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The Co-Development and Interrelation of Proof and Authority: The Case of Yana and Ronit

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Students' mathematical lives are characterized not only by a set of mathematical ideas and the engagement in mathematical thinking, but also by social relations, specifically, relations of authority. Watching student actions and speaking to students, one becomes cognizant of a 'web of authority' ever present in mathematics classrooms. In past work, it has been shown how those relations of authority may sometimes interfere with students' reflecting on mathematical ideas. However, "...by shifting the emphasis from domination and obedience to negotiation and consent..." (Amit & Fried, 2005, p.164) it has also been stressed that these relations are fluid and are, in fact, a *sine qua non* in the process of students' defining their place in a mathematical community. But can these fluid relations be operative also in the formation of specific mathematical ideas? It is my contention that they may at least coincide with students' thinking about one significant mathematical idea, namely, the idea of *proof*. In this talk, I shall discuss both the general question of authority in the mathematics classroom and its specific connection with students' thinking about proof in the context of work done in two 8th grade classrooms.

Relations of authority are at work in the classroom shaping students' mathematical world just as they are at work in students' everyday lives shaping their understanding of the social world. Indeed, Amit and Fried (2005), showed that a 'web of authority' is ever present in mathematics classrooms and that those relations of authority may sometimes interfere with students' ability or willingness to reflect on mathematical ideas. At the same time, they noted that "...by shifting the emphasis from domination and obedience to negotiation and consent..." (Amit & Fried, 2005, p.164), relations of authority may be understood also as fluid and, in fact, a *sine qua non* in the process of students' defining their place in a mathematical community.

The complexity and dynamism of community formation within the microcosm of the mathematics classroom—and, with that, how classroom communities prefigure students' eventual participation within the greater mathematical community—is, of course, at the heart of the many sociologically oriented studies in mathematics education (e.g., Goos, 2004; Lerman, 2000; Dowling, 1996; Cobb & Bauersfeld, 1995; Mellin-Olson, 1987). But is defining a place in a mathematical community one thing and developing actual mathematical conceptions another? Is finding a place in a mathematical community only a kind of general precondition for doing and learning mathematics? Can the sort of fluid relations of authority alluded to above, by contrast, be operative also in the formation of specific mathematical ideas? In this paper, we suggest that a developing sense of authority may at least coincide with students' thinking about one significant mathematical idea: the idea of proof.

That proof should be our focus is not an accident. Modern ideas in the philosophy of mathematics, particularly those of Imre Lakatos (1976), suggest that the idea of proof rests on a broader foundation than pure logic: It may also be the product of social relations in a living mathematical community.[1] In the first part of the paper, accordingly, we shall discuss the fluid character of proof in a social and historical setting, as well as in students' own mathematical experience. We begin the first part, however, by looking at general notions of authority and how authority develops into a shared entity in a community in general, and in a mathematical community in particular. Together, the idea of authority as a part of community life and the social aspects of proof form the two main components of our theoretical framework.

Following this theoretical discussion, we shall provide some empirical findings suggesting that students' own thinking about proof may go *pari passu* with thinking about authority. This evidence comes from a microanalysis of a revealing conversation on proof with two 8th grade students, Yana and Roni. That conversation occurred in the context of a wider international study of 8th grade mathematical classrooms known as the *Learners' Perspective Study* (LPS) (Clarke, Keitel, & Shimizu, 2006). In the methodological section preceding the data section, we shall give a brief background of the LPS. The data section proper concerns the conversation between Yana, Roni, and the researchers; however, we shall also provide there additional data highlighting the sense of authority both among other students in Yana and Roni's classroom and in another classroom studied within the LPS framework.

Proof and Authority in Theory: *Authority*

Authority is a subtle and complex idea. Its complexity is born out by the varied contexts in which it appears in the history of ideas: jurisprudence in classical times, religion in the middle ages, and politics and society in modern times (Krieger, 1973). Authority is related to power, but, though command and obedience are essential to both, authority also requires, crucially, a sense of legitimacy. This is the common thread in the varied contexts in which authority is discussed. Thus, for example, in section 34 of Augustus' *Res Gestae* [a passage which Leonard Krieger describes as "the most revealing pronouncement in the whole history of the idea of authority"

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[1] This point of view has been developed with regards to mathematics in general, including proof, in Kitcher (1984) and Hersh (1998), among others.

(Krieger, 1973, p.144)], Augustus tells how he relinquished power (*potestas*), by which he meant specifically legislative power, whereas, because of his deeds, fame, and virtue, he possessed greater authority (*auctoritas*) than anyone else in Rome.[2] Augustus possessed authority, and not just power, because those over whom he had authority recognized that he *deserved it by right*; his own legitimacy as a leader, rather than any kind of coercion, assured the obedience Augustus commanded among the Romans.

The modern theory of authority may be said to have begun with the work of Max Weber.[3] Authority (*Herrschaft*), for Weber, is a kind of power; however, again, what is salient about authority for Weber is that it is power based on legitimacy. In fact, the idea of legitimacy is the central notion in Weber's conception. It is only by understanding legitimacy that one can understand a person's consent to obey, that "certain minimum of voluntary submission," as Weber puts it (Weber, 1947, p. 247), without which authority is not truly authority. Weber, therefore, analyzes the kinds of authority according to their sources of legitimacy. He distinguishes three forms of authority: Charismatic authority arises where there is a recognition of extraordinary, sometimes superhuman qualities in the authority figure; traditional authority is founded on the stable, time-tested structures of society, what Weber (1947, p. 247) refers to as the "sanctity of everyday routines"; and bureaucratic-rational authority derives from the acceptance of the rules of a rationally organized society. Charismatic authorities include religious figures and great leaders; they are singular figures inspiring awe and reverence. Traditional authorities are, first of all, parents and elders; a religious figure such as a priest, rabbi, or imam can also be a traditional authority to the extent that obedience towards them is commanded by the sanctity of their position rather than their possession of any special abilities unavailable to ordinary human beings. Bureaucratic-rational authorities are policemen, office managers, bank tellers—people whom one respects and obeys because of the necessary organization of society; such authorities are sanctioned neither by their extraordinary powers nor by traditional and cultural norms, but by their licenses or diplomas on the wall.

It must be stressed that these types of authority are 'ideal types', that is, a given figure of authority may manifest one or more of these types to various degrees: They are categories into which a given authority may be

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[2] The relevant passage of *Res Gestae*, 34 is: Post id tempus auctoritate omnibus praestiti; potestatis autem nihilo amplius habui quam ceteri qui mihi quoque in magistratu conlegae fuerunt.

[3] Weber's ideas on authority were set out in detail in his posthumous work *Wirtschaft und Gesellschaft* (1920/21) (trans.by Henderson & Parsons as *The Theory of Social and Economic Organization*—hereafter, Weber (1947)), which is still really the *locus classicus* for contemporary accounts of the subject.

analyzed. A teacher in a classroom, for example, may certainly possess traditional authority as well as bureaucratic-rational authority; a teacher's authority may even have some of the characteristics of charismatic authority, as we shall see below. Even the kind of authority, which one might think of as the special province of teachers, namely, expert or intellectual authority (Levin & Shanken-Kaye,1996; see also French & Raven, 1959, who speak of expert *power*), can actually be analyzed as a kind of bureaucratic-rational authority, for such authorities are generally recognized to have "specialized training"(Weber, 1947, p. 304). Their titles, diplomas, or licenses attest such 'specialized training'; indeed, being 'a Ph.D', professor, or Nobel Prize Winner is often precisely the reason one is called an expert.

Naturally, the concept of expert authority is very much pertinent to the general question of authority in education. But the heart of that question has more to do with the role authority plays in students' learning and students' intellectual lives—in our case, students' mathematical lives. One must ask, in particular, how a mean can be found between a stifling, persistent appeal to authority, on the one hand, and a disavowal of all authority, on the other. The former, it is worth pointing out, is both a kind of learning behavior and also a technical logical fallacy having to do with what Toulmin (1958) refers to as the 'warrant' of an argument. The logical fallacy, also known as *argumentum ad verecundiam* (literally, "arguing on the basis of respect"), is essentially arguing that q follows from p because a figure of authority says so. The authority is usually, but not necessarily, an expert authority. Thus, it happens that, say, political views of a Nobel Prize winner are taken seriously, even though the prize was awarded for some field having nothing to do with politics. That said, the appeal to authority cannot be completely discounted. As Copi (1972, p. 80) says, "This method of argument is not always strictly fallacious, for the reference to an admitted authority in the special field of his competence may carry weight and constitute relevant evidence," and, in this connection, one need only think of citations in professional papers, like this one. The appeal to authority, however, runs against the tendency in education towards authenticity and autonomy, towards students' thinking for themselves. This tendency, which is encouraged and given foundation by constructivist pedagogies, brings into focus the other extreme, the disavowal of all authority [this is touched on in Dowling (1998)].[4] Without quite reaching that extreme, one,

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[4] In broader contexts, this is the extreme to which neo-anarchism of the New Left tends. For, as Krieger (1973, p.160) points out, whereas the older anarchism of Godwin, Proudhon, and Bakunin, and Kropotkin denied authority in the political sphere, the neo-anarchism of the New Left denies authority altogether for the sake of utter equality—natural, moral, social, intellectual, and personal—, openness, and spontaneity.

nevertheless, feels its proximity in the devaluation of the role of teachers expressed in such statements as this: "We won't meet the needs for more and better higher education until professors become designers of learning experiences and not teachers" (Spence, 2001, p. 12).

A first step towards finding a path between the continual appeal to authority on the one hand, and the complete denial of authority, and, therefore, teachers, among other things, on the other, is the notion of 'shared authority' found particularly in studies on classroom management and discipline.[5] In treatments of the latter, such as that by Levin and Shanken-Kaye (1996), discipline is viewed as a matter of changing or managing students' behavior (so that ultimately students manage their own behaviour) rather than of repressing disruptive behavior. To this end, being flexible and responsive to students' needs is assumed to be preferable to being rigid and despotic. However, this requires that teachers bend to the will of the student *without themselves losing authority*. Hence, rather than one-sidedly dominating the students, teachers achieve discipline by allowing the students to have a part in the authority structure of the classroom, to share authority. Authority, in this view, becomes itself flexible and fluid.

Shared authority, although introduced here in the specialized setting of classroom management, hints at a more general conception of authority in which authority is not so much a force for preserving a rigid social structure but as way of mediating the growth and development of a community, including an intellectual community. Thus shared authority is, ultimately, a kind of authority which is non-localized, that is, in which there is no immovable division between the subject and agent of authority (a possibility that may be compared, by the way, to Foucault's views on power, specifically, that it "must be analyzed as something which circulates" (Foucault, 1980, p. 98)). For this more general, non-localized authority, we take the work of the educational theorist, Kenneth Benne (Benne, 1970) as an example.

Benne's (1970) basic thesis is that, understood correctly, authority can be approached as a force for liberation and social change; it need not be the mark of a despot. Thus, from the start, Benne situates authority not in a context characterized primarily by power and struggle but by dependence and interdependence, not by domination but by negotiation; an authority relationship arises, in Benne's (1970, pp. 392-393) view, where one person has a need or purpose that another has the power to satisfy (this is the one restricted sense in which power is connected to authority—power that is not coercion):

[5]Wolfe (1959) uses the term 'shared authority' in a slightly different sense in speaking about the way authority is shared in a household.

> Authority is always a function of concrete human situations however large or complex the situation may be. It operates in situations in which a person or group, fulfilling some purpose, project, or need, requires guidance or direction from a source outside himself or itself. The need demarcates a field of conduct or belief in which help is required. The individual or group grants obedience to another person or group (or to a rule, a set of rules, a way of coping, or a method) which claims effectiveness in mediating the field of conduct or belief as a condition of receiving assistance. Any such operating social relationship—a triadic relation between subject(s), bearer(s), and field(s)—is an authority relationship.

Conceiving the exercise of authority (rather than mere power) in terms of negotiation and of actions intending to bring a group into accord is a logical extension of Weber's (1947) focus on legitimacy. For Benne (1970, p. 391), it leads to a form of authority which he calls "anthropogogical authority."

Benne (1970, p. 391, note) coined the word "anthropogogy" as an alternative to "pedagogy" to remind us "…of the need of human beings at all chronological ages to be reeducated…." Thus, it is directed towards the continuous formation and renewal of the entire community. The nature of anthropological authority and how it develops is exemplified by the relationship between a doctor and a medical student. At the start, the relationship between a doctor and a medical student appears similar to an authority relationship based on expertise. As the relationship persists, however, the distinction between the doctor and medical student fades, until finally, when the medical student's studies have been completed, the distinction has vanished completely, and the two become colleagues. This is why Benne (1970, p. 401) says, "All anthropogogy is at once a mothering and a weaning, a rooting into ongoing authority relations and a pulling up of roots." With the distinction between the doctor and the once medical student blurred, the phrase "ongoing authority relations" begs the question of the identity of the bearer of authority. The doctor is no longer the clear bearer of authority; it is now the entire medical community—not only the people who make it up, but also the norms by which it functions and the collective knowledge and skill it contains—into which the medical student is inculcated.[6] And the community is fluid and ever developing. As Benne (1970, p. 401) puts it:

> The ultimate bearer of educational authority is a community life in which its subjects are seeking fuller and more valid membership. Actual bearers and subjects of this authority must together build a proximate set of mutual relationships in which the aim is the development of skills, knowledges, values, and commitments which

[6] There is some resemblance here to Lave and Wenger's (1991) notion of "legitimate peripheral participation." (We owe this observation to Norma Presmeg).

> will enable the subjects to function more fully and adequately as participants in a wider community life which lies beyond the proximate educational associations.

So, to reiterate, authority can be seen in this light as a component in the formation of communities, particularly intellectual communities, rather than in the reproduction of rigid relations of power. For full members of a community, authority is always a flexible double relation: One asserts oneself (which can take the form of making an assertion) as an authority and, simultaneously, submits oneself to the greater authority of the community, creating a dynamism in which true growth can take place. Keeping this aspect of community formation in mind, we must now turn to the mathematical community, which is our main concern. Specifically, we now turn to one of the central activities within that community, namely, proof.

Proof and Authority in Theory: *Proof as a Social Activity*

Mathematics education has given extensive attention to proof (see Weber, 2003 for a concise survey)—and rightly so, since proof, as has so often been stressed, is truly at the heart of mathematics. It is therefore natural that research in mathematics education should aim to uncover the competencies and difficulties involved in students' actually constructing proofs (e.g., NCTM, 2000) and to find ways for promoting mathematical understanding by means of proof (Hanna, 2000). But it has also been long-recognized that these goals depend on how (and if) students understand what proof is (e.g., Bell, 1976; Galbraith, 1981; Vinner, 1983).

A typology of students' understandings of proof was attempted by Harel and Sowder (1998). Their classification of students' 'proof schemes' takes in not only mathematically sophisticated approaches to proof and justification but also well-known educational phenomena concerning proof, such as students' viewing proof as a matter of producing evidence (Chazan, 1993; Fischbein, 1982). More importantly for the present paper, Harel and Sowder (1998) also recognize a category of proof schemes based on external conviction, which includes a subcategory of authoritarian proofs. Typical behavior associated with this proof scheme is that students "...expect to be told the proof rather than take part in its construction" (Harel & Sowder, 1998, p. 247).

Harel and Sowder (1998) also recognize the importance of context in applying their categorizations to actual student behavior, that students may manifest different proof schemes in different circumstances and that proof schemes can change, indeed must change into more sophisticated forms when teaching is successful. So, an authoritarian proof scheme, from their perspective, may well give way, for example, to the more desirable 'transformational' proof scheme.

Whether proof schemes may be different in different contexts or manifested by students at different stages in their development, Harel and Sowder's (1998) proof schemes have a certain fixity about them: one at one time or one place. This of course is characteristic of any typography and not completely unreasonable or unenlightening in this case. Harel and Sowder's approach is certainly consistent, say, with perspectives seeing understandings of proof as hierarchical (e.g., Hoyles, 1997). It seems likely, however, that students' understandings of proof as well as their proof-behaviors are much less fixed, defined, and univocal than characterizations as 'empirical' or 'explanatory' might imply, and that these understandings are likely to be continually shaped by circumstances and interactions in very dynamic ways.

That proof schemes cannot be understood simply as stages leading to a stable single transformational or axiomatic scheme is dramatically clear even in the case of authoritarian schemes: These, for example, have been shown to be at work (not necessarily negatively) even among research mathematicians (Inglis & Mejia-Ramos, 2006). And that students' understandings of what proof is may be shaped continually by interaction in social contexts has, needless to say, has been documented in mathematics education research for many years now (e.g., Yackel & Cobb, 1996). Moreover, evidence from science education has shown how interactions connected to students' 'meaning making' involve a tension between dialogic and, precisely, authority (Scott, Mortimer, & Aguiar, 2006).

The dynamic character of students' developing ideas of proof is, in a way, a reflection of what has become increasingly clear over the course of the last century among philosophers of mathematics, namely, that the nature of proof is not exclusively logical or formal,[7] but also historical, cultural, and social, and, therefore, it is not itself a settled, fixed notion. Historical considerations, for one, show great changes in what has been thought to constitute a proof or even what has been understood as rigorous (e.g., Hanna, 1983; Kleiner & Movshovitz-Hadar, 1997), and mathematicians themselves have been aware, historically, that these are notions that are not rock-bottom: Whether or not, for example, indivisibles, motion, and the method of contradiction should be viewed as legitimate means of proof was hotly debated in the 17th century (Mancosu, 1996), as much as the status of 'computer proofs' (Tymoczko, 1985) have been in our day. Historical developments such as these make it difficult to say that there is a privileged perspective from which one can judge absolutely whether a proof is a proof.

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[7] One should not get the impression that this older view of proof is outmoded: "proof theory" is still a subfield of mathematical logic. The point is only that we have come to realize that a *complete* theory of proof must extend its borders beyond logic.

That proof is a fluid idea, not only from a historical perspective but also within the context of a given mathematical community at a given moment, has brought the philosophy of mathematics to recognize that the nature of proof is, perhaps, better investigated as the nature of an activity rather than of a concept. In other words, proof is something that the mathematical community *does*, and it is for this reason that proof takes on a social character. This insight was spelled out in a particularly insightful and colorful way by Lakatos (1976) in his *Proofs and Refutations*. Lakatos (1976), one recalls, begins with a theorem and a proof, namely, for all polyhedra, V-E+F=2, where V is the number of vertices, E the number of edges, and F the number of faces, and, in the light of criticisms, counterexamples, and refutations, proceeds to modify the theorem and the proof, until it becomes unclear exactly what is being proved and what is the content of the proof. What is particularly important is that Lakatos' (1976) text takes the form of a dialogue—the appropriate setting for developing the idea of proof, that is, a social one. No less important for us is that the dialogue takes place in a classroom and that it is the teacher, the dominant authority figure in the classroom, who presents the proofs criticized by the students. This, in a nutshell, represents the double relation of authority in a community mentioned above: The teacher presents proofs to the classroom, but in doing so submits herself to the greater authority of the mathematical community, symbolized by the students who present counterexamples and other refutations.[8]

So, proof is a fluid idea, and it is one defined as much by the activity and social relations of a mathematical community as by logical and formal relations. Moreover, from our considerations of authority, we recognize that a notion of legitimacy is as crucial to authority as it is to proof. But more importantly for the present paper, we have seen that authority has a part in the formation and functioning of a dynamic community, and, therefore, also of an intellectual community like a mathematical community: Authority works not only to preserve hierarchies of power but to shape just the kind of idealized setting of Lakatos' (1976) mathematical classroom, where one asserts positions, submits to criticism, and deepens collective knowledge. Thus, we have a partial answer to the question of whether relations of authority may be operative in shaping a mathematical idea like

[8] Lakatos himself referred to authority, but not as we have done. Rather, he saw authoritarianism as the stifling projection of infallibility. With that, he remarks that "...present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought" (Lakatos, 1976, p.142-3, note 2). But this represents a simple view of authority as rigid, localized, and unidirectional, whereas we have shown that authority can be understood as a more complex and non-localized relation.

proof. However, whether any of this plays out in students' own thinking about proof remains to be seen. That is, while we may assume that students' ideas of proof are not yet clear and crystallized, and that they are, rather, in a process of formation, we still need to ascertain whether ideas of authority are active, or even present, in that process. Otherwise, the theoretical connections between authority, community, and proof, though correct, will be of little help to teachers seeking to improve their students' understanding of proof. With that, we turn to some empirical results.

Research Background

As mentioned in the introduction, the work presented here was undertaken within a broader international framework known as the *Learner's Perspective Study* (LPS). The LPS is an ongoing international effort involving 15 countries. Its goal, in general, is

> to document both the practices of eighth-grade mathematics classrooms and the meanings, mathematical and social associated with those practices and to utilise the data collected to draw conclusions, both locally and internationally situated, concerning those practices most likely to lead to the optimisation of learning (Clarke, Keitel, & Shimizu, 2006, p.9).

Like the TIMSS video study the LPS focuses on eighth grade classrooms; however, unlike the TIMSS study, which examined exclusively teachers and only one lesson per teacher (see Stigler & Hiebert, 1999), the LPS focuses on student actions within the context of whole-class mathematics practice and adopts a methodology whereby student reconstructions and reflections are considered in a substantial number of videotaped mathematics lessons.

As specified in Clark (2001), classroom sessions were videotaped using an integrated system of three video cameras: one recording the whole class, one the teacher, and one a 'focus group' of two or three students. In general, in a given classroom, every lesson over the course of 3 weeks was videotaped, a period comprising about 15 consecutive lessons. The extended videotaping period allowed every student at one point or another to be a member of a focus group.

The researchers were present in every lesson, took field notes, collected relevant class material, and conducted interviews with each student focus group. Teachers were also interviewed once a week. Although a basic set of questions was constructed beforehand, in practice, the interview protocol was kept flexible (along the lines of Ginsburg,1997; see also Patton, 1990) so that particular classroom events could be pursued. This also meant that the interviews often had a conversational character lending themselves to the kind of discourse analyses described by Roth (2005).

Two classrooms in Israel were studied in this round of the LPS. The specific classroom to which we shall principally refer in this paper was taught by a teacher, whom we call Sasha.[9] Sasha came originally from the former Soviet Union. He has had several years' experience teaching in Israeli schools and much experience teaching in Russian schools. His mathematical background is strong, having completed advanced studies in applied mathematics. His 8th grade class is a high-level class and comprises 30 students. The lessons observed in Sasha's class all concerned geometry and, therefore, were particularly appropriate for examining students' ideas of proof.

The second classroom, to which we shall refer briefly below, was taught by a well-trained and experienced teacher, whom we call Danit. Danit teaches in a comprehensive high school. Her eighth grade class is heterogeneous and comprises 38 students, mostly native-born Israelis, but also new immigrants from the former Soviet Union and one new immigrant from Ethiopia. It is also notable that Danit, who is presently engaged in doctoral studies in mathematics education, is well informed about theoretical educational ideas, such as constructivism, which are important background ideas for the Israeli mathematics program. The 15 lessons observed in Danit's classroom belonged to a unit on systems of linear equation.

The data that we will present below came from a particularly striking interview with two very bright, spirited, and talkative girls from Sasha's class. As we mentioned in the introduction, we call these girls Yana and Ronit. The two are very good friends: They are generally attentive to one another, but, as good friends do, they also allow one another the independence to disagree and qualify one another's remarks. This is evident in their discourse style, and it says much about the character of their own interactions.

Yana and Ronit, we admit, are not typical students—they are not 'good' data in a statistical sample. But it is precisely this that made them so interesting: Their intelligence, openness, and loquaciousness made them atypically revealing. Naturally, we are well aware of the methodological questions arising from this kind of qualitative microanalysis—of the problem of 'generalizability' for example (Patton, 1990; Yin, 1994; Green et al., 2006). Yet, the case of Yana and Ronit serves very well to illustrate the possibility of a sense of authority coinciding with students' developing thinking about proof and, more importantly and very much in line with our qualitative approach, the particular manner in which development may occur.

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[9] Sasha, of course, is a pseudo name, as are all the other names in the study, including the names Yana and Roni. We do, however, preserve the gender and a hint of the background of the students in the pseudo names we choose.

Data

Before we turn to Yana and Ronit, it is important to present some data concerning the sense of authority in their classroom taught by Sasha as well as in the second LPS classroom taught by Danit [more detailed data and analysis may be found in Fried and Amit (2005)].

Keeping in mind Benne's (1970) statement, quoted above, that authority "...operates in situations in which a person or group, fulfilling some purpose, project, or need, requires guidance or direction from a source outside himself or itself," we asked students whom they turn to for help when they run into difficulties. To this, students consistently cited their teacher, their friends, their parents, or their siblings. Teachers, friends, parents, and siblings form a web of sources of assistance; when one source is unavailable or unable to help, one turns to another. For example, if Yara in Sasha's class cannot get help from Sasha, for one reason or another, she turns to one of her friends:

Interviewer: And if your friend doesn't know?

Yara: If my friend doesn't know, I ask someone else—or my father.

The web forms a hierarchy according to the degree of authority possessed by the sources, where by the latter we mean the degree to which a person's statements are taken unchallenged (this sense, in a way, is already built into the word 'source'; indeed, the Greek word for 'source', *arche*, also means 'sovereignty', and in the plural, *hai archai*, 'the authorities'). Conversely, turning to an authority means turning to a person for an answer or for instructions, not, by contrast, for a discussion. Moreover, we use the word 'authority' rather than, say, 'expertise' because the reason a person's statements are not to be challenged is, as we shall soon see, not always dependent on the degree of the person's knowledge, though it may be perceived that way. Now, in this hierarchy, there is no question, the teacher comes first.

The predominance of the teacher's authority was apparent in all of the student interviews, both in Danit's and Sasha's class. For example, at one point in our interview with two students, Michael and Saul, in Danit's class, we asked whether a graphical method or algebraic method of finding the solution to a system of equations was more reliable. Here is the exchange:

Michael: If I get a answer for one and a different answer for the other, then you've got to check. If I get the same answer, then I'll believe it's correct. But if there's, maybe, still some doubt in my mind, I ask Danit.

Interviewer: What does Danit have that other people don't?

Michael: She's a teacher, she can help; if you make a mistake, she corrects it!

Interviewer: And if she errs?

Michael: She doesn't err.

Saul: She studies everything at home before she comes to class.

Michael: Otherwise she couldn't correct—she's a teacher!

Interviewer: But she did make a mistake at the board.

Saul: She got mixed up because she substituted wrong.

Michael: Those are nonsense things she gets mixed up about, but real things (gestures to show the weightiness of the things he has in mind)—if two exercises are supposed to get the same answer or not, it doesn't seem to me she'd get mixed up about that.

What is striking in this exchange is how Michael and Saul are willing to see Danit as nearly infallible, and how far they are willing to defend her authority, even when she is seen to make a mistake. The students view her, apparently, not only as one who knows more than they do, but also as a strong figure with powers they lack; she possesses not only expert authority, but also, to use Weber's (1947) term, charasmatic authority, that is, authority whose power is supernatural and which commands devotion more than mere obedience. Thus, when Michael says, "She's a teacher, she can help; if you make a mistake, she corrects it!" he sounds as if he is speaking of a healer, a miracle worker, rather than of his 8th grade math teacher. Similarly, when we asked Sylvia and Shari, also Danit's students, what exactly do they expect from the teacher when they ask her for help, Sylvia said simply "That she will explain to us better," to which Shari added immediately, "When she comes over to me, when she explains to me, suddenly I understand better…" Consistent with this image of Danit, was the importance the students seemed to place on the mere fact of Danit's coming over to help them when they worked on exercises. When we asked what the climax of the lesson was, Elana, in the same interview in which Sylvia and Shari participated, answered, "When I was having trouble with the book and I called (Danit)." In a different interview, another girl in Danit's class, Gila, answered the same question in precisely the same way. Conversely, on two different occasions we came across a student in Danit's class who also appeared to be having trouble with the exercises, but who did not ask Danit for help. When we asked them why not, we received the same response both times: "The teacher doesn't want to help me." Such a statement presents a picture in which the attention the students receive from the teacher is dependent on the teacher's whim. The teacher becomes, in this interpretation, a dictator, though, surely, for most students, a beneficent one who willingly helps them when they need help. Nevertheless, conceiving the teacher as a creature of whim is to conceive the teacher as a creature with terrific power.

The extent of teachers' authority from the students' perspective was significant enough; however, we were more surprised to discover how easily students would see *other* people as authorities to the same degree. For example, we were interested in how students understood the requirement of "showing their work," whether this was only a requirement of students or of mathematics itself. So, we asked whether a salesperson explaining to customers how much they should pay given such and such a discount should be required to show his/her work. To this, Ben, again from Danit's class, replied: "No, I can rely on him…I can rely on him—for sure lots of people come to him—there must be those who know percentages and things, and they rely on him, so I can rely on him too." It is worth noting here that the Hebrew word Ben uses for "rely" is *somech* which is closely related to the word *somchute* meaning, literally, "authority."

In the students' world, then, authorities are ubiquitous; sources of assistance are invariably sites of authority - authority in its highly localized, non-shared form, to use the terminology of section one of this paper - and the most striking fact is that this applies also to the students' friends and classmates. As we already remarked, friends in the class are a dominant source of help. But when the students turn to their friends they tend to turn to them only for answers. And, as we saw with Sasha's student Yara, when one friend does not know, she turns to another. In one interview in Danit's class, we asked a student why he did not ask his friend for help at a certain point during the lesson. He replied, "I knew Yuri wouldn't know the answer…" Thus, when students are perceived by their fellow students as knowing the answer to some question they are treated for that instant as an authority, that is, the answer is accepted and not discussed. When students are not perceived as knowing the answer, they are usually not asked. In fact, in the classroom videos it can be seen quite often (though less so in the geometry classes) that students sit together, occasionally speak together, but do not really work together, even though they are not necessarily encouraged to work individually.

This tendency of students to treat one another as authorities *ad hoc* was brought into relief by the contrasting behavior of Ronit and Yana. In the lesson when Yana and Ronit were our focus group, we watched how the two girls in a truly collaborative spirit worked on a problem given to the class: Ronit showed her diagram to Yana; Yana commented and pointed to her own diagram; they discussed the problem together, and, finally, came to a solution. In general, during the lesson and throughout the interview, we noted how different Yana and Ronit's manner was from that of the other students we observed: They consulted with one another, raised possibilities on their own, revised opinions, and seemed to arrive at common conclusions. In short, Yana and Ronit treated one another as intelligent interlocutors who could work together to make progress on the question at hand. We should stress that this was, indeed, behavior different not only from that of students in Danit's class, but also from that of other students in Sasha's class. For instance, at one point in our interview with Yara, we asked if she could draw a triangle having two acute exterior angles; she said she could, and she proceeded to draw a diagram, which, obviously, could not be correct. When we asked Panina, the second girl in that focus group, whether Yara's diagram was ok, she assented immediately with no further comment. The degree to which Yana and Ronit differed in this way from the other students, made the considerations of authority that entered their discussion of proof in the interview particularly interesting. So, let us now move on to that segment of the interview with them.

The interview with Yana and Ronit took place after a series of lessons centred on the first basic theorems in geometry including the first congruence theorems for triangles. In those lessons, Sasha discussed these theorems and proofs of propositions related to them. He also tried to highlight the idea of proof. For example, in the lesson just before the interview, one student (prompted by Sasha) suggested that the exterior angle of a particular triangle was equal to the sum of the opposite interior angles. Sasha commended the girl, but added: "Look, this may be (true), but maybe it is only by chance, for we have not yet proved it." The workbooks used in the class contained, moreover, proofs to be completed by the students. So, although the first 5 minutes[10] of the interview with Yana and Ronit concerned the nature of students' notebooks and workbooks, we were easily able to shift the conversation to the nature of proof. The initial response of Ronit and Yana was one of incomprehension:

Interviewer: (38 min.) Tell me now, are there also proofs in the book (the workbook), things you have to prove?

Ronit: To prove?

Interviewer: Yes.

Ronit: Umm.

Interviewer: Did you meet up with something you had to prove yourself?

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[10] This being the 2nd interview that day, the time notations carry over. The time indication for the interview, therefore, begins with "33 minutes."

Yana: There are exercises here, what do you mean? I don't understand, like, prove what...like what was on the board?

At this point, Ronit offers what might be called a first definition, which we coded, accordingly, d0.

Ronit: (Referring to interviewer's question above) Like correct and not correct.

Interviewer: Yes?

Ronit: But you have to write if it is correct and not correct and to prove why this is correct and why this is not correct.

Yana: Explain what you say.

So, d0 is that "Proof is saying whether something is correct or incorrect and explaining what you say." But, d0 subtly introduces another characteristic of the discourse, which we playfully coded TW, 'They & We'. Ronit begins by telling us what 'you have to do', in other words, what the book or teacher, that is, 'they' expect you to do; Yana's refinement, that you explain what you say, adds that part of the proof must come from you, our contribution. Therefore, we have here a first hint that the discussion of proof is connected with external authority and individual agency: what they require or do or expect and what we do and think. But this only becomes substantiated as we move on to the next 'definition' of proof, which is made in contradistinction to 'argument'.

Interviewer: Is 'to argue' and 'to prove' the same thing?

Yana: Uh...depends on the case

Ronit: No, 'to argue' is to say why you think this way.

Yana: //No, it depends...

Interviewer: (~39.5 min.) Hold on, Yana. Roni (indicating to her to go on).

Ronit: 'To argue'...

Yana: All right (laughs)

Interviewer: No, it's ok, yes.

Ronit: 'To argue' is to say why you think that way, and 'to prove' is, umm, to find something to support what you say.

Yana: Something that (you) already...

Ronit: Something existing, something you already learned.

That Ronit is referring to something learned from an external source becomes clear when, restating her position, she adds to "something already learned," "something written." Definition d1, then, was this: "To argue is to say why you think something, while to prove is to show how something is supported by what you have already learned." This second definition makes it very clear that the distinction between 'they' and 'we' hinted at in d0 coincides with that between 'proof' and 'argument'; what is proved rests on what has been learned (recall, in the first transcript quotation above, Yana related proof to what was written on the board by the teacher), that is, what came from an external authority, while what is argued rests on what you yourself think.[11]

At this point, Yana questioned Ronit's definition, making a comment wonderfully reminiscent of the 'learning paradox':

Yana: (40 min.) But if you try to prove something new? Then that's not something that's written...

Ronit: Yes.

Interviewer: I don't understand.

Yana: No, if you want to prove something new, like, that no one's ever proven before, then that can't be written, so...I don't know

Interviewer: That is, what you are doing then is...?

Ronit: You (4 secs. Pause) prove. (Ronit and Yana laugh)

Just as in Plato's *Meno*, where the 'learning paradox' first appears, this interchange signals a new turn in the conversation towards one's own proving and concrete instances, that is, a move away from the TW distinction and towards 'we' alone.

In the examples we presented in this phase of the conversation, we had other motives beyond the students' understanding of what proof is—for example, we wanted to see how they understood the logical import of

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[11] We might remark here that especially in this context, the TW distinction is reminiscent of what Harel and Sowder (1998) refer to as "persuading" and "ascertaining," the acts by which one removes someone else's doubts, in the first case, and one's own doubts, in the second. For Harel and Sowder, persuading and ascertaining are the crucial sub-processes of proving; one sees, therefore, how closely related is the very notion of proof with a sense of oneself and others, and, thus, also of agency and authority.

counterexamples and contrapositive statements and the place of diagrams in proofs—but in the course of the examples new definitions of proof arose.

An example (ex. 1) we discussed at length was the following: "If one of the angles in a triangle is right, then the two other angles are acute. Argue yes or no."

Interviewer: (44 min.) "If one of the angles in a triangle is right, then the two other angles are acute." True?

Yana: Yes

Interviewer: You wrote yes.

Yana: We didn't argue (the point).

Interview: What is the argument?

Ronit: The argument is...

Yana: Umm...

Ronit: That...(Ronit and Yana, at this point, laugh)

Yana: If, wait a minute, if one of the angles, one of the angles of the triangle is right...

Ronit: Since, if one is right and the other is obtuse, then this will go over 180 and then it won't make sense.

Yana: Also it won't come out a triangle, one angle is right//

Ronit: //It won't come out a triangle, exactly, one angle is right.

Yana: And one is obtuse, so if you join (the sides opposite the right and obtuse angles), it comes out a quadrilateral, because it comes like this, right.

Now, throughout this whole discussion (which continued beyond what is quoted here), Yana and Ronit referred only to their own thoughts and never once mentioned something 'learned', even when they were relying on things learned—for example, that the angle-sum of a triangle is 180°. But this was consistent with the distinction they made in d1: They were 'arguing' the point here, not proving, so they were only setting out their own reasons for their conclusion. We pressed the issue, therefore, and asked for a proof of what they were saying, reviving in this way the questions as to what is a proof and what is the difference between a proof and an argument.

Interviewer: (~45.5 min.) Suppose I be nasty and tell you to prove what you've been saying. What do you think?

Yana: What do mean, 'to prove'?

Interviewer 2: Supposing that ('to prove') was written here instead of 'argue'.

Interviewer: 'Argue' (or) 'Prove' your words. Will your answer be any different?

Yana: (4 secs.) Here, I proved it (referring to what we quoted above)

So, for Yana, at least, the distinction between proof and argument seems to have dissolved, and, with that, also the concomitant distinction between 'they' and 'we'. Eventually, Yana says explicitly that there is no difference between proof and argument:

Yana: ...For me, I don't know what the difference is between an argument and a proof.

Interviewer: Any conjecture, then?

Yana: If you write for me 'argue' or 'prove', I will write the same thing.

Interviewer: (~51.5 min.) The same thing?

Yana: Yes

That proving and arguing are the same thing, we referred to as d3. Between d3 and d1, there was another definition and yet another afterwards, both Yana's: "If I explain, I think, that if I explain in words and with a diagram I prove (~47 min.)" (d2); "The proof of a proposition is the claim facing (sic.) the argument (52 min.)" (d4). 'Definition' d4 recalls the two-column proofs that Yana has seen both in class and in her workbook. Yet, like the theorem concerning the triangle angle-sum, there is nothing in the way she frames these 'definitions' to suggest an exterior source. The 'they' has disappeared—or has it?

One might expect that having worked on their own, felt their own ability to think about a proof, and reflected on their thinking—for example, after the discussion of ex. 1—Yana and Ronit would no longer see proof as something done under another's authority, that is, that definition d1 would be discarded, or, alternatively, d3 would now represent a true harmonizing of one's own thoughts or agency ('argument' of d1) and the authority of books, teachers, and mathematicians ('proof' of d1). But it turns out that the situation is not so straightforward. For with Yana's statement of d3 there ensues a discussion between Yana and Ronit in which d1 returns in force (and not just as the position of Ronit), with the TW distinction playing an explicit part:

Yana: The argument is your opinion, what you think, and the proof is...

Roni: //That is what I think (what I do)

Yana: And the proof is what they write? Like, what others write?

Ronit: No, in fact when you are asked why you think that way, so, umm...

Yana: You are not asked why you think that way, they ask you, argue (!)

Ronit: Come on (Nu! in Hebrew) that's the same thing. So in fact when you are asked you answer, umm, you think this way because of what you have learned, I think. So, it comes out the same thing since in proof you write what you've learned before. (54 min.)

Yana: No, for an argument you write, like, what you say (i.e., what you mean)—that for an argument, that you think this way because of what you have learned and in a proof you write what you have learned...that's what I understood.

With this return of d1 (and, in fact, d3 as well, for recall there was another definition before this exchange), the time has come for us to sum up.

Concluding Remarks

The last exchange quoted above was followed by Yana and Ronit's laughing, partly, perhaps, because of Yana's not altogether clear last remark. But although the conversation continued a few more minutes along the lines of that exchange, their laughter seemed also to mark some kind of conclusion or summary of the situation. It was a slightly embarrassed laughter. It seemed to betray a sense of going in circles and of discomfiture regarding the questions what is proof, what is argument, and are proof and argument the same thing. And in a way that's right: Yana and Ronit do not yet have a settled understanding of proof, and, yes, in a way, they are going round and round. What we have been looking at is one turn in their continually spiraling process of coming to understand proof. We have also seen that this process coincides with a debate about 'they' and 'we', about the authority of a discipline, of their teacher, of their textbook, and their own agency, their own legitimate authority, their own ability to say why they think what they do. Their understanding of these things is also unsettled.

When we consider, on the one hand, Benne's (1970) ideas of educational authority, where authority, when it is fully developed, informs the growth of an intellectual community and Lakatos' (1976) ideas of proof as an activity occurring in a social setting, on the other, we realize that there is also going round and round, like a dance,[12] in a mature mathematical community fully at work on what can be said and proved about mathematical objects and about proof itself. That dance-like relationship between authority and proof in such a mathematical community can be represented and specified diagrammatically as follows:

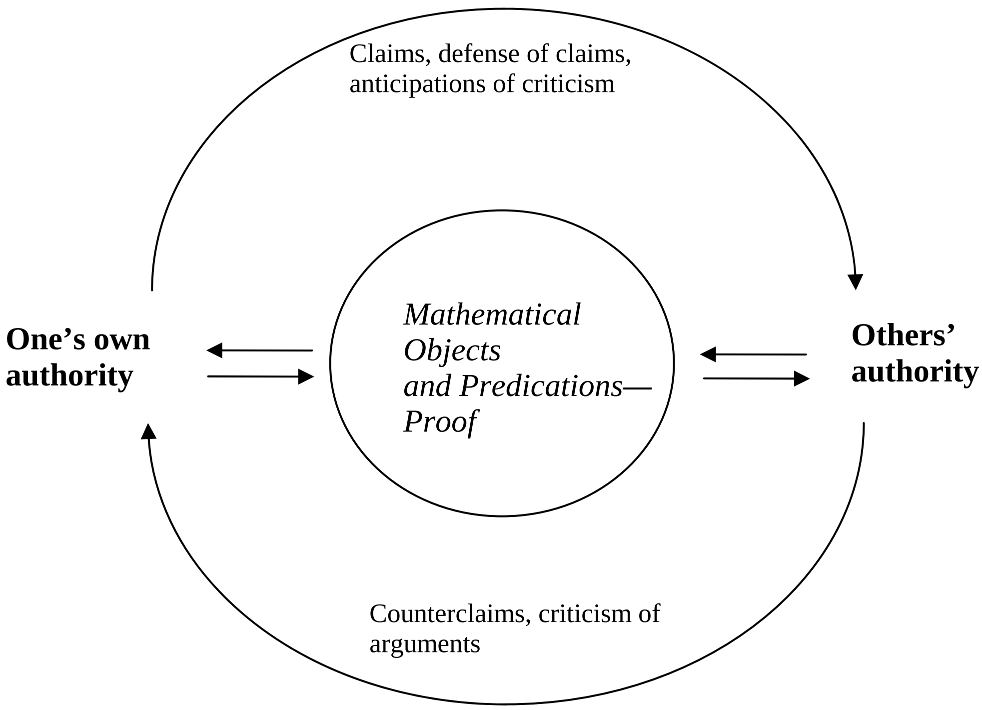

Figure 1. Proof and authority in a mature mathematical community.

12 Jo Boaler (2003), relying on the sociologist of science Andrew Pickering's work, stressed that in their thinking about mathematical ideas, mathematicians are engaged in a 'dance of agency', balancing their own agency with the authority of the discipline. Pickering, himself, it is worth mentioning, mentions Lakatos' work in connection to his "dance of agency"; in his conception of the dance, the "dialectic of resistance and accommodation," Lakatosian counterexamples are like resistances (Pickering, 1995, p.119, note 7).

Yana and Ronit's going round and round has not yet this mature form, but there are hints of it in their discussion of the geometrical problem in Sasha's lesson, which we alluded to above, and there is a reflection of it in their discussion of what proof is, some first murmurings of a sense that proof is something they engage in within a community and something that gives a mathematical community life and makes it grow.

With that, we must ask how teachers can make use of what we have described in this paper. First, we must make it absolutely clear that by emphasizing proof as a community activity resting on shared authority, we do not mean to suggest that teachers ignore logical aspects of proof and justification. This is essential in proof; research with this logical emphasis (e.g., Selden & Selden, 1995) is still of enormous value and should not be put aside. What we do suggest is that teachers learn to see that proof is more than its logic, so that students' problems with proof are not necessarily rooted in their failure to grasp logical principles; beyond those logical principles, proof also requires a certain posture with respect to a mathematical community, a certain way of asserting oneself. We would like to see, in other words, that a teacher involved in a conversation such as that with Ronit and Yana will not deem those remarks, which we coded "They and We," only marginally important or even completely nugatory, but will take hold of them as levers towards a deeper understanding of proof and more fruitful engagement in the activity of proof. Indeed, being aware of such levers may well help teachers find a bridge between the process of classroom discourse and its mathematical content, the lack of which has been known to be a source of tension in teachers' practices (e.g., Sherin, 2002).

The insights gained from the conversation with Yana and Ronit, we believe, are also transferable to other kinds of dialogues teachers might pursue with their students. For example, in other interviews with the LPS students, we asked the students about "doing mathematics in their heads." The students did not always see the necessity of going beyond that, that is, they did not see the need to give an account of their thinking. The need to give an account goes hand in hand with the need for proof—both involve making assertions and being accountable for them, and, again, words like "assertiveness" and "accountability" form a bridge between ideas of proof and ideas of authority and community (compare figure 1)—the latter being because accountability implies a community to which one is accountable. So, the question of "doing mathematics in one's head," though not explicitly about proof, is deeply related to proof as we have spoken about it here.

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