The Spirit of Investigation: Modifying Pascal and Fibonacci

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Introduction

Using investigations in teaching mathematics has for many years become an established feature of most curricula around the world. And with good reason. Investigations can be a vehicle for enabling children to experience the genuine excitement that comes from mathematical discovery. Of course, most of the investigations used in teaching are instigated by the teachers themselves, who already have prior knowledge that they will lead to specific, desirable discoveries or conclusions. This is perfectly valid because, for the individual child who discovers a particular relationship or connection or rule, it really does not matter that it was already known by others. The point is that it is new for them.

But the true spirit of inquiry and investigation lies in a mind-set that continually asks questions about a given situation. And it is always going to be more interesting if you yourself, rather than a third party, is the one to ask the questions. This is what, as a loftier purpose, we should ultimately be trying to encourage in our pupils. This is idealistic of course and I was recently brought back down to earth by some of my students (training to be teachers) when we were discussing these ideas. I was suggesting that one of the most fruitful questions we can ask about a mathematical situation is "What would happen if…?" This kind of question gives us the freedom to change anything we like in the original situation (e.g., some parameters or conditions) and is also likely to lead us into uncharted territory. I was also claiming that it is easy to ask this question in any mathematical situation.

One of my students then commented that topics like Pascal's Triangle, the Fibonacci Sequence, Golden Ratio, etc. must be among the most explored topics in mathematics and are often used for investigations in teaching. How is it possible to come up with any new questions on such topics? An Internet search using Google, for example, gives thousands of hits for each of these phrases (in fact, well over a million for the Golden Ratio!) So what can possibly be left to investigate? Of course, I made the point again that this does not really matter as long as what you are investigating is new to you. However, the challenge was implicitly there: use your question to come up with something we have not seen before.

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Let us look at Pascal's Triangle and first remind ourselves of some of the reasons why it is interesting. Largely it is because of the many patterns that emerge from the triangle and the links with other areas of mathematics (binomial expansion etc.) For example, in the diagonals we find the sequence of natural numbers and the triangle numbers, and the sums of "oblique" diagonals even yield the Fibonacci sequence. Perhaps the most striking pattern concerning sums is found in the rows of the triangle and this pattern is easily found by children:

Row	1	2	3	4	5	6	…
Sum	1	2	4	8	16	32	…

This gives us the general formula 2^{n-1} for the nth row.

Let us go back to the rule for generating Pascal's Triangle. It is simply adding two adjacent terms in a row to produce an element in the next row. So, what happens if we start with different numbers? Since it is with the second row that we actually have a pair of numbers to add, then for the purpose of our investigation the first row is essentially redundant. We could say that we are really looking at Pascal Trapeziums for which the original one starts:

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Interestingly, this immediately gives us a slightly different, and simpler, generalisation for the sums of rows. That is, we now have the formula 2^n for the nth row.

Now, let us start with 1,2 for the first row. (It does not matter that this is not connected to any specific mathematical relationship, as with binomial expansions; we are simply playing with the numbers here.) The Pascal Trapezium becomes:

1 2

1 3 2

1 4 5 2

1 5 9 7 2

1 6 14 16 9 2

1 7 20 30 25 11 2

Interesting patterns again appear, as with the original Pascal Triangle. Of course, we still have a diagonal of the natural numbers (now starting at 2) which is generated by the 1 in the first row. In addition, we now have a diagonal of the odd numbers (generated by the 2 in the first row) and this naturally gives rise to a diagonal of the square numbers next to it. However, let us keep our focus on the sums of the rows. We now have:

Row	1	2	3	4	5	6	…
Sum	3	6	12	24	48	96	…

This is just as striking as the original Pascal trapezium situation and just as easily generalisable. That is, we have 3.2^{n-1} for the nth row. But notice, in passing, that we have curiously returned to the term 2^{n-1} in this formula.

Let us try another starting point, for example 2,3 for the first row and denote this trapezium by Pascal (2,3).

2 3
2 5 3
2 7 8 3
2 9 15 11 3
2 11 24 26 14 3
2 13 35 50 40 17 3

The table of row sums gives us:

Row	1	2	3	4	5	6	…
Sum	5	10	20	40	80	160	…

Again, the pattern is striking and the generalisation becomes 5.2^{n-1} for the nth row. Moreover, the overall pattern of these results for different starting rows is also becoming clear. We appear to have a constant term of 2^{n-1} for each formula and the multiplier of this term appears to be simply the sum of the two numbers in the first row. Going back to our first Pascal Trapezium, which we now designate Pascal (1,1), we see that this also fits in with our new generalisation. That is, the formula can be written as 2.2^{n-1} for the nth row and it is only because the multiplier happens to be 2 that it simplifies to 2^n as before.

The explanation for the doubling feature of these results, although simple enough in the original Pascal Triangle, is perhaps even clearer in these generalised versions. For example, let us consider the transition from the 4th to the 5th row in the Pascal (2,3) pattern. If we write it without actually performing the additions but simply recording the numbers involved, we have:

2 9 15 11 3

2 2,9 9,15 15,11 11,3 3

Now we see clearly that the 5th row is just duplicating the 4th row twice. Because Pascal's Triangle is symmetric, a similar illustration for that involves a confusing number of repetitions of individual numbers. This also helps us to see the reason for the multiplier. If the first row consists of a,b

then the consecutive row sums become:

Row	1	2	3	4	5	6	…
Sum	$a+b$	$2(a+b)$	$4(a+b)$	$8(a+b)$	$16(a+b)$	$32(a+b)$	…

Our final generalised result then becomes, for a Pascal (a,b) Trapezium:

$$\text{Sum of terms in the } n\text{th row} = (a+b).2^{n-1}$$

Fibonacci

Let us turn our attention now to another popular source of investigations. What is often referred to as the Fibonacci sequence begins with 1,1 and subsequent terms are recursively defined by $a_n = a_{n-1} + a_{n-2}$. Thus we have:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

As with the Pascal triangle situation, many interesting patterns and relationships can be found in the sequence. For example, given any ten consecutive terms in this sequence, the sum of this subsequence is equal to 11 times the 7th term of the subsequence. To illustrate, consider the subsequence 5, 8, 13, 21, 34, 55, 89, 144, 233, 377. The sum of these terms is 979. The 7th term of the subsequence is 89, and 979 = 89 × 11.

A less well-known relationship is found with any four consecutive terms. For example, consider 3, 5, 8, 13. Construct the product of the two outer terms (i.e., 3 × 13) and twice the product of the two inner terms (i.e., 2 × 5 × 8). Then we have 39 and 80. These two numbers form two parts of a Pythagorean Triple: $39^2 + 80^2 = 89^2$. It turns out that the third number of this triple (in this case 89) is also a member of the Fibonacci sequence. (Readers may like to check this rather surprising result with other sequences of four consecutive terms).

However, perhaps the best known property of the sequence is the fact that the ratios of consecutive terms themselves form a sequence that converges to a limit and this limit is the Golden Ratio ϕ. In this case, correcting the ratio values to 3 decimal places, the limit 1.618 already appears with the 9th and 10th terms 34 and 55, which is a rapid convergence. (One has to be a little careful here since it is an oscillating sequence. That is, consecutive terms are alternately above and below the limit of the sequence).

It is also well-known that many of the patterns and properties arising in the Fibonacci sequence re-occur whatever two numbers are chosen to start the sequence. (Do the properties mentioned earlier still hold? This would be worth investigating.) In particular, the limit of the ratios of consecutive terms is always ϕ. Changing the initial numbers of the sequence is of course what we have just done with Pascal's Triangle. So this time let us focus on the recursive rule itself. That is, instead of generating the sequence by adding the two previous terms, let us ask what happens if we add the three previous terms? For simplicity, we start in the same way as the original Fibonacci sequence, that is, with ones. The new sequence, which we shall denote by Fibonacci (3), becomes:

1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201, …

The obvious question is: do the ratios of consecutive terms for this sequence also converge to a limit? In fact, again rounding to 3 decimal places, in this case we find that it does indeed tend to a limit. This time the limit is 1.839 which settles down after the 11th and 12th terms 193 and 355; a slightly slower rate of convergence than before.

We can clearly continue this investigation by considering Fibonacci (4) and Fibonacci (5) sequences and so on. The Fibonacci (4) sequence is:

1, 1, 1, 1, 4, 7, 13, 25, 49, 94, 181, 349, 673, 1297, 2500, …

In this case the limit of the ratios is 1.928 to 3 decimal places and this settles down after the 14th and 15th terms. Similarly, Fibonacci (5) gives us:

1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 129, 253, 497, 977, 1921, 3777, …

This time the limit of ratios is 1.966 to 3 decimal places which is reached, somewhat surprisingly, after the 13th and 14th terms 497 and 977. We can tabulate our results so far, denoting Fibonacci (3), etc. as F(3):

Sequence	F(2)	F(3)	F(4)	F(5)
Limit of ratios	1.618	1.839	1.928	1.966

It appears as though the sequence of these limits is itself converging to a limit which we could guess is 2. But what sequence is such that the ratio of consecutive terms is exactly 2? Of course, this is simply the powers of 2. (An interesting connection here with the earlier Pascal results?) That is, the following sequence:

1, 2, 4, 8, 16, 32, 64, …

How can this sequence emerge as a limiting process of the Fibonacci sequences that we have defined? In fact, it is clear from the definition of the sequences that they are all increasing sequences (after the first few terms) since each new term is defined as the sum of a number of previous terms. But if each new term is the sum of m previous terms say ($m \geq 2$) then obviously the ratio of consecutive terms must always be less than 2. Now, for each Fibonacci sequence we have investigated above, we have been increasing the number of previous terms to be added in the recursive definition. Let us consider a slight variation of this. Suppose we define a new sequence such that each new term is the sum of all previous terms in the sequence. Now we can start the sequence with just the number 1 (or indeed any number) which will give us, for example:

1, 1, 2, 4, 8, 16, 32, …

or 3, 3, 6, 12, 24, 48, … etc.

and these sequences have exactly the ratio 2 for successive terms (apart from the first two terms) and are also Fibonacci-type sequences. It is now intuitively clear that our earlier sequence of limits tends to 2 because increasing the number of previous terms to be added is taking us closer to the situation of adding all previous terms.

Conclusion

These are not "mind-blowing" conclusions but they did have the effect of convincing my students that the "What if…?" question is powerful, simple to use, and may yield some interesting results. This was all the more meaningful because these conclusions were new to them, and to me. I have not the slightest doubt that somewhere among those thousands of search engine hits precisely these ideas (or something similar) are to be found, although I have not come across them. It is also important to go back to the phrase I used earlier, that is "may yield some interesting results," because another important element of the true spirit of investigation is that one really does not know where the investigation will lead, unlike the normal classroom situation where the students do "know" that it is leading somewhere because the teacher has given them the starting point. If we can encourage students to come up with their own questions, and therefore their own starting points for an investigation, we can be fairly confident that any discoveries they do make will be that much more satisfying and, dare I say, magical for them.