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Ninth Grade Students Studying the Movement of Fish to Learn about Linear Relationships: The Use of Video-Based Analysis Software in Mathematics Classrooms

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The use of technology to create multiple representations of a concept has become one of the significant instructional environments that the National Council of Teachers of Mathematics (2000) suggests strongly for mathematics teachers to consider. One example of this type of environment is educational software with linked multiple representations. An activity for both linked and semi-linked versions of multi-representational software which was used in a dissertation study is presented along with two ninth grade algebra students' responses in order to provide an example of possible uses and effects of semi-linked and linked computer software in mathematics classrooms. It was also aimed to make connections between practice and research. The conclusion of this study was that semi-linked representations could be as effective as linked representations and that there was a role for each in different situations, at different levels, and with different mathematical concepts.

> All aspects of a complex idea cannot be adequately represented within a single notation system, and hence require multiple systems for their full expression, means that multiple, linked representations will grow in importance as an application of the new, dynamic, interactive media. (Kaput, 1992, p. 530)

The utilization of technology for exploring multiple representations has received increased attention in mathematics education in the last decade. The National Council of Teachers of Mathematics (NCTM, 2000) states, "new forms of representation associated with electronic technology create a need for even greater instructional attention to representation" (p. 67). By implementing advanced technologies, like movies, new forms of representations are possible in mathematics classrooms. Interactive and dynamic linkages among multiple representations provide new capabilities that traditional environments, such as blackboard and paper-and-pencil, cannot (Ainsworth & Van Labeke, 2004). Linked multiple representations are a group of representations in which altering a given representation automatically updates every other representation to reflect the same change; semi-linked representations are defined as those for which the corresponding updates of changes within the representations are available only upon request and are not automatically updated (Rich, 1996). Educational software is one environment that allows for these linkages (Hegedus & Kaput, 2004).

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VideoPoint (Luetzelschwab & Laws, 2000) is a video-based motion analysis software tools that allows users to collect data from digital movies and perform calculations with that data, such as finding the distance between points (see Figure 1). To accommodate the request of this author, the software developer made changes to create the fully linked and semi-linked versions of VideoPoint. VideoPoint links traditional representations–graphs, tables, equations–but also offers a novel representation–the movie. Although VideoPoint was designed as linked representational software, the linkage for the table representation was not two-way. When the user makes changes in one representation, the table as an example, another representation, like the graph, is highlighted to reflect the change. At this stage the linkage between the table and the graph is one-way. In order to make this linkage two-way, the user should also be able to make a change in the graph and see its effects on the table. The graph, table, and movie representations are linked two-way in the fully linked version of the software. Thus, when the user clicks on a point in those representations, the corresponding data points in the other two representations are highlighted. This can be observed in Figure 1 among the table, the graph, and the movie.

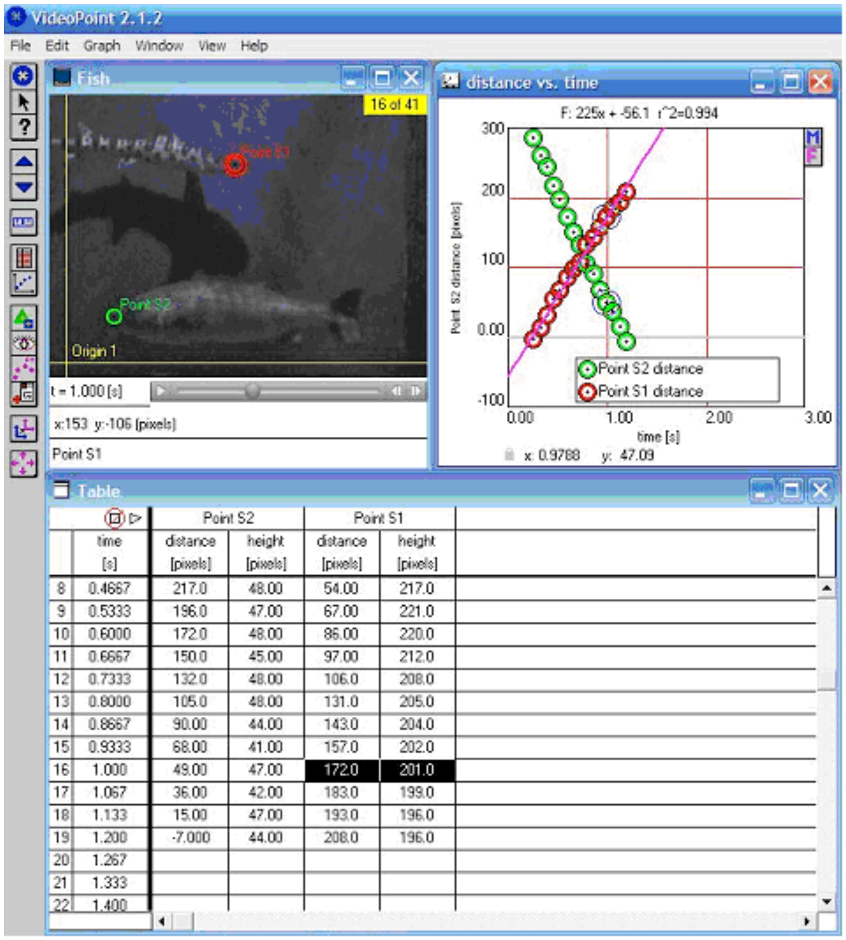


Figure 1. A screenshot from VideoPoint (Fish movie is obtained from Graph Action Plus)

In this study, a movie of two fish swimming towards each other from opposite sides of the screen was used. The distances between the fish's head and the left hand side of the screen were measured over time and reported in the four representations. The movie, graph and table show the fish's positions at one second. When the user makes a change in any of these representations, all other representations are updated to reflect the change. For instance, when one clicks on a different cell in the table or advances to the next frame in the movie the corresponding points at the new position are highlighted in all other representations.

When the user of the linked version clicks to see the algebraic form (the equation of best fit) of the phenomena, the line of best fit is graphed in the graph window and its equation appears above the graph automatically (see Figure 1). On the other hand, the user of the semi-linked version is not able to see any updates when he or she clicks on one representation or makes changes in any representation. The only linkage that is available in the semi-linked version is between the graph and equation forms. When the user estimates the coefficients in the algebraic form, he or she has an option to see the graph of the predicted equation (see Figure 2).

In this example, the student is creating a best-fit line for one set of data points (with a positive slope) on the graph by modeling. She changed the slope from 70 to 100 in order to obtain a steeper slope. Being able to see the graph of the previous model with the current model helped the student relate the algebraic form and the graph more effectively by comparing before and after pictures.

Review of Literature

Research studies and practitioner articles indicate that the use of multiple representations with or without technology may help students to construct mathematical concepts in more empowering ways. Articles without the use of technology emphasize the use of various representations during instruction (Clement, 2004; Harel, 1989; Knuth, 2000; Suh & Moyer, 2007). Technology oriented studies, on the other hand, utilized computer (Harrop, 2003; Hegedus & Kaput, 2004; Jiang & McClintock, 2000; Noble,

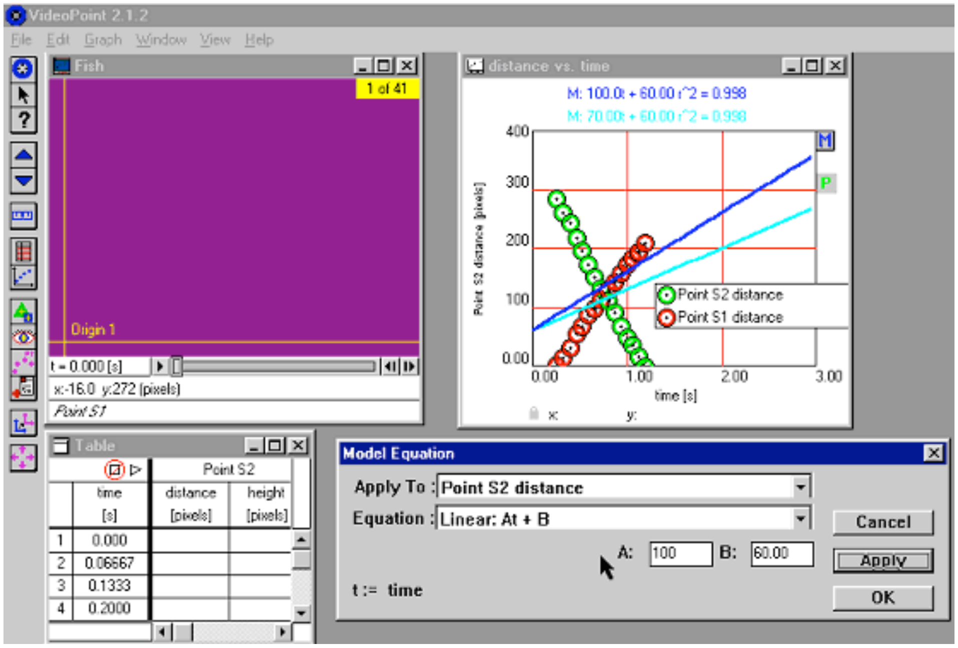


Figure 2. A screenshot from VideoPoint—Estimating the Coefficients

Nemirovsky, Wright, & Tierney, 2001; Suh & Moyer, 2007; van der Meij & de Jong, 2006) or calculator (Herman, 2007; Piez & Voxman, 1997; Ruthven,1990) technology in order to investigate the effects of multiple representations on learning.

Goldenberg et al. (1988) argue that "multiple linked representations increase redundancy and thus can reduce ambiguities that might be present in any single representation" (p. 1). Therefore, multiple representations can facilitate understanding. Kaput (1986) also notes,

> By making visually explicit the relationships between different representations and the ways that actions in one have consequences in the others, the most difficult pedagogical and curricular problem of building cognitive links between them becomes much more tractable than when representations could be tied together only by clumsy, serial illustration in static media. (p. 199)

Goldenberg (1995) describes other advantages of computer-based multiple linked representations:

> - The interactive nature of the computer allows students to become engaged in dialogue with themselves.
> - Raising conflict and surprise [which leads to more thinking].
> - Affirming (if not paralleling) students' own internal multiple representations.
> - Helping us [educators and researchers] distinguish between students' expressed models and the ones they act on.
> - [Providing] immediacy and accuracy with which the computer ties two or more representations together.
> - [Helping] students themselves multiply represent their concepts. (pp. 159–161)

In these kinds of environments, the computer performs the required computations, thus leaving the student free to alter the representations and to monitor the consequences of those actions across representations. Moreover, the ability to represent the same mathematical concept using many representations and to make explicit the relationships among these representations by dynamically linking them to each other have been discussed as reasons for the effectiveness of these learning environments (Kaput, 1986, 1994).

Research utilizing linked multiple representational software creates two groups of studies: comparative studies and case studies. The former ones (Rich, 1996; Rosenheck, 1992) compared groups of students by using different technologies in treatments. Due to the crucial differences in the environments (e.g., computer versus non-computer or calculators versus computers), it is difficult to draw clear conclusions. In fact, results of these studies showed no significant differences between groups. On the other hand, the case studies indicate more encouraging results because of the use of linked multiple representation software (Borba, 1993;

Borba & Confrey, 1993; Lin, 1993; Rizzuti, 1992; Yerushalmy & Gafni, 1992). The linkage among representations in the computer-based environment was a powerful tool that provided a visual environment for students to develop and test their mathematical conjectures. However, Ainsworth (1999) and van der Meij and de Jong (2006) discussed possible disadvantages of multiple linkage representations: dynamic linking may put students in a passive mode by doing too much for them and a cognitive overload may result from providing too much information.

Theoretical Framework

Dienes' *multiple embodiment principle* is a prominent theory emphasizing multiple representations in mathematics education. The multiple embodiment principle suggests that conceptual learning of students is enhanced when students are exposed to a concept through a variety of representations (Dienes, 1960). Additionally, Kaput (1995) notes the relationship between external and internal representation: when one moves from mental operations (internal representations) to physical operations (external representations), "one has cognitive content that one seeks to externalize for purposes of communication or testing for viability" (p. 140). On the other hand, in moving from physical operations to mental operations, "processes are based on an intent to use some existing physical material to assist one's thinking" (p. 140). Students' pre-existing knowledge structures influence the external representational tools they use to perform mathematics tasks and to communicate mathematically. Conversely, the representational tools available to students influence their mathematical knowledge.

Now the question is how understanding across multiple representations can be improved with educational technology. Kaput (1994) notes that physical links, such as those provided by a graphing calculator or a computer, could be beneficial in supporting students' construction of cognitive links:

> The purpose of the physical connection is to make the relationship explicit and observable at the level of actions in order to help build the integration of knowledge structures and coordination of changes. (p. 389)

Furthermore, Goldin and Kaput (1996) note,

> By acting in one of the externally linked representations and either observing the consequences of that action in the other representational system or making an explicit prediction about the second representational system to compare with the effect produced by actions in the linked representation, one experiences the linkages in new ways and is provided with new opportunities for internal constructions. (p. 416)

According to Piaget's theory (Piaget, 1952; Piaget & Inhelder, 1969), cognitive development is described as a process of adaptation and organization driven by a series of equilibrium-disequilibrium states. If everything is in equilibrium, we do not need to change anything in our cognitive structures. Adaptation occurs when the child interacts with his or her environment. The child is coping with his or her world, and this involves adjusting. Assimilation is the process whereby the child integrates new information into his or her mental structures and interprets events in terms of the existing cognitive structure, whereas accommodation refers to changing the cognitive structure. Adaptation is achieved when equilibrium is reached through a series of assimilations and accommodations. Organization is a structural concept used to describe the integration of cognitive structures.

Linked representational software gives students immediate feedback on the consequences of their mathematical actions with machine accuracy, but it may or may not engender the disequilibrium necessary for learning. Semi-linked software, by not showing the corresponding changes in other representations, forces students to resolve the dissonance in their cognitive structures by giving time to reflect or to ask questions about what kind of changes could result from a change in any representation. If their organization of knowledge is well established, they can deal with the question. However, if not, then they will need accommodations in their cognitive structures. Thus, a semi-linked representational environment puts students in a more active role as learners.

Purpose and Rationale

The studies reviewed above investigated various effects of multiple linked representation software. However, the present study focused directly on the effects of the linking property of the software on students' learning. Two groups of students in a classroom environment used different versions of the same computer software: fully linked and semi-linked. The goal was to see how this feature of the software affected their learning and understanding of the relationships between the representations and the mathematics content itself.

The major aim of this paper, however, is to present an activity for both linked and semi-linked versions of the software in order to demonstrate the use of the software with the aim of connecting practice with research. The purpose here is to offer an activity using video-based motion analysis software in a mathematics classroom and to suggest how to handle multiple representations within the activity. While doing that, the results of the dissertation research study are also provided in order to emphasize the connection between research and practice.

Methodology

In an eight-week period, 20 Algebra I students, separated into two groups, used VideoPoint: one group used linked representation software and the second group used semi-linked representation software. Four computer lab sessions were spaced out during the data collection period. Because this particular school schedules its classes for 78 minutes, one group was taken out of the classroom for a 35-minute computer lab during the first part of the class; then during the second part, the other group went to the computer lab.

This study used a mixed method design. Its aim is to "provide better (stronger) inferences and the opportunity presenting a greater diversity of divergent views [explanations]" (Tashakkari & Teddlie, 2003, pp. 14–15). Tashakkari and Teddlie (1998) note that "the term mixed methods typically refers to both data collection techniques and analyses given that the type of data collected is so intertwined with the type of analysis that is used" (p. 43). Data collection methods included mathematics pre-and post-tests, follow-up interviews after the mathematics post-test, clinical interviews in the computer lab at the end of the treatment, classroom and lab observations and document analysis (classroom materials, computer dribble files, exams, and assignments). A grounded survey was conducted at the end of the study in order to see students' opinions about mathematics, representations in general, and the computer environment.

A panel of experts assured the researcher of both the content and face validity of the instruments. Instruments were continuously updated according to feedback from students both during the pilot and throughout the actual study. As Tashakkari and Teddlie (1998) argued, the use of a mixed method design led this study to have data and methodological triangulation. Other techniques used to provide trustworthiness of this study were member check and peer debriefing.

The data obtained through clinical interviews will be used in this paper to provide an example of possible uses and effects of semi-linked and linked computer software in mathematics classrooms. The analysis of the data obtained from these clinical interviews was based on categorizing in order to investigate the emerging themes throughout.

The Use of Semi-linked and Linked Software in Mathematics Classroom

This section presents an activity using multiple-representational software. The parallel tasks for the linked and semi-linked versions are displayed in two columns. The activity included five main sections: namely an introduction section; three sections that focus on the graphical, tabular, algebraic representations separately; and a general questions section at the end. Two students' responses are provided, one using the linked version, the other student using the semi-linked version of the software. Even though just two students' responses are displayed below in the tables, general conclusions from the larger study and general comments about the use of different versions are also included in the narrative.

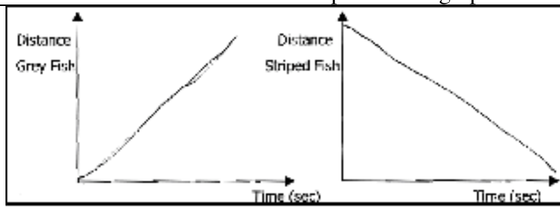
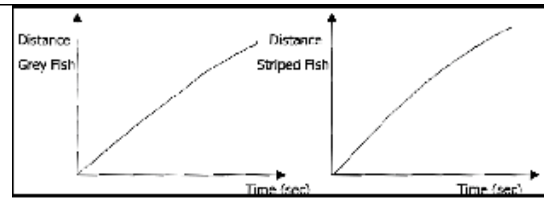
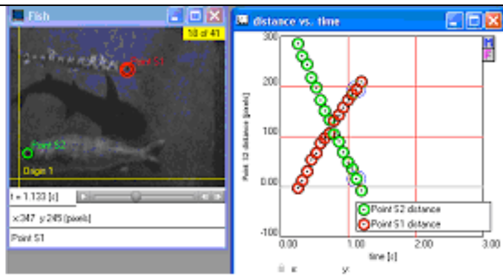
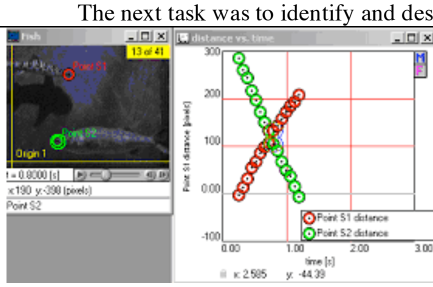
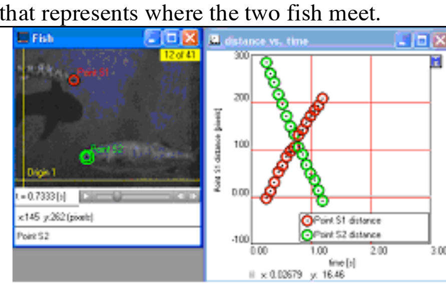
The lab activity started by watching a movie: two fish swimming at a constant rate across the screen towards each other. The fish movie was obtained from Graph Action Plus. A grey fish (the fish at the bottom of the screen) swims from right to left and a striped fish (the fish at the top of the screen) swims from left to right (see Figure 1). Students were asked general questions about the movie, such as, "How does the distance between the two fish change as they swim?" At this point, only the movie window was open on the screen. Typical responses included, "At first they got closer and closer together and then they got farther and farther away."

The second part of the activity focused on the graphical representation (see Table 1). First students were asked to create the graph of the phenomenon by paper and pencil after watching the movie. As Goldin and Kaput (1996) mentioned, asking students to "make an explicit prediction" (p. 416) before seeing the computer-produced result could be very effective in creating environments for students to construct linkages among representations (i.e., between the movie and the graph in this case). This approach was used throughout the activity with all representations.

When creating a graph after watching this movie or, more generally, when making predictions about a representation by using another representation, students need to recognize the outside information, select an appropriate schema, and create an answer to the question. This assimilation is described as recognitive assimilation in Piaget's theory, defined as

Table 1

Graphical Part of the Activity for Linked and Semi-Linked Versions of Video Point

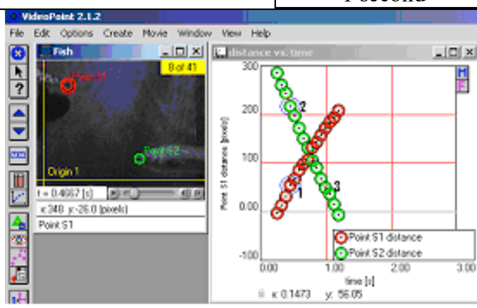
Linked Version	Semi-Linked Version
Students were asked to predict the graphs of each fish's distance from the left-hand side of the screen versus time.	
 Her graphs are switched. "As time went, [the grey fish] started farther away and it got closer and closer and the striped fish started out really close and it got farther and farther away."	 "As the time increase, both of the fish's distance increases actually. They both increase."
After sketching their graph, students opened a window to see what the computer-produced graph looked like and compared their graph to the computer's graph.	
 When she opened the computer graph, she used the linkage to find out which line belongs to which fish. She saw that her prediction was not correct by clicking on a data point on the graph. Then VideoPoint showed her a movie screen where the fish's labels were shown. She needed to accommodate this new information.	After seeing that the graphs were not as expected, we started discussing what was happening. The student focused on the distance between the fish instead of their distances from the left-hand side of screen separately. After this new information he said, "The striped fish's distance increases. This would be the decreasing one [showing the gray fish's graph]."
The next task was to identify and describe the point on the graph that represents where the two fish meet.	
 To find out where two fish meet on the graph, she used the linkage. There she could see that at the intersection of the two graphs, the movie frame showed that the two fish met.	 In this version, when the student clicked on the frame where the two fish meet, he could not see the corresponding data points on the graph.

considering reality and choosing an appropriate scheme (Montangero & Maurice-Naville, 1997). After making their predictions, students opened and observed the computer-produced graph. This gave them a chance to check their work; differences existed between both students' hand-produced graphs and the computer-generated graphs (see Table 1). The linked group student's graphs were switched. The grey fish was the one whose distance was decreasing whereas the striped fish was the one with increasing distance. The semi-linked group student, on the other hand, provided increasing graphs for both fish. Because of these discrepancies, the students may have experienced disequilibrium and needed to accommodate this new information. With the help of the software, the linked group student used the linkage and accommodated the new information. Even if the version of the software did not provide linkages among representations for the semi-linked group student, the information provided by the representations helped him to re-think his prediction and compare them with the computer-produced representation (see Table 1).

In the next task, students were asked to identify and describe the point on the graph that represents

Table 2

Tabular Part of the Activity for Linked and Semi-Linked Versions of VideoPoint

<table>
<tr><th>Linked Version</th><th>Semi-Linked Version</th></tr>
<tr><td colspan="2">First, students were asked their predictions about the table of values.</td></tr>
<tr><td>“The gray fish’s distance will decrease. The striped fish’s will increase.”</td><td>“The gray fish’s distance is going to decrease. The striped fish’s distance will increase because it is going away from the starting point.”</td></tr>
<tr><td colspan="2">Students were asked to fill out a table by using the graph.
<table>
<tr><th>Time (Seconds)</th><th>Grey Fish Distance</th><th>Striped Fish Distance</th></tr>
<tr><td></td><td></td><td>50 pixels</td></tr>
<tr><td></td><td>130 pixels</td><td></td></tr>
<tr><td>1 second</td><td></td><td></td></tr>
</table></td></tr>
<tr><td>

She had trouble reading values from the graph. After finding when the striped fish was 50 pixels away [point #1], to find what the distance of the other fish was at that time [point #2], she moved horizontally to the point #3 and read the distance of the other fish at another time. The use of linkage could be helpful. If she clicked on the point #1, she would see other fish’s data point highlighted [point #2].</td><td>He had the same trouble as the linked group student when reading the values from the graph. At this point, the linkage could be helpful for him to correct that mistake.</td></tr>
<tr><td colspan="2">We checked their answers with the computer-produced table in order for them to have feedback.</td></tr>
<tr><td colspan="2">Students were asked to identify and describe the point in the table which represents where the two fish meet.

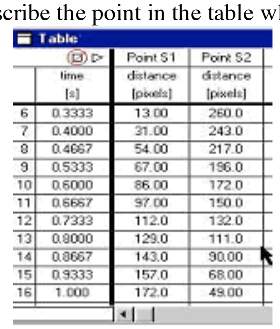



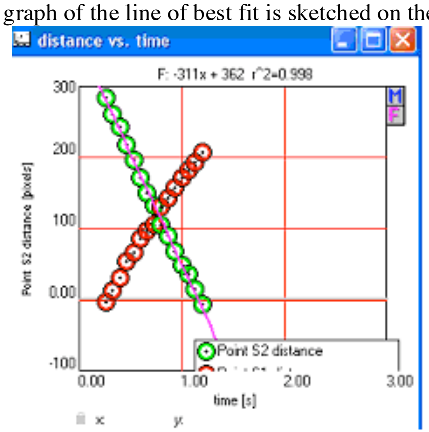
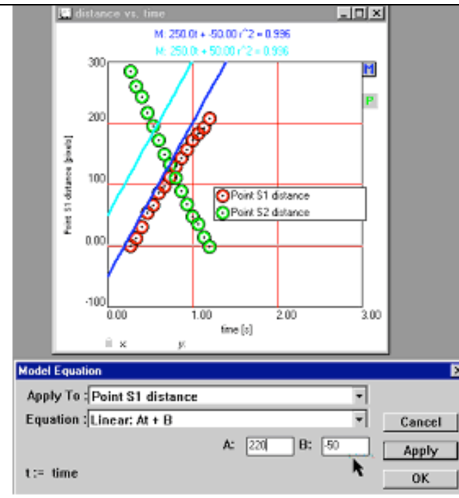
Table
<table>
<tr><th></th><th>time [s]</th><th>Point S1 distance [pixels]</th><th>Point S2 distance [pixels]</th></tr>
<tr><td>6</td><td>0.3333</td><td>13.00</td><td>260.0</td></tr>
<tr><td>7</td><td>0.4000</td><td>31.00</td><td>243.0</td></tr>
<tr><td>8</td><td>0.4667</td><td>54.00</td><td>217.0</td></tr>
<tr><td>9</td><td>0.5333</td><td>67.00</td><td>196.0</td></tr>
<tr><td>10</td><td>0.6000</td><td>86.00</td><td>172.0</td></tr>
<tr><td>11</td><td>0.6667</td><td>97.00</td><td>150.0</td></tr>
<tr><td>12</td><td>0.7333</td><td>112.0</td><td>132.0</td></tr>
<tr><td>13</td><td>0.8000</td><td>129.0</td><td>111.0</td></tr>
<tr><td>14</td><td>0.8667</td><td>143.0</td><td>90.00</td></tr>
<tr><td>15</td><td>0.9333</td><td>157.0</td><td>68.00</td></tr>
<tr><td>16</td><td>1.000</td><td>172.0</td><td>49.00</td></tr>
</table></td></tr>
<tr><td>“Because those numbers are where they were the closest, like at .7333 seconds, the striped fish was still like closer to the…[left hand side of the screen] than the grey fish was and by .8 seconds it was farther away than the grey fish was.”</td><td>“So like right there [showing .8 second on the table] 130 [pixels]. Right on the graph they crossed about right here [he reads the distance 130 from the graph] and then I just looked that closer to that [on the table].” At this point he constructed the linkage between the table and the graph by himself.</td></tr>
</table>

where the two fish meet. The linked group students could use the linkage and identify the point on the graph without needing more explanation (see Table 1). They sometimes only referred to the movie, saying something like, “Just look at the movie. This is the point where the two fish meet,” after double clicking on the graph. The semi-linked group students, on the other hand, did not have this kind of opportunity. This lack of linkage between the movie and the graphical representation seemed to force some of the semi-linked group students to provide richer explanations such as, “They are at the same place at the same time.”

A similar approach was followed for the tabular representation (see Table 2). First students were asked their predictions about the table of values. Then they were asked to complete a table by reading values for the graph (see Table 2). Both students in Table 2 had the same trouble—reading values from the graph. For instance, after the linked group student located the time that the striped fish was 50 pixels away on the graph successfully (labeled as point # 1 on the graph in Table

Table 3

Symbolic Part of the Activity for Linked and Semi-Linked Versions of VideoPoint

Linked Version	Semi-Linked Version
In the third section, students' were asked to make predictions about the slope and y-intercept of the algebraic form.	
"The grey fish has negative slope and the striped fish has positive slope" What about the y-intercepts? "I do not know"	He thought that the striped fish would have positive slope and the grey fish would have negative slope. Predictions for the y-intercept were not clear.
She accessed the equation of the line of best fit immediately with the F button next to the graph. The equation is shown at the top of the graph and also the graph of the line of best fit is sketched on the data points.  	 He predicted the coefficients of the equation with the modeling button. It did not take long for him to predict the equations; he used the computer feedback and proceeded accordingly. He could interpret how the changes in the algebraic form would affect the graph.
The next task was interpreting the differences in the equations of the two fish.	
She thought that the slopes of the two fish had different signs, "because one keeps getting closer to the point and the other one keeps getting farther away."	"The striped fish went away…so it [the distance] increased that has positive [slope] but the grey fish's distance decreased; that has negative slope."
Students were also asked to use the equations to determine the time that represents where the two fish meet.	
"I am not sure how to do it"	"I do not get [understand] the equation."

2), she was asked to locate the distance of the grey fish at that time (approx. 0.42 seconds). Instead of moving vertically to the point labeled # 2, she moved her cursor horizontally to the point labeled as #3 to read the distance of the grey fish at 1 second. The linkage could be helpful to both students. If the linked group student used the linkage and clicked on the point # 1, point #2 would be circled. However, the linked group student did not think of using the linkage at this point, and the semi-linked group student did not have this option.

When students were asked to identify and discuss the point in the table that represents where the two fish meet, the semi-linked group student was able to construct a linkage by using the information provided by the multiple representations. He used the graphical representation (that he used previously to answer a similar question) in order to interpret the tabular representation more effectively. The linked group student, on the other hand, attended to the distances of both fish from the left-hand side of the movie screen. At one data point, one fish was closer to the left-hand side of the screen and then at the next data point the other fish was closer to the left-hand side of the screen. So she concluded that the fish should meet between those data points.

The third section of the activity focused on the algebraic representation (see Table 3). Students were asked to make predictions about the symbolic representation of this phenomenon. This part was the most difficult section for the students. The two students in Table 3 were representative of many students who could not predict or even start to think about the symbolic form. Whereas linked group students had easy access to the algebraic form, the semi-linked group students needed to predict the coefficients of the equation. When the semi-linked group students entered their predictions, the line for their last two predictions appeared on the graph window. This feature of VideoPoint showed students how well their predictions fit the data points and how the changes in the algebraic form affected the graph (see Table 3). Because the computer software creates linkages between the symbolic and graphical representations, students can focus on how manipulating the algebraic form in a specific way causes changes in the graph (Kaput, 1995).

Many students had difficulties interpreting the algebraic form and using the equations to predict the time where the two fish meet. When finding the time that the two fish meet by using the graph or the table, students were able to make connections to the context (movie) more easily than when asked to use the algebraic form. The interpretation of the algebraic model and the symbolic manipulation required are possible reasons that students struggled more in this part of the activity.

The final section included a general question, such as identifying the distance between the two fish at the beginning of the movie. The students were allowed to use any representation they wished to answer this question. Students reported the representation they used and were encouraged to use other representations to answer the same question. The linked-group student used a table effectively to answer this question. When asked to use the graph, she used the linkage, clicked on the data point on the graph and saw both points circled at the same time (see Figure 3). The semi-linked group student also approached the question using a tabular representation: “You could subtract these two distances [pointing to the first distances at the table].” When he was asked to use the graph, he said, “You would take the beginning from right here [pointing to the first data point of the striped fish on the graph] and that beginning right up there [pointing to the first data point of the gray fish on the graph] and subtract them.”

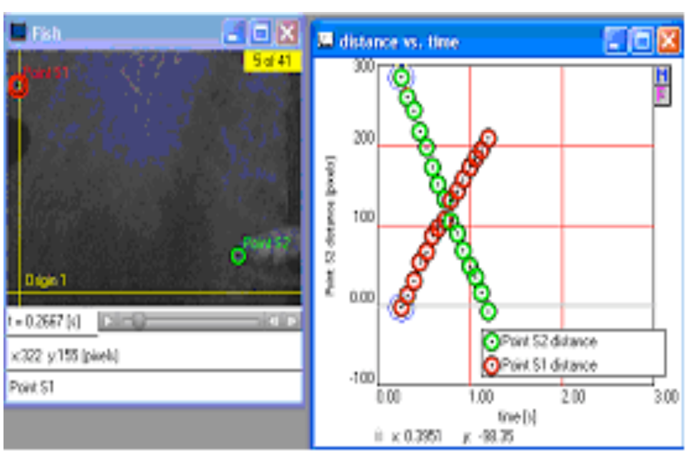


Figure 3. Using linkages to find two fish’s distance at the beginning

Conclusions and Discussion

This study focused on the effects of the use of a multiple representational computer environment on students’ learning. In the linked computer environment, students either used the linkage directly to answer the question or they assimilated the information provided through the linkage, using their previous knowledge to choose an existing, appropriate schema to answer the question. If they used the linkage directly, the software was the basis of their explanation. Students who chose not to use the linkage provided explanations for their answers based more on the mathematical aspects of the question. In either case, when the computer feedback contradicted their predictions, disequilibrium occurred, and the students needed to re-interpret this new information through their existing knowledge; that is, they assimilated the new information. If they could not interpret this new information with their present schema, they needed to accommodate their preexisting knowledge in order to reach equilibrium; that is, in Piaget’s words, they modified “internal schemes to fit reality” (Piaget & Inhelder, 1969, p. 6).

There were students who trusted their own knowledge and answers. They did not use the linkage at all. An interpretation of Kaput’s theory (1995) discussing the relationship between external and internal representations could be helpful in interpreting this issue. Students who trusted their internal representations (mental operations) might not need to test their knowledge for viability with the software; they preferred not to use the linkage. However, there were students who ignored the linkage or did not use the linkage when they could have benefited from using the linkage. Now, the other direction in Kaput’s theory, moving from physical operations (external representations) to mental operations, could be used. Here, the linkage, if used, could have served as the “existing physical material” (p. 140) to help students further construct their incomplete schema.

Results suggested that in a semi-linked environment, students seemed to rely mainly on their own existing knowledge with the help of the software to respond to a question. Although this environment did not provide such rich feedback as in the linked environment, ready-made graphs or tables presented powerful visual information/feedback for students to use while answering the questions. The software could have served as helper, record keeper, or representation provider for the students. Without the linkage, students seemed to provide more mathematically based explanations rather than movie-based explanations and constructed the linkages between representations for themselves. They were seen to be in a more active role mentally as learners. However, some students were not able to discern the relationships among representations. They could have used the linkage, if it had been available, in order to construct more empowering concepts.

Having access to multiple mathematical representations provided by VideoPoint enabled students to choose the types of representation with which they were most comfortable. Another advantage was the increased attention to the relationships among representations and the mathematical content instead of computation, manipulation, or drawing. Moreover, the software offered an environment with resources and constraints for students to construct new schema or change their existing ones by passing through a series of equilibrium-disequilibrium states.

Semi-linked representations can be as effective as linked representations for mathematical concept development. Being able to switch between the linked and semi-linked versions would be invaluable because the linked and semi-linked versions have their own benefits and limitations. Mathematics teachers might prefer linked or semi-linked versions of software for different age groups or grade levels. The most beneficial usage could come from using a linked version to introduce a mathematical idea and help students construct their schema. Once accomplished, the linkage could be removed and the semi-linked version could be turned on in order to make students use their newly constructed schema. This emphasizes the importance of the teacher's role in the classroom. Technology, if used appropriately, is a very effective tool in the process of teaching and learning of mathematics. However, there are many important decisions to be made by the teacher, such as when and how to use technology and with whom.

This article provides an example of utilizing linked and semi-linked representational software in mathematics teaching. The existing theories and the results of this research study were used to discuss the advantages, disadvantages, roles and effects of both types of technological environments in students' learning of linear relationships. Research in mathematics education allows us to improve the teaching and learning processes in mathematics classrooms. When strong bridges are constructed between the practice of teaching mathematics and research in mathematics education, they might serve educators, researchers and teachers in more empowering ways.

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