An Analysis of the Use of Graphical Representation in Participants' Solutions

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InterMath participants spend time in workshops exploring technology-rich mathematical investigations and completing write-ups. These write-ups include a written explanation of their problem solving process, screen captures of files that they generated while completing the investigation and links to these files. This paper examines the use of graphical representations in write-ups that included incorrect mathematics and/or incorrect solutions. Our findings indicate that a large number of incorrect write-ups included a graphical representation for cosmetic purposes, meaning that it was not used to explain or justify participants' solutions.

InterMath[1] is a professional development effort designed to strengthen teachers' mathematical content knowledge through the exploration of mathematical investigations using various technologies. It was created to address a critical problem encountered in many middle schools: the mathematics teachers are deficient in content knowledge or in content-specific pedagogy because they do not have adequate preparation (Wilson, Hannafin, & Ohme, 1998). InterMath addresses this goal by engaging teachers in open-ended explorations that simultaneously allow them to develop their own mathematical understandings and learn to use technologies to support their mathematical thinking.

InterMath participants have the opportunity to select which problem(s) they want to work; the approach they use to solve the problem; and, ultimately, the depth of learning they take from the class by choosing the appropriate difficulty level of the problems they worked. Consistent with many current professional development guidelines (Hawley, 1999; Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003; National Partnership for Excellence and Accountability in Teaching [NPEAT], 2000), InterMath allows the participating teachers to identify their own needs and direct their own learning in a supportive environment. In a sense, InterMath participant teachers determine what they need to succeed as mathematics teachers and learners. The primary deliverable of this process is a series of documents called "write-ups" in which participants "communicate and synthesize investigations involving exploration, solving a problem, or working with an application" (Wilson et al, 1998, p. 18). The key elements of a write-up include a restatement of the problem, the writer's initial plan for solving the investigation, an explanation of how the investigation was actually approached, and a statement of the findings. Write-ups may also include justifications of solution processes, answers, extensions, or ideas of how the problem might be modified for classroom use. Most InterMath write-ups include screen captures of relevant technology-enhanced work the participants engaged in, and links to files that they created, for example, Microsoft Excel or Geometer's SketchPad® (GSP; Jackiw, 1993) files, as they solved the problems. For each InterMath course in which we have collected data, the participants were asked to complete approximately 10 of these write-ups.

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In our current analysis, we have focused on a subset of these write-ups that include either incorrect mathematics and/or incorrect answers. Our analyses began with an effort to make sense of what participants were learning, and quickly became focused on how participants used graphical representations (defined here as any kind of graphical representation including graphs and GSP sketches) in their work.

Theoretical Framework

InterMath, as a professional development experience, is grounded in theory and research that suggests that teaching mathematics should be something other than "chalk and talk" models in which the teacher transmits his or her knowledge to the students (e.g., NCTM, 2000). Specifically, InterMath takes Cohen and Ball's (1990) question to heart: "How can teachers teach a mathematics that they never have learned, in ways that they never experienced?" (p. 238). To this end, InterMath engages teachers in learning to teach differently by engaging them in a variety of mathematical activities as learners.

Consistent with the NCTM (2000) vision, the InterMath workshops engage teachers in making conjectures, communicating mathematically, reasoning, and problem solving, while simultaneously strengthening their content knowledge. By involving participants in these activities as engaged and reflective owners, InterMath seeks to foster a sense of comfort and confidence for using similar teaching approaches so that the teachers will begin to change their own practices.

InterMath is also strongly based on the notion that technology should be used in mathematics to support problem solving and reasoning, as well as to reduce the tedious aspects of certain calculations. We recognize the potential of technology for promoting reform-oriented approaches to mathematics, but also realize that simply using technology is not reform in itself; e.g., Kaput (1992). After all, mathematical learning is not fundamentally different if students are using a drill and practice program as opposed to a worksheet. However, if the learners are engaging with a technology to explore an aspect of mathematics or to tie together different understandings, then that technology offers an innovative learning opportunity.

Building from this vision of mathematics and the belief that, through InterMath, teachers are experiencing mathematics in a different way and exploring some new understandings for the role of technology in their classrooms, the research team has set out to understand what the impact of this learning

is. To this end, we have begun considering the mathematics the participants in the workshops seem to be learning and/or applying to the investigations presented in InterMath. In evaluating a subset of the write-ups from a set of courses, we recognized a trend that a high number of write-ups that included incorrect mathematics and/or incorrect solutions[2] also featured the use of a graphic element such as a graph or a geometric construction. Inspired to understand why there were a high number of incorrect write-ups that included a graphical representation, we turned to the literature to determine the different ways graphic elements might be used.

According to the NCTM Standards (2000), mathematical representations are useful tools for building understanding and for communicating both information and understanding. Given this, representations are critical elements in write-ups as they relate to mathematical communications. But, what is the role of the graphic? Do the teachers participating in InterMath understand the power of representations for both solving problems and communicating their understandings?

Graphical Representations

There has been a shift in recent years in mathematics educators' views on the role of drawn representations. As presented in Monk (2003), graphs can be viewed in two distinct ways. First, and more traditionally, a graph is a tool for communication. That is, graphs describe a set of data or a solution of a problem to the reader. However, Monk introduces the notion that there is a second way to use graphs – as tools for generating meaning. Monk elaborates saying, "Whereas a graph had earlier been seen exclusively as a *conduit*, a carrier of information, for example, about the motion of a car, it can now also be seen as a *lens* through which to explore that motion." (p. 251, emphasis in the original). Monk continues to point out that these are not opposites, nor is one preferable, rather that they are two different approaches to using tools that look the same. Consistent with InterMath's goals and vision, it was expected that participants would use graphs (and other visual representations) in both of these ways. Further, it had been assumed that the participants were using the representations as problem-solving tools because that was the approach modeled for them in the course.

More specifically, by using visual representations as problem-solving tools, participants would be able to see some benefits – particularly in their abilities to solve the kinds of complex problems they were often

faced with in InterMath. Consistent with Monk's views, the InterMath team considered a number of benefits to using graphs and graphic elements in this way. These included

- Using graphics to explore aspects of a context that might otherwise not be apparent;

- Developing a deeper understanding of a context through the use of graphics that elicit particular questions about those contexts; and

- Developing a deeper understanding of the kinds of information that can be conveyed through graphics (Monk, 2003).

Additionally, building on Gagatsis and Shiakalli (2004), we assert that it is most important for teachers to be able to work with these representations in both ways – as communicating and problem solving. While Gagatsis and Shiakalli were more concerned with moving between representations, their point applies to InterMath teacher participants. That is, translating between representations and within representation systems is a vital aspect of teaching. If a teacher is unable to interpret a graphic representation that has been developed by her students, she or he has lost one way of making sense of (a) whether the student understands a concept and (b) where the student may still need additional support in refining his or her understanding. In their assertion that students often need nonstandard representations in order to support their mathematical problem solving, Greeno and Hall (1997) highlighted this need for teacher development even more. If teachers are to fully support their students, they need to be able to understand how students are using graphical elements to not only explain their answers but also to solve problems.

We believe that InterMath provides participants with opportunities to develop these kinds of dispositions toward graphical representations as well as to refine their ability to interpret a wide range of representations. While the investigations and technologies used in InterMath do inherently support more traditional forms of representation, they also do promote multiple forms of representation. In classes, participants are encouraged, but not required, to use one or more technologies for their investigations; InterMath instructors often demonstrated two or three different approaches to solving the investigations, each with their own use of representations. Participants experienced the same kinds of teaching and learning opportunities we hope they will develop for their students.

It is our view that the use of graphical elements should greatly enhance the problem solver's ability to successfully complete an investigation. Yet in our sample, this was not necessarily true. This study, therefore, considers why teachers who were using one or more visual representations in their write-ups used mathematically inappropriate approaches and/or got wrong answers. For the purposes of this study, we consider the following questions: How did participants use graphical representations in their problem-solving processes? How did the graphical representations allow the participants to stray from correct or appropriate mathematical approaches and/or fail to reach correct or appropriate solutions?

Methods

This post-hoc study examined InterMath participants from five InterMath courses taught between 2001 and 2004. The workshops lasted between 1 and 15 weeks and included 3 to 24 participants each. The content in each course varied; in some cases all four strands (Number Sense, Algebra, Geometry, Data Analysis) were included, whereas in others only one strand was emphasized.

We examined all of the participants' write-ups in the smaller classes (n < 10). In the larger classes, 25–30% of the participants were randomly selected. In all, 236 write-ups from 27 participants were coded into 4 categories: correct math/correct answer (CM/CA), correct math/incorrect answer (CM/~CA), incorrect math/correct answer (~CM/CA), and incorrect math/incorrect answer (~CM/~CA). See Table 1 for the breakdown of the write-ups in terms of mathematics and answers.

Table 1

Distribution of Write-Ups in Terms of Correctness of Mathematics and Answers

	Correct Mathematics (CM)	Incorrect Mathematics (~CM)
Correct Answer (CA)	170	22
Incorrect Answer (~CA)	23	21

Each write-up was examined by two researchers independently and coded based on the elements above. The two analyses were then compared and a consensus was reached when there was a disagreement. In all cases, there was 100% inter-rater agreement before the analysis proceeded.

Table 2

Descriptions of Representation Categories and Coding Strategies.

Type of Graphical Representation	Definition (Adapted from Monk, 2003)	Coding
Communicate	Graphic is used to convey a meaning, or express one's ideas	Write-up could have been done without graphic; discussion of graphic occurs before graphic; graphic is referred to, i.e. "as you can see from the diagram"
Make sense	Graphic is used in the understanding of the problem or in the process of finding a solution	Reference to "I" or "we" implying a collaborative effort; graphic referred to throughout investigation; discussion of graphic occurs after graphic; graphic used to find solution
Cosmetically enhanced	Graphic is neither appropriate nor relevant to the investigation or solution	Randomly placed; graphic without explanation or reference; graphic not appropriate/relevant; used for organization purposes only; used as filler

Of the 236 write-ups, over one-fourth of them, 66, had incorrect mathematics and/or an incorrect answer. Further study showed that of these 66 write-ups, 62 (94.93%) used some sort of graphical representation (i.e. any kind of graph or diagram). By contrast only 48 (28.24%) of the remaining 170 write-ups (CM/CA) used graphical representation. Therefore, we focused our attention on these 62 write-ups that included some level of incorrectness.

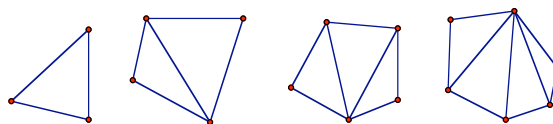*Identifying Types of Graphical Representations*

Aside from Monk's (2003) two roles of graphical representations as mentioned above, we recognized a third role from our analysis, one in which the graph was used to cosmetically enhance their write-ups. Each write-up was coded into only one category: *communication, make sense*, or *cosmetically enhanced*. Table 2 displays the definitions of each category and describes how the representations were coded. All four members of the research team were trained on a subset of the write-ups (approximately 15) to reach consensus on definitions of categories. Then one member of the team coded each remaining write-up. A sampling of the codes underwent inter-rater reliability and in all cases where there were initial inconsistencies a 100% agreement was reached.

The following examples chosen from the 62 write-ups provide further details about the meanings of the categories. These write-ups were taken verbatim from the participants' portfolios, including any misspelled words, grammatical errors, and inconsistencies with the graphic and the discussion of the graphic. They are taken from 2 of the 5 courses and vary in their level of correctness (one shown with CM/~CA, one shown with ~CM/CA, and one shown with ~CM/~CA).
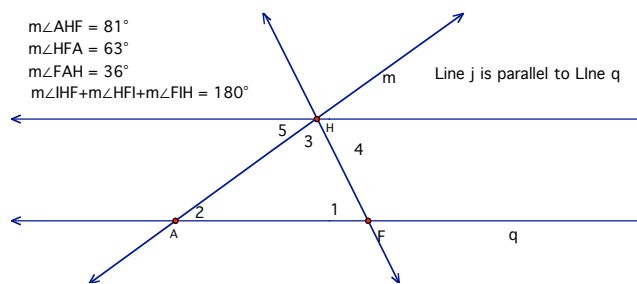
Communication example. The following example by participant G8[3] was coded as having incorrect mathematics and an incorrect answer resulting from the

author's assumption that the given polygons are regular. The write-up illustrates the communication category due to the discussion of the graphic coming after the graphic. The graphic is referred to as the participant is trying to communicate to the reader what can be seen in the graphic (see Appendix A for the entire write-up).

Investigation: What is the sum of the angles of a triangle? Of a quadrilateral? Of a pentagon? Of a hexagon? What is the sum of the angles in convex polygons in terms of the number of sides?

Write-up: When using Geometer Sketchpad to create a triangle that is formed by having two transversals intersect a set of parallel lines, students can then use the properties that they have learned about angles to determine the sum of the angles of a triangle.

m∠AHF = 81°
m∠HFA = 63°
m∠FAH = 36°
m∠IHF+m∠HFI+m∠FIH = 180°

Line j is parallel to Line q

Because lines j and q are parallel to one another, then lines x and m are transversals to these parallel lines. Thus, angle 1 and angle 4 are congruent because they are alternate interior angles. Similarly, angle 2 and angle 5 are congruent. Therefore, the following angles are congruent:

Angle 1 = Angle 4, Angle 2 = Angle 5, Angle 3 = Angle 3

Conclusion: We can conclude that Angle 5 + Angle 3 + Angle 4 = 180 degrees since these three angles form a straight angle. From the above conclusion, we see that: Angle 1 + Angle 2 + Angle 3 = Angle 4 + Angle 5 + Angle 3 = 180
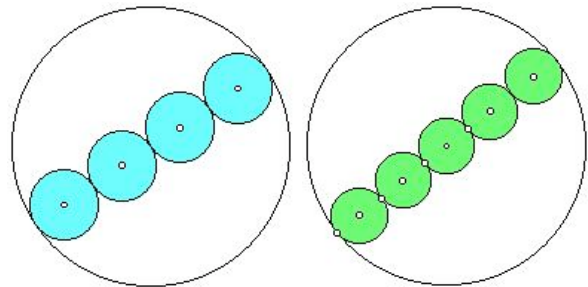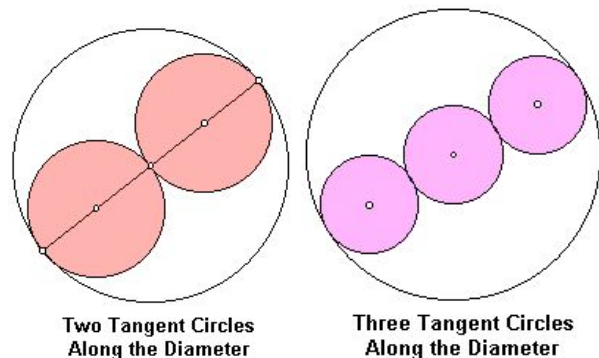
Thus, we can conclude that the sum of the measures of the three vertex angles in a triangle is 180 degrees.

G8 uses the graph of two parallel lines cut by two transversals as a means to illustrate to the reader that the angles are congruent due to alternate interior angles. Since the graphic is not used as a means to make sense of the mathematics, and it appears that it is appropriate for the discussion that follows in the write-up, it was coded as *communication*.

Making sense example. The following example by participant G7 was coded as having incorrect mathematics and a correct answer because the answer is written as a ratio of 1:2 and is said to be equivalent to 0.496, showing that there is lack of understanding of rounding and truncating numbers. The write-up illustrates making sense because the participant uses the first person to reason through the investigation by creating a representation and then "talking" through it (see Appendix B for the entire write-up).

Investigation: …How does the combined area of all of the shaded circles relate to the area of the entire circle?

Write-Up: I constructed circles using 2, 3, 4, and 5 smaller tangent circles along the diameters in Geometer's Sketch Pad. Those constructions are shown below:

Two Tangent Circles Along the Diameter   **Three Tangent Circles Along the Diameter**

Four Tangent Circles Along the Diameter   **Five Tangent Circles Along the Diameter**

I used Excel to record the data about the areas of each circle and its tangent circles. That table is below and is colored to correspond to the constructions above:
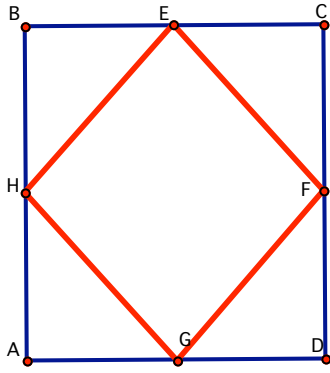
Area of Large Circle	Area of Small Circle	# of Small Circles	Small Circles Comb. Area	Small/Lrg Circle Ratio
3.14	0.78	2	1.56	0.**496**815287
3.14	0.35	3	1.05	0.**334**394904
3.14	0.2	4	0.8	0.**25**477707
3.14	0.13	5	0.65	0.**207**006369

In each situation the relationship formed between the circle and its tangent circles along its diameter could be closely described in terms of the number of tangent circles. When there were two tangent circles, the area relationship was 1:2 or .496. When three, the relationship was 1:3 or .33, etc.

Cosmetically enhanced example. The following example by participant C6 was coded as having correct mathematics and an incorrect answer. Despite the fact that the participant refers to the graphs in such a way that might imply that the graphical representations are being used as a tool to communicate, further analysis revealed that what the participant wrote and what was displayed were not in alignment (i.e. no measures were taken to show that the quadrilateral referred to in the graphic was actually a rectangle). Therefore, the write-up depicts a cosmetically enhanced write-up (see Appendix C for the entire write-up).
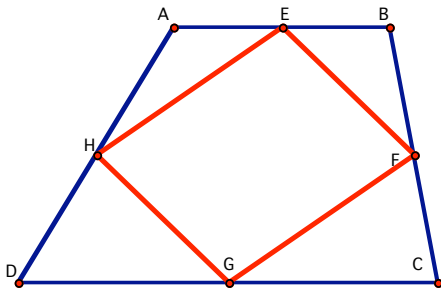
Investigation: A number of investigations can be done involving quadrilaterals. One investigation that can be explored deals with drawing an original quadrilateral. Then by marking the midpoints of the sides and connecting the midpoints with line segments to create an inscribed quadrilateral.

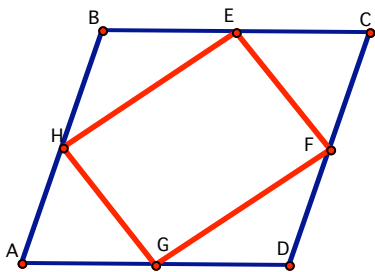Write-up: This can be done using GSP. An example can be seen in the sketch below.

One idea that can be explored is if the shape of the new quadrilateral depends on the shape of the original quadrilateral. One could determine if the two quadrilaterals will have the same shape or different shapes. This could be explored using GSP. In GSP, one could drag the different vertices of the original quadrilateral to form different shapes.

Examples of this using GSP are shown below.

The image above seems to show that if the original quadrilateral is a trapezoid, the smaller quadrilateral will be a rectangle.

The image above seems to show that is the original figure is a rhombus; the second quadrilateral will again be a rectangle.

It appears in the above write-up by C6 that in the construction of the various quadrilaterals little discussion was presented for proving the internal quadrilateral is a rectangle. Since C6 showed little indication of using the graphics as a means for making

sense of the mathematics or as a means to communicate a point, the write-up was coded as cosmetically enhanced.

Findings and Discussion

Table 3 shows the distribution among the types of incorrect write-ups as well as the uses of graphical representations in the incorrect write-ups.

Table 3

Distribution of Types of Incorrect Write-Ups and Uses of Graphical Representations

	Communication	Making Sense	Cosmetically Enhanced	Total
Correct Mathematics Incorrect Solution	6	8	9	23
Incorrect Mathematics Correct Solution	6	7	6	19
Incorrect Mathematics Incorrect Solution	2	7	11	20
Total	14	22	26	62

Communication

Fourteen of the 62 incorrect write-ups (22.6%) included graphical representations for the purpose of communicating mathematics. These graphical representations were more likely than the other types of representations to result in either correct mathematics or a correct solution. Only 2 out of the 14 write-ups using graphics for communication (14.3%) had both incorrect mathematics and an incorrect answer. Participants that used these representations included an explanation to either the investigation or the solution prior to the graphical representation in their write-up. The representation in these instances did not help them complete the investigation, but rather it served as an additional way of representing the investigation. However, in these write-ups either the discussion of the mathematics or the solution was incorrect.

Making Sense

Twenty-two of the 62 incorrect write-ups (35.5%) included representations that were used to make sense of the mathematics and generate a solution to the investigation. Fifteen of the write-ups in this category (68.2%) had either incorrect mathematics or an incorrect answer only. In these 15 write-ups, participants portrayed an understanding that the

representation was going to be used as a tool to help them reach a solution. In most of these write-ups, participants correctly explained the mathematics in the investigation or the process of finding a solution, but used the representation erroneously or did not correctly interpret the representation. In the seven write-ups that had both incorrect mathematics and an incorrect answer, participants explained that the representation would lead them to a solution, but their discussion of mathematical concepts and their solution were incorrect or incomplete.

Cosmetically Enhanced

Twenty-six of the 62 incorrect write-ups (41.9%) included cosmetically enhanced graphical representations. Eleven of those 26 write-ups (42.3%) had both incorrect mathematics and an incorrect solution. In these cases, participants' write-ups were brief, as they did not explain their process or discuss their solution, and the write-ups included a graphical representation that did not seem to enhance their work with the investigation. We speculate that participants felt obligated to use technology to create graphical representations of the investigations (e.g. a graph in Excel, a Geometer's SketchPad sketch), and so technology was being used just for the sake of using technology.

Teachers' Use of Representations

One of InterMath's goals is to shift participants' thinking to a more constructivist view of mathematics by using technology to generate representations that will help participants make sense and communicate the mathematics embedded in the investigations. We speculate that in the "cosmetically enhanced" write-ups, the participants felt compelled to include technology, and therefore a representation, because of the emphasis on technology in InterMath courses. We assert that in these cases, the participants created a representational graphic without a clear sense of the type of representation that should be created or how it should be used to reach a solution.

The idea that students think that technology (e.g. calculators and computers) will provide them with answers is a concern many educators share. It is our experience that teachers are concerned that technology does the work for the student and/or that the student accepts the answer without question because the technology generated it. We believe that this tendency occurred for the InterMath participants in the form of cosmetically enhanced representations. These write-ups provide evidence that participants without a sense

of the mathematics or a way of finding a solution used technology to generate a representation in an effort to miraculously come to one.

Write-ups that used representations for the purposes of communicating and making sense of mathematics were not flawless either. Participants who used representations for communication did not always have an accurate grasp of the mathematical concepts, made careless errors while reaching a solution, or wrote a solution based on something not visible in the graphic. Representations for making sense also led to incorrect write-ups. Participants used these representations to lead them towards a solution, but the representations were often incorrect (e.g. dimensions of a geometric figure, pattern in an Excel spreadsheet). Further, the interpretation of these representations led to incorrect solutions.

Implications for Professional Development

The data discussed in this article illuminates dilemmas concerning professional development for mathematics teachers. First, professional developers need to be more explicit in guiding teachers through the use of graphical representations. InterMath is designed to be very learner-centered. Participants have the freedom to select investigations, their approach to completing investigations, the technology that they employ and how they write up their solution. As seen in this paper, this approach can be problematic. While instructors serve as a model and guide students through a few investigations, our findings suggest that more guidance and explicit attention should be given to the use of representations. In our view, this is not a paradox, but a very real part of learning how to structure a learner-centered professional development program. That is, the learners need to own aspects of their learning, however, the instructor needs to be sensitive to the scaffolds the teachers need in order to be successful.

Second, participants need opportunities to engage in the process of effectively using mathematical representations. This process extends from choosing the representation that will be created to interpreting the representation to find a solution. This builds on Greeno and Hall's (1997) contention that teachers need professional development that prepares them to support their students' use of representations to solve mathematical problems. Greeno and Hall focus on the use of non-standard representations, which do not include Geometer's SketchPad sketches and Excel spreadsheets. Still, we posit that teachers need to know how to effectively generate and use these

representations in order to support their students' use of them.

Our findings indicate that, while teachers did use technology to generate representations, their use of the representations was not always what the InterMath team had hoped. Specifically, representations were used for the sake of using them or were included as an add-on at the end of their write-up. Further, efforts are needed to help teachers understand the value of representations as a tool for communicating and making sense of mathematical concepts.

Implications for Research

While this study highlights the use of graphical representations in incorrect mathematical write-ups, the findings only show half of the picture. An examination is needed of correct write-ups to see how graphical representations were used in those write-ups. While we hypothesize that the correct write-ups included mostly representations for the purpose of sense making, we have not conducted the necessary analysis.

Further analysis is also needed to examine the incorrect write-ups included in this paper. More information is needed about whether and how participants justified the use of a representation in their write-up. Our hypothesis is that the participants were attempting to fit the use of representations into their belief structures about mathematics. For example, many write-ups that had cosmetically enhanced representations or used representations for communication included algebraic work and an explanation of how to use paper and pencil to solve the problem. This suggests that the participants may have believed that they had to include a graphic even though they were not using it in a way that promoted understanding. Another examination of the write-ups might help us better understand this trend.

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[1] For a more detailed description of InterMath, refer to the InterMath website at http://intermath.coe.uga.edu.

[2] By *incorrect mathematics*, we mean that the participant's write-up contains some mathematical errors, which may or may not lead to the correct answer. By *incorrect answer*, we mean that part or all of the participant's answer is incorrect.

[3] Each participant has been assigned a unique identifier. The letter refers to the course and the number refers to which person this is within that course.
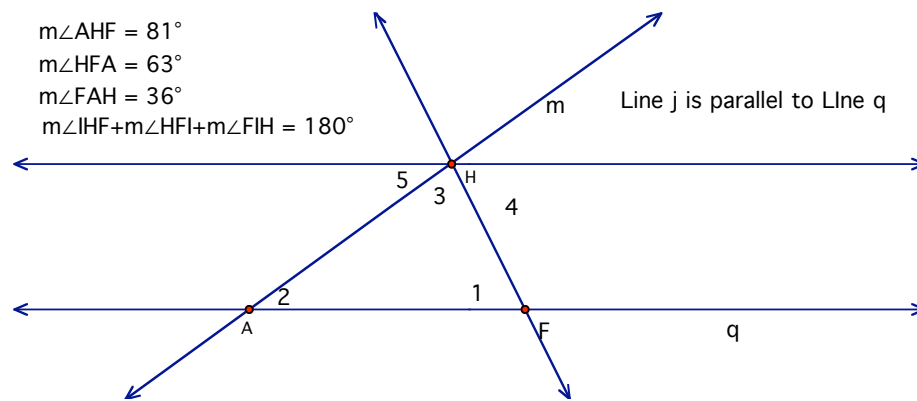
<u>Investigation</u>: What is the sum of the angles of a triangle? Of a quadrilateral? Of a pentagon? Of a hexagon? What is the sum of the angles in convex polygons in terms of the number of sides?
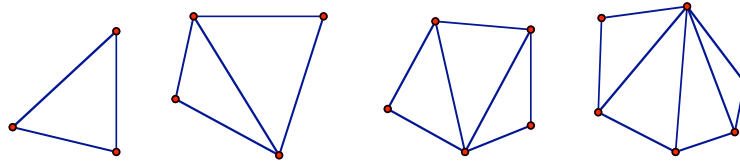
<u>Write-up</u>: When using Geometer Sketchpad to create a triangle that is formed by having two transversals intersect a set of parallel lines, students can then use the properties that they have learned about angles to determine the sum of the angles of a triangle.

Click here for InterMath dictionary if unfamiliar with terminology.

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m∠HFA = 63°
m∠FAH = 36°
m∠IHF+m∠HFI+m∠FIH = 180°

Line j is parallel to LIne q

Because lines j and q are parallel to one another, then lines x and m are transversals to these parallel lines. Thus, angle 1 and angle 4 are congruent because they are alternate interior angles. Similarly, angle 2 and angle 5 are congruent. Therefore, the following angles are congruent:

        Angle 1 = Angle 4   Angle 2 = Angle 5   Angle 3 = Angle 3

Conclusion: We can conclude that Angle 5 + Angle 3 + Angle 4 = 180 degrees since these three angles form a straight angle. From the above conclusion, we see that: Angle 1 + Angle 2 + Angle 3 = Angle 4 + Angle 5 + Angle 3 = 180
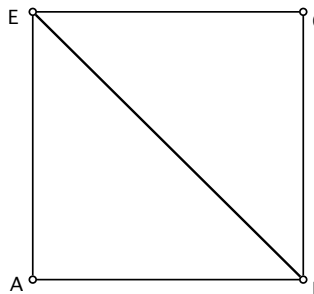
Thus, we can conclude that the sum of the measures of the three vertex angles in a triangle is 180 degrees.

We can use the angle sum in a triangle property to find the measure of the vertex in a regular n-gon. By forming triangles within a polygon, we can find the sum of the vertex angles in any polygon.

In a quadrilateral, we formed two triangles. Since each triangle has 180 degrees, and there were two triangles formed in the quadrilateral, we can conclude that the sum of the measures of the vertex angles is 2 x 180 = 360. Furthermore, the individual angle measures of each vertex angle within the quadrilateral would be 360/4 = 90 degrees.

Click here for interactive sketch (shown below).

m∠EBA = 45°
m∠AEB = 45°
m∠EAB = 90°
m∠EBA+m∠AEB+m∠EAB = 180.00°
m∠ECB = 90°
m∠CEB = 45°
m∠CBE = 45°
m∠ECB+m∠CEB+m∠CBE = 180.00°

(m∠EBA+m∠AEB+m∠EAB)+(m∠ECB+m∠CEB+m∠CBE) = 360.00°

In a pentagon, we formed three triangles. Since each triangle has 180 degrees and there were three triangles formed in the pentagon, we can conclude that the sum of the measures of the vertex angles is 3 x 180 = 540 degrees. Furthermore, the individual angle measure of each vertex angle within the pentagon would be 540/5 = 108 degrees.

In a hexagon, four triangles can be formed. Since each triangle has 180 degrees and there were four triangles formed in the hexagon, we can conclude that the sum of the measures of the vertex angles is 4 x 180 = 720 degrees. Furthermore, the individual angle measures of each vertex angle within the hexagon would be 720/6 = 120 degrees.

Using the relationship shown above, we created a spreadsheet to show how to calculate the sum of the angles in convex polygons in terms of the number of sides.

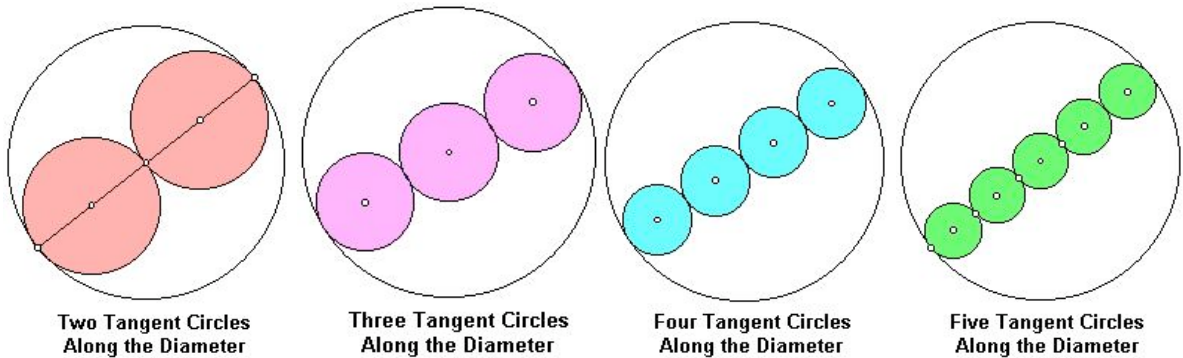Click here to view spreadsheet (shown below).

n	Sum of Angles	Vertex Angles
3	180	60
4	360	90
5	540	108
6	720	120
7	900	128.5714286
8	1080	135
9	1260	140
10	1440	144
11	1620	147.2727273
12	1800	150

In each instance, the number of triangles formed in each polygon is equal to the number of sides in the polygon minus two. Therefore, to find the sum of the angles in any convex polygon in terms of the number of sides, the formula $180(n - 2)$ can be used.

Problem: Along the diameter of a circle you can construct circles with equal radii that are tangent to each other. The outermost circles in the string of circles will be tangent to the large circle. (Tangent means that the circles touch each other but do not cross each other, nor do they leave gaps.) How does the combined area of all of the shaded circles relate to the area of the entire circle?

Solution: I constructed circles using 2, 3, 4, and 5 smaller tangent circles along the diameters in Geometer's Sketch Pad. Those constructions are shown below:

**Two Tangent Circles
Along the Diameter** **Three Tangent Circles
Along the Diameter** **Four Tangent Circles
Along the Diameter** **Five Tangent Circles
Along the Diameter**

I used Excel to record the data about the areas of each circle and its tangent circles. That table is below and is colored to correspond to the constructions above:

Area of Large Circle	Area of Small Circle	# of Small Circles	Small Circles Comb. Area	Small/Lrg Circle Ratio
3.14	0.78	2	1.56	0.496815287
3.14	0.35	3	1.05	0.334394904
3.14	0.2	4	0.8	0.25477707
3.14	0.13	5	0.65	0.207006369

In each situation the relationship formed between the circle and its tangent circles along its diameter could be closely described in terms of the number of tangent circles. When there were two tangent circles, the area relationship was 1:2 or .496. When three, the relationship was 1:3 or .33, etc.

Extension: If you were to walk along the circumferences of all the small circles, there is almost a 1:1 relationship to the circumference of the original circle. Examine the table below. Again it is color-coordinated with the constructions above.
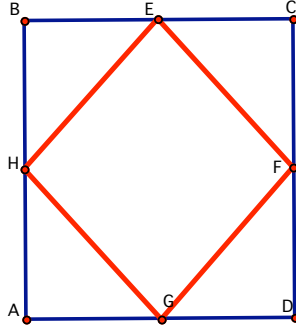
Large Circle Circumference	Small Circle Circumference	# of Small Circles	Small Circles Total Circum.	Small/Lrg Circle Ratio
6.28	3.14	2	6.28	1
6.28	2.1	3	6.3	1.003184
6.28	1.57	4	6.28	1
6.28	1.26	5	6.3	1.003184

It is interesting that when there are an even number of tangent circles along the diameter, the relationship is precisely1:1. However, when there are an odd number of tangent circles the relationship is not precisely one to one. The sum of the circumferences of the tangent circles is slightly more than the original circle, but appears to always be the same difference.
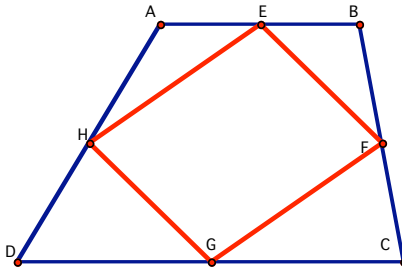
Investigation: A number of investigations can be done involving quadrilaterals. One investigation that can be explored deals with drawing an original quadrilateral. Then by marking the midpoints of the sides and connecting the midpoints with line segments to create an inscribed quadrilateral.

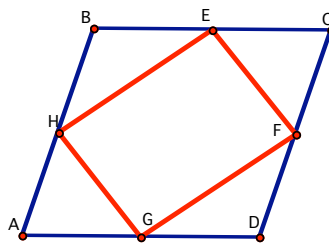Write-up: This can be done using GSP. An example can be seen in the sketch below.

One idea that can be explored is if the shape of the new quadrilateral depends on the shape of the original quadrilateral. One could determine if the two quadrilaterals will have the same shape or different shapes. This could be explored using GSP. In GSP, one could drag the different vertices of the original quadrilateral to form different shapes.

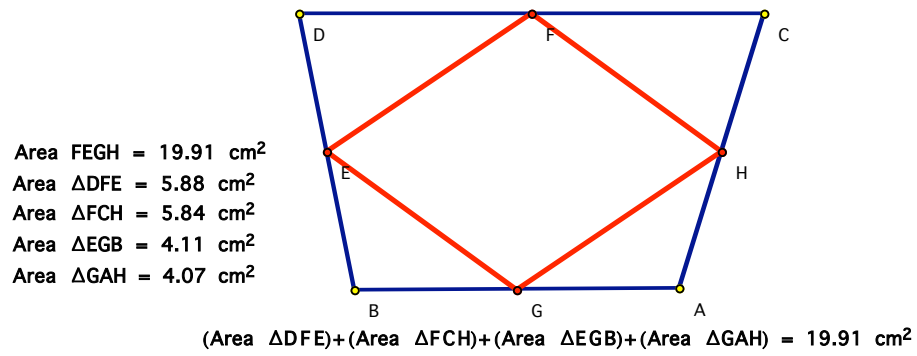Examples of this using GSP are shown below.

The image above seems to show that if the original quadrilateral is a trapezoid, the smaller quadrilateral will be a rectangle.

The image above seems to show that is the original figure is a rhombus; the second quadrilateral will again be a rectangle.

Another idea that could be explored is if the four smaller triangles created near the vertices of the original quadrilateral have the same area. This can be explored using the measure and calculate features of GSP. An example of this can be seen below.

Area FEGH = 19.91 cm²
Area ΔDFE = 5.88 cm²
Area ΔFCH = 5.84 cm²
Area ΔEGB = 4.11 cm²
Area ΔGAH = 4.07 cm²

(Area ΔDFE)+(Area ΔFCH)+(Area ΔEGB)+(Area ΔGAH) = 19.91 cm²

The calculations show that the four small triangles do not have the same area. However, the calculations do show that the sum of the areas of the four triangles is equal to the area of the smaller quadrilateral.

GSP could be used to explore a number of different aspects of this problem. Examples would include, is there a constant ratio between the perimeter of the smaller quadrilateral and the larger quadrilateral. Another investigation could be done comparing the areas of the two quadrilaterals.