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How Structure Sense for Algebraic Expressions or Equations is Related to Structure Sense for Abstract Algebra

Jarmila Novotná
Charles University, Czech Republic

Maureen Hoch
Tel Aviv University, Israel

Many students have difficulties with basic algebraic concepts at high school and at university. In this paper two levels of algebraic structure sense are defined: for high school algebra and for university algebra. We suggest that high school algebra structure sense components are sub-components of some university algebra structure sense components, and that several components of university algebra structure sense are analogies of high school algebra structure sense components. We present a theoretical argument for these hypotheses, with some examples. We recommend emphasizing structure sense in high school algebra in the hope of easing students' paths in university algebra.

The cooperation of the authors in the domain of *structure sense* originated at a scientific conference where they each presented the results of their research in their own countries: Israel and the Czech Republic. Their findings clearly show that they are dealing with similar situations, concepts, obstacles, and so on, at two different levels—high school and university.

In their classrooms, high school teachers are dismayed by students' inability to apply basic algebraic techniques in contexts different from those they have experienced. Many students who arrive in high school with excellent grades in mathematics from the junior-high school prove to be poor at algebraic manipulations. Even students who succeed well in 10th grade algebra show disappointing results later on, because of the algebra. Specifically, some students drop out of advanced mathematics in 11th grade due to an inability to apply algebraic techniques in different contexts (Hoch & Dreyfus, 2004, 2005, 2006).

Similarly, university lecturers involved in training future mathematics teachers often notice their students' difficulties in developing a deeper understanding of mathematical notions that they meet in their mathematics courses. We refer to experiences from Novotná and Stehlíková's longitudinal observation of university students—future mathematics teachers—during the course *Theoretical Arithmetic and Algebra* (Novotná, Stehlíková & Hoch, 2006). Students enter the course having experience with number sets and with linear and polynomial algebra (Novotná, 2000), but they often have problems with basic algebraic concepts.

We would like to emphasize the danger of assuming that students know or can do certain things just because they were in the school curriculum. For example, some high achieving students in 11th grade of high school were unable to factor the expression $81 - x^2$ (Hoch & Dreyfus, 2006) and others were unsure whether $(xy)^2$ is the same as x^2y^2. It is worthwhile also noting the low level of manipulative ability found in some high achieving students (those most likely to continue to university). In an algebra questionnaire distributed to 176 high achieving high school students we found that the majority did not manage to do more than half of the exercises accurately, or even with only minor errors (Hoch & Dreyfus, 2006). We had expected these students to maintain a high level of instrumental proficiency (manipulation skills), and that any difficulties they displayed would be on a relational level (structure sense). Yet these students performed neither at a high instrumental level nor at a high relational level.

The term structure is widely used and most people feel no need to explain what they mean by it. In different contexts the term structure can mean different things to different people (e.g., Dreyfus & Eisenberg, 1996; Hoch & Dreyfus, 2004; Stehlíková, 2004). The term algebraic structure is usually used in abstract algebra and may be understood to consist of a set closed under one or more operations, satisfying some axioms. Hoch (2003) discussed and analysed structure in high school algebra, considering grammatical form (Esty, 1992), analogies to numerical structure (Linchevski & Livneh, 1999), and hierarchies (Sfard & Linchevski, 1994), culminating in a description of algebraic structure in terms of shape and order.

Structure Sense in High School Algebra

We explain students' difficulty with applying previously learned algebraic techniques in high school as a lack of structure sense, a term coined by Linchevski and Livneh (1999). They suggested that students' difficulties with algebraic structure are in part due to their lack of understanding of structural notions in arithmetic. Their conclusions were based on research on students just before and just after beginning algebra. The structure they examined is the order of operations in arithmetic expressions. There was no discussion of structure or structure sense in terms of what that might mean beyond the initial stage.

We consider structure sense to be an extension of symbol sense, which is an extension of number sense. Number sense can be described as an intuition for numbers that includes such things as an eye for obviously wrong answers and an instinct for choosing the arithmetic operation needed to solve a given problem (Greeno, 1991). Arcavi (1994) suggested that symbol sense is a complex feel for symbols, which would include appreciation for the power of symbols, ability to manipulate and to interpret symbolic expressions, and a sense of the different roles symbols can play in different contexts. Arcavi talked about algebraic symbols displaying structure, and Zorn (2002), for example, talked about unpacking the symbolism to reveal meaning and structure.

A sense for structure has only been hinted at in the literature. Kieran (1992) discussed students' inability to distinguish structural features of equations. Linchevski and Vinner (1990) suggested that one of the components of success in school mathematics is the ability to identify *hidden structures* in algebraic terms. Kirshner and Awtry (2004) discussed visual salience of algebra rules: "Visually salient rules have a visual coherence that makes the left- and right-hand sides of the equation appear naturally related to one another" (p. 229). For example, the rule $\frac{w}{x} \cdot \frac{y}{z} = \frac{wy}{xz}$ is visually salient while the rule $\frac{w}{x} \div \frac{y}{z} = \frac{wz}{xy}$ is not. Tall and Thomas (1991) indicated that versatility of thought is necessary to switch from an analytical approach to a global one, giving as an example the ability to see $2x + 1$ as a common factor in the expression $(2x + 1)^2 - 3x(2x + 1)$. Pierce and Stacey (2001) defined and investigated algebraic expectation, which includes identifying form and linking form to solution type. Structure sense requires anticipation, which Boero (2001) considered to be crucial for directing the transformation of mathematical structure when attempting to solve an algebraic problem.

Our definition of high school (HS) structure sense is an operational definition that will enable us to determine by observation whether a student is using structure sense. This definition was developed as follows. A preliminary definition was formulated according to theoretical considerations and empirical observations. Five experts in mathematics education research were interviewed to acquire their views on structure sense. These views were used as a basis for an intermediate definition of algebraic structure sense that was used as a guideline to design questionnaires. These questionnaires, administered to students near the end of 10th grade, were analysed and found to be inadequate to identify some aspects of structure sense. It was thus considered necessary to refine the definition further. The idea was developed and refined by Hoch (2007) who arrived at the following definition.

Students are said to display structure sense for high school algebra if they can:

1. Recognise a familiar structure in its simplest form
2. Deal with a compound term as a single entity and through an appropriate substitution recognise a familiar structure in a more complex form
3. Choose appropriate manipulations to make best use of a structure

The following are examples for each type of structure sense, as related to the structure $a^2 - b^2$ (difference of squares).

- Structure sense 1: Factor $81 - x^2$ —recognise difference of squares and factor accordingly
- Structure sense 2: Factor $(x - 3)^4 - (x + 3)^4$—deal with $(x - 3)^2$ and $(x + 3)^2$ as single entities, recognise difference of squares of these entities, and factor accordingly
- Structure sense 3: Factor $24x^6y^4 - 150z^8$—see the possibility of difference of squares, extract common factor to get $6(4x^6y^4 - 25z^8)$, deal with $2x^3y^2$ and $5z^4$ as single entities, recognise difference of squares of these entities, and factor accordingly

An important feature of structure sense is the substitution principle, which states that if a variable or parameter is replaced by a compound term (product or sum), or if a compound term is replaced by a parameter, the structure remains the same.

Structure Sense in University Algebra

Novotná et al. (2006) adapted structure sense and defined it for university algebra. University algebra (UA) structure sense was developed by using longitudinal observations of future mathematics teachers, by analysis and classification of these students' mistakes, and by looking for analogies with HS structure sense (Novotná, et al., 2006). We distinguish two main stages of UA structure sense, SSE and SSP, each of which is further subdivided into components. In this paper, we define them as follows.

SSE: Structure Sense as Applied to Elements of Sets and the Notion of Binary Operation

Students are said to display SSE if they can:

- (SSE-1) Recognise a binary operation in familiar structures
- (SSE-2) Recognise a binary operation in non-familiar structures
- (SSE-3) See elements of the set as objects to be manipulated, and understand the closure property

The vague terms 'familiar and non-familiar structures' can be explained as structures that must be seen by students as conceptual entities, so that they that can take these objects as inputs to procedures (Harel & Tall, 1991). What will be 'familiar' depends on how the individual student was introduced to abstract algebra.

SSP: Structure Sense as Applied to Properties of Binary Operations

Students are said to display SSP if they can:

- (SSP-1) Understand identity element in terms of its definition (abstractly)
- (SSP-2) See the relationship between identity and inverse elements
- (SSP-3) Use one property as a supporting tool for easier treatment of another: (e.g. commutativity for identity element, commutativity for inverse element, commutativity for associativity)
- (SSP-4) Keep the quality and order of quantifiers

SSP involves attending to interrelationships between objects that are consequences of operations. We, as teachers, "would like our students to attend not to the particular objects and operation, but to the fact that imposing the operation on the set of objects creates interrelationships which are important, such as associativity, inverse elements, etc." (Simpson & Stehlíková, 2006, p. 350). SSP can be analysed only for students who have at least partial SSE. The situation is more complicated here, involving both objects (identity and inverse elements) and properties (commutative, associative, and distributive—in the case of two operations). Moreover, there are two focus points: the first on individual properties and objects, the second on the role of quantifiers in the definition (their type and order). For the subdivision of SSP, we looked into mutual relationships among objects.

The following are examples for types of structure sense, as related to binary operations and their properties:

- (SSE-1): Students display SSE-1 if they can determine whether the following are binary operations (N is the set of natural numbers, Z is the set of integers, R is the set of real numbers):
- $(N,\circ)$: $x \circ y = x + y$; $(N,\triangleright)$: $x \triangleright y = x - y$; $(Z,\oplus)$: $x \oplus y = x + y$;
- $(Z,*)$: $x * y = x - y$; $(Z,\otimes)$: $x \otimes y = x \cdot y$; $(R,\bullet)$: $x \bullet y = x \div y$;
- $(R,\succ)$: $x \succ y \Leftrightarrow \exists k \in R : x = y + k$.
- (SSE-2): Students display SSE-2 if they can determine whether the following are binary operations:
- $(Z,\oplus)$: $x \oplus y = x + y - 4$; $(R,*)$: $x * y = x \cdot y - 2$ $(Z,\otimes)$: $x \otimes y = 5x - 6y$;
- $(Z,\bullet)$: $x \bullet y = 3x + xy$; $(R,\circ)$: $x \circ y = x^y$; (R, Δ): $x \,\Delta\, y = \dfrac{9x^2 - 16y^2}{6x - 8y}$.
- (SSE-3): When students are asked to find the identity element in $(F, \circ)$, where F is the set of real functions defined on R and $\circ$ is the composition of functions, they display SSE-3 if they start working with properties of mappings and discover that $n(x) = x$ is the identity element in this structure. Students who start working with numbers

display a lack SSE-3; later they may answer that the identity element is 1 without taking into consideration the nature of objects in the set.

- (SSP-1): Students display SSP-1 if they answer that 99 is the identity element in (Z_{99}, +), where Z_{99} = {1, 2, ... 99} and + is addition in congruence modulo 99. They lack SSP-1 if they answer that there is no identity element because there is no 0 in the set.
- (SSP-2): Students display SSP-2 if, given (O, +), where O is the set of odd numbers and + is the addition of integers, they answer that it is not meaningful to look for inverse elements because there is no identity element in the structure ($0 \in Z$ is not an element of O)[1]. They lack SSP-2 if they say that the inverse element of 3, for example, is –3 because both are odd.
- (SSP-3): In the case of (R^+, $\circ$), where R^+ is the set of positive real numbers and
- $x \circ y = x^y$, students display SSP-3 if, due to the absence of commutativity, they start checking both equalities $x \circ n = n \circ x = x$. They show a lack of SSP-3 if they say that identity element is $n = 1$ because $x^1 = x$ (correct answer: n does not exist – $1^x \neq x$).
- (SSP-4): Students display SSP-4 if, given (L, $*$), where L is the set of all positive rational numbers, $x * y = \frac{x}{2} + \frac{y}{2} + x$, they answer that there is no identity element because although for any x, $x * n = \frac{x}{2} + \frac{n}{2} + x$ and therefore the equation $x * n = x$ has the solution $n = \frac{x}{1+2x}$, i.e. n depends on x. Students do not display SSP-4 if they answer that $n = \frac{x}{1+2x}$ is the identity element: instead of "there exists n such that for all x ...", they use "for all x there exists n such that ...".

Note: On the other hand, here the students display SSP-3 by using commutativity for inverse elements.

The model presented here accounts for only binary operations and their properties. A model for groups, for example, would have to be far more complex (see e.g., Dubinsky et al., 1994).

Research Question and Hypotheses

In this paper we examine how structure sense for algebraic expressions or

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[1] This can also be interpreted in terms of students' concept image of inverse element. The number –3 could have simply been chosen because their concept image of inverse element is a negative number. It is widely accepted that students tend to rely on their images from number theory when studying and applying group theory (e.g., Stehlíková, 2004). They often hold a deeply rooted image of the additive identity element in numerical contexts necessarily being 0, and the additive inverse element a negative number.

equations is related to structure sense for structures in abstract algebra. We discuss two types of relationship between high school (HS) and university algebra (UA) structure sense:

1. HS structure sense components as sub-components of UA structure sense components.
2. UA structure sense components as analogies of HS structure sense components.

Our hypotheses are:

For relationship type 1:

- R1-1: A student who does not have a high level of structure sense 2 (HS) cannot display a high level of SSE-2 (UA).
- R1-2: SSP (UA) cannot be developed without a high level of structure sense 1 (HS) and structure sense 3 (HS).

For relationship type 2:

- R2-1: SSE-1 (UA) is an analogy or generalisation of structure sense 1 (HS); a student who does not have a high level of structure sense 1 (HS) cannot display a high level of SSE-1 (UA).
- R2-2: SSP (UA) (mainly SSP-2 and SSP-3) is an analogy or generalisation of structure sense 3 (HS).

Theoretical Analysis of the Hypotheses

We present here theoretical justifications for the four hypotheses.

R1-1: Let M be a set, and $\circ$ a relation between M x M and M. Often, determining whether $\circ$ is an inner binary operation on M—the mapping M x $M \rightarrow M$—requires simplifying the formula (e.g. by factoring some algebraic expressions or other editing of the formula). To perform the necessary steps requires structure sense 2 (HS).

Example: Let $x \,\Delta\, y = \dfrac{9x^2 - 16y^2}{6x - 8y}$. Decide whether Δ is a binary operation on the set of real numbers and if so, determine its properties. The formula defining Δ is in non-standard form. $x \,\Delta\, y$ is not defined on R. Examining its properties requires structure sense 2 (HS); if we want to simplify the formula we need to factor the numerator and denominator.

Remark: Students may solve the task in this example without factoring; in this case, it is sufficient if they remember the existence condition (denominator not equal 0) and then they do not need to use HS structure sense 2.

R1-2: SSP deals with the properties of structures with binary operations. To find out if structure (M, $\circ$) has an identity element, which elements of M have an inverse element, or if $\circ$ is a commutative or associative operation requires, in most cases, a treatment of algebraic expressions and solution of equations. Similarly to R1-1, this cannot be done without developed structure senses 1 and 3 (HS).

Example: Let us define (D, $\circ$) as follows: $D = R^+ \cup \{0\}$, $x \circ y = x + y + xy$. It is obvious that $\circ$ is a binary operation on D. The following are examples of questions that require SSP (UA):

- Does an identity element exist? If yes, find it. In order to answer this question, the solver has to decide if there exists $n \in D$ such that for each $x \in D$ the following is satisfied: $x + n + xn = x$ (the operation is commutative therefore it is sufficient to solve one equation instead of two); this leads to solving the equation $n + xn = 0$ with the unknown n. A student lacking structure sense 1 (HS) cannot find the answer $n = 0$.

- Find all $x \in D$ which have inverse elements in $(D, \circ)$ and determine them. In order to answer this question, the following equation has to be solved for each $x \in D$: $x + x^{-1} + x\, x^{-1} = 0$. Structure senses 1 and 3 (HS) are needed for the solution.

R2-1: Structure sense 1 (HS) deals with the simplest forms of algebraic expressions and equations. It can be reformulated as being able to recognise the affinity of simple formulas or equations in standard form, when the solver knows the standard forms. Several step factoring or substitution is not required; the affinity is "transparent". SSE-1 (UA) requires an analogical treatment in the following sense: The general definitions (analogy to the general formula/equation/…) are given in their standard form, and checking the concrete "simple" operation characteristics and properties is done for the simplest ("transparent") algebraic structures. See the examples of SSE-1 above.

R2-2: Structure sense 3 (HS) deals with choosing appropriate manipulations to make best use of a structure. SSP (UA) can be considered as analogies because here, the choice of appropriate operation properties as tools for recognising or finding others is required. For example, in case of SSP-2, students may be familiar with the definition of identity and inverse elements, they might even be able to apply the general definition for the property in question, but do not pay attention to the existence of all necessary prerequisites, such as the existence of identity element in the structure. In the case of SSP-3, students can know the formal definitions of properties to be checked, can even be able to apply the definitions in individual cases, but do not make use of the consequences of already confirmed properties. Let us illustrate this in the case of identity element and commutativity: If an operation is proved to be commutative, then one equation only is sufficient to determine the identity element. In the case of commutativity and associativity this is even more efficient, as seen in the following example.

Example: Is the operation $\circ$ in the structure $(M, \circ)$, where $M = \{e, a, b, c\}$ and $\circ$ is defined in the table below, associative?

Table 1
Structure (M, ○)

○	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

To verify that associativity is valid in a structure with n elements, (maximum of) n^3 equalities are to be checked. Here 4^3 equalities would be required for $\circ$ [$(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in M$], without the benefit of commutativity. Using commutativity reduces considerably the number of necessary calculations. With the use of commutativity, the order of elements does not change the results. Applying commutativity can be seen as the best use of the structure's properties.

Illustration from Future Teachers' Work

We present here some examples from our teacher trainers' practice illustrating the theoretical conclusions. These examples were selected from a test solved by 31 students attending the pre-service secondary mathematics teacher-training course *Algebraic Structures* (usually a 3rd year course) at the Faculty of Education in Prague, in the school year 2005/06.

The Faculty of Education of Charles University in Prague offers mathematics teacher training for both primary and secondary mathematics. The future lower and upper secondary school teachers (from the 6th till the 12th grade) take a five-year course. Mathematics is combined with other subject (e.g. physics, descriptive geometry, computer science, chemistry, biology, geography, arts, or foreign languages).

Future teachers entering faculties of education were taught a varied range of mathematics at secondary schools of different types. We assume that students—future mathematics teachers—have positive attitudes towards this subject. Unfortunately, this attitude is not always accompanied by sufficient knowledge of mathematical concepts and skills. Students' structure sense (HS) is at different levels. Before the 3rd year of their pre-service mathematics teacher training at the Faculty of Education of Charles University, the future teachers successfully completed mathematical courses of algebra, geometry, calculus, and problem solving. They are several university level textbooks available for each course; properties or binary operations are included (e.g. in Novotná & Trch, 1993).

The aim of test was to acquire feedback about students' knowledge and skills in the domain of introduction to abstract algebra. The four tasks below focused directly on the relationships between structure sense at HS and UA levels. (Altogether there were eight tasks, others dealing with inequalities at the HS level and binary relations at the UA level.)

1. Solve in *R*: $(x+3)^2 \le 6x+18$.
2. Simplify in *R*: $\dfrac{(x-2)^4-(x+2)^4}{14x^2-2x}$.
3. Let $x \Delta y = \dfrac{9x^2-16y^2}{6x-8y}$. Decide whether Δ is a binary operation on the set of real numbers and if so, determine its properties.
4. Consider the following structure: (Z, •), where Z is the set of integers, and $x \bullet y = x + y - 4$. If • is a binary operation on Z, determine its properties. If the neutral element exists, find the inverse elements of all integers for which they exist.

Tasks 1 and 2 are standard HS level tasks, and Tasks 3 and 4 are UA level tasks. In Task 1, the use of a mechanical solving algorithm (opening brackets, transferring sides, collecting like terms) results in the inequality $x^2-9\le 0$. This can be easily solved using structure sense 1 (HS). Alternatively, the solver could choose to factor to obtain $(x+3)^2-6(x+3)\le 0$, using structure sense 3 (HS). Simplification (factoring numerator and denominator) in Task 2 requires structure sense 2 (HS). Theoretically, the numerator $(x-2)^4-(x+2)^4$ could also be simplified by opening brackets and collecting like terms, but this is a long and cumbersome method, inviting calculation errors.

Tasks 3 and 4 are UA level tasks. The formula defining the operation in Task 3 is in non-standard form. $x \Delta y$ is not defined on R. Examining its properties requires structure sense 2 (HS). Task 4 is constructed as an analogy between structure sense 1 (HS) and SSP (UA).

In these test items, the relationships among HS and UA structure senses can be clearly traced. Tasks 1 and 2 were used as control tasks for checking HS structure sense.

Let us denote success in solving a task by 1, and failure by 0. We will record students' answers in the test by an ordered foursome (x_1, x_2, x_3, x_4) where $x_i = 1$ in the case of a correct solution to Task i, and 0 in the case of an incorrect solution.

Theoretically, there are 2^4 possible foursomes, some of them being more probable and some nearly impossible (according to our hypotheses). For example the following explanations of the success/failure in the tasks are considered:

- (1, 1, ,) represents a typical result of a student with a good command of HS structure sense
- (, , 1, 1) indicates UA structure sense
- (0, 0, ,) represents a result of a student lacking HS structure sense
- (, , 0, 0) indicates lack of UA structure sense

Examples from the test:

(1, 1, ,) occurred in 18 cases divided as follows:

- (1, 1, 0, 0): 6 students [indicating HS structure sense developed, no UA structure sense]
- (1, 1, 1, 0): 8 students [indicating HS structure sense developed, SSE-3 (UA) developed, SSP (UA) lacking]
- (1, 1, 1, 1): 4 students [indicating HS and UA structure sense developed]

(0, 0, ,) occurred in 2 cases divided as follows:

- (0, 0, 0, 0): 1 student [indicating neither HS nor UA structure sense]
- (0, 0, 1, 0): 1 student [we explained this case in the remark in R1-1: The student remembered that a denominator cannot equal 0 and did not need to use HS structure sense—he did not simplify the expression]

Although there were other foursomes, most of them were unique cases; we mention only those that occurred more frequently:

- (1, 0, 0, 0): 3 students [no UA structure sense]
- (1, 0, 1, 0): 4 students [SSE-3 (UA) developed, SSP (UA) lacking]

All, (bar one, explained above) of these cases could be explained by our hypotheses.

Concluding Remarks

In this paper, we presented theoretical arguments for each of four hypotheses. We used examples from the pre-service mathematics teacher-training course *Algebraic Structures* at the Faculty of Education in Prague as supporting arguments for our hypotheses. Mathematical backgrounds of students from their upper secondary studies[2] differ. On the one hand, the knowledge and skills of students

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[2] In the Czech educational system, pre-university studies are divided into primary (age 6-11), lower secondary (age 12-15), and upper secondary (age 15-19). In Israel they are divided into primary (age 6-12), junior-high (age 12-15), and high school (age 15-18). We consider

coming from technical schools are practically oriented. On the other hand, the knowledge of those who attended general upper secondary schools and specialised in mathematics and sciences is of a theoretical, more abstract nature.

We have characterized HS and AS structure sense and shown how they are manifested in the high school and university algebra environment. The roots of our perspective and the related literature are described in the theoretical background. The transition from HS to AS could be analysed in terms of Tall's (2007) framework of long-term learning, which consists of three distinct 'worlds of mathematics'—conceptual embodiment, proceptual symbolism and axiomatic formalism. HS structure sense is based on symbolic thinking (referring to the use of symbols that arise from performing an action schema, such as counting, where the symbols used become thinkable concepts, such as number), whereas UA structure sense belongs to the formal thinking "world" (based on formal definitions and proof).

If we attribute students' difficulties to their lack of structure sense, we can concentrate on developing their structure sense. Our results indicate that HS symbolic world structure sense could be a prerequisite for UA formal world structure sense. Obviously more research is required to verify this, but the implication is clear. Encouraging teachers in high school to place more emphasis on algebraic structure could help to ease students' transition from school to university mathematics. The relationships between the UA and HS structure senses studied in our paper can serve as a basis for a teaching programme explicitly addressing the problematic issues.

We see an attention to structure as being an important part of mathematics in general, and the learning of algebra in particular. The view of mathematics that students build up during their school career survives long after they leave secondary school. If we do not develop an awareness of structure in students during their teacher training at the faculty, this lack of awareness may return with the teachers back to the schools. The teaching of mathematics only as a set of precepts and instructions to be learned by rote can lead to ever-deeper formalism in the teaching of mathematics. This can result in a lack of understanding of the conceptual structure of the subject, and an inability to use mathematics meaningfully when solving real problems.

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Hersonissos, Crete, Greece: University of Crete.

Authors

Jarmila Novotná, Department of Mathematics and Mathematical Education, Faculty of Education, Charles University, Czech Republic. Email: <jarmila.novotna@pedf.cuni.cz>

Maureen Hoch, Tel Aviv University, Israel. Email: <hochfam@bezeqint.net>