a number of prompts, such as a diagram like the one in Figure 5 which accompanies the following problem:

> Raju and Samy shared $410 between them. Raju received $100 more than Samy. How much money did Samy receive? (*Primary Mathematics* volume 5A, page 23, problem 1)

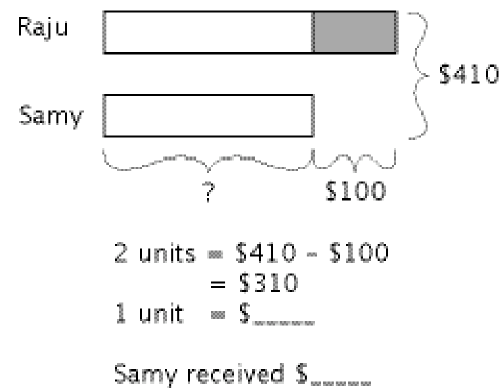

Figure 5: Raju and Samy Split Some Money

Notice that the manipulations we perform with strip diagrams usually correspond to the algebraic manipulations we perform in solving the problem algebraically. For example, to solve the previous Raju and Samy problem, we could let S be Samy's initial amount of money. Then,

$$2S + 100 = 410$$

as we also see in Figure 5. When we solve the problem algebraically, we subtract 100 from 410 and then divide the resulting 310 by 2, just as we do when we solve the problem with the aid of the strip diagram.

Strip diagrams make it possible for children who have not studied algebra to attempt remarkably complex problems, such as the following two, which are accompanied by diagrams like the ones in Figure 6 and Figure 7 respectively:

> Encik Hassan gave 2/5 of his money to his wife and spent 1/2 of the remainder. If he had $300 left, how much money did he have at first? (*Primary Mathematics* volume 5A, page 59, problem 6)
>
> Raju had 3 times as much money as Gopal. After Raju spent $60 and Gopal spent $10, they each had an equal amount of money left. How much money did Raju have at first? (*Primary Mathematics* volume 6B, page 67, problem 1)

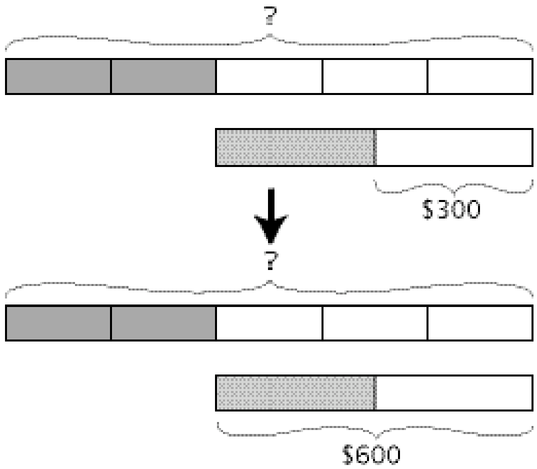

Figure 6: How Much Money Did Encik Hassan Have at First?

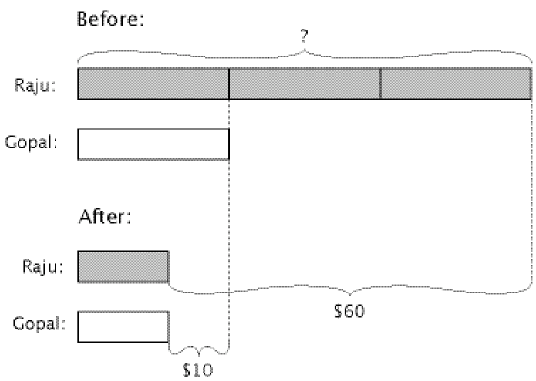

Figure 7: How Much Did Raju Have at First?

Performance of 8th Graders on TIMSS

In light of the complex problems that children in Singapore are taught how to solve in elementary school, the strong performance of Singapore's 8th graders on the TIMSS assessment is not surprising. Among the released TIMSS 8th grade assessment items in the content domain "Fractions and Number Sense" classified as "Investigating and Solving Problems," Singapore 8th graders scored higher than U.S. 8th graders on all items. These released items included the following problems (see NCES, 2003):

> Laura had $240. She spent 5/8 of it. How much money did she have left? (Problem R14, page 29. Overall percent correct, Singapore: 78%, United States: 25%).

> Penny had a bag of marbles. She gave one-third of them to Rebecca, and then one-fourth of the remaining marbles to John. Penny then had 24 marbles left in the bag. How many marbles were in the bag to start with?
>
> A. 36
>
> B. 48
>
> C. 60
>
> D. 96
>
> (Problem N16, page 19. Overall percent correct, Singapore: 81%, United States: 41%)

These problems are similar to problems in *Primary Mathematics*. The strong performance of Singapore 8th graders on these problems indicates that the instruction children receive in solving these kinds of problems is effective. Similarly, among the released TIMSS 8th grade assessment items in the content domain "Algebra" classified as "Investigating and Solving Problems," Singapore 8th graders scored higher than U.S. 8th graders on all items.

But the strong problem-solving abilities of Singapore's 8th graders in fractions and number sense and in algebra does not necessarily result in factual knowledge in other mathematical domains in which the children have not had instruction. For example, U.S. 8th graders scored higher than Singapore 8th graders on the following item in the content domain "Data Representation, Analysis and Probability" classified as "Knowing":

> If a fair coin is tossed, the probability that it will land heads up is 1/2. In four successive tosses, a fair coin lands heads up each time. What is likely to happen when the coin is tossed a fifth time?
>
> A. It is more likely to land tails up than heads up.
>
> B. It is more likely to land heads up than tails up.
>
> C. It is equally likely to land heads up or tails up.
>
> D. More information is needed to answer the question.
>
> (Problem F08, page 74. Overall percent correct, United States: 62%, Singapore: 48%)

The mathematics texts used in Singapore through 8th grade do not address probability. Thus the difference in performance in fraction, number sense, and algebra problem-solving versus knowledge about probability can reasonably be attributed to effective instruction.

Conclusion

The mathematics textbooks used in elementary schools in Singapore show how to represent quantities with drawings of strips. With the aid of these simple strip diagrams, children can use straightforward reasoning to solve many challenging story problems conceptually. The TIMSS 8th grade assessment shows that 8th graders in Singapore are effective problem solvers and are much better problem solvers than U.S. 8th graders. Although cultural factors probably also affect the strong mathematics performance of children in Singapore, children in the U.S. could probably strengthen their problem-solving abilities by learning Singapore's methods and by being exposed to more challenging and linguistically complex story problems early in their mathematics education.

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