DISCOVERY

with Neville de Mestre

Experiments with patterns

Here is a hands-on experiment that covers many areas of high school mathematics. Included are the notions of patterns, proof, triangular numbers and various aspects of problem solving.

Consider the following problem which involves the arrangements of a school of fish. Suppose that the fish are swimming along in a triangular pattern as shown in Figure 1.

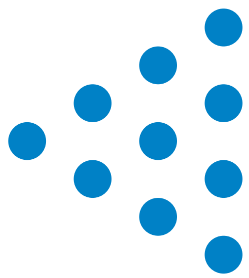

Figure 1

We want to reverse this pattern by moving only three fish. Your students could use split peas or buttons to represent the fish, and you could use plastic counters on an overhead projector to illustrate the appropriate moves.

Once your students have attempted the problem, you will notice that some have succeeded quickly, while others may still be struggling after some time. The ones who have solved it quickly could be asked to write out a clear description of their solution. They may find this rather difficult to do, but it will illustrate to them how important it is to have unambiguous statements in their description. This should occupy them for a short time while you attend to the remainder.

For those struggling with the problem, you could now introduce them to the well-known problem-solving technique recommended by George Polya (1945). He said that if you cannot solve a problem immediately you should consider a related, but simpler, problem. In this case a simpler related problem would be to consider only six fish in a triangular pattern, as in Figure 2.

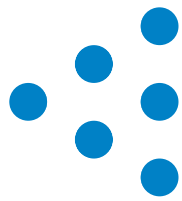

Figure 2

It is extremely easy to reverse this pattern by moving three fish: just move column 2 to the right of column 3, and then move column 1 to the right of them all. Now ask your students to try to reverse the pattern in Figure 2 by moving only two fish. Hopefully some may see that this can be achieved by moving the top and bottom fish in column 3 to equivalent positions in column 1. It should then be suggested that they try this technique on the original ten-fish problem.

Some will move the top and bottom fish of column 3 to column 1, but eventually they will move the top and bottom fish of column 4 to column 2. It is then a simple matter to move column 1 to the far right and "Eureka! We have solved it." What a sense of achievement this is for many students.

Returning to the six-fish problem of figure 2, you could ask your students to reverse the pattern by moving only one fish. Let them try for a short time, and wait to see if one of them suggests that this is impossible. If not, you can tell them that it is impossible, and then raise the question of how to prove this. The notion of proof is important in many mathematics courses, and you should indicate that later on you will give one proof, but state that there may be others.

At this stage you should introduce the concept of the minimum number of moves needed to reverse the triangular fish patterns. Start with the simple three-fish triangular pattern formed from one fish in column 1 and two fish in column 2. Clearly the minimum number of moves will be one. Make a table of the experimental information obtained so far, as indicated below.

Columns (C)	Fish (F)	Minimum moved (M)
2	3	1
3	6	2
4	10	3

Your students should satisfy themselves that three is the minimum number of moves for the ten-fish problem. Next add a fifth column containing five fish to the triangular pattern of Figure 1. The total number of fish (F) will now be 15, and you can talk about triangular numbers at this stage. Your students should then be asked to make a conjecture (this is what mathematicians frequently do) about the minimum number needed to be moved to reverse the pattern. Many of them will look at the table and say four, and you should ask them to check this by experiment with their buttons or peas. Soon they will let you know that it can be done by moving five, but no one will be able to do it by moving only four. You can tell them that four is impossible and this will also be proved shortly.

Now add a sixth column with six fish and seek a conjecture to extend the corresponding M value in the table. They will look for a pattern and those who have come across the concept of Fibonacci numbers will invariably say eight is the minimum number for M. Again ask them to check this by experiment, and they will all be able to reverse the pattern in eight moves. Wait for a moment, as someone may tell you that they can do it in seven moves. This is possible and the solution is indicated in Figure 3, where O denotes the original pattern positions and x denotes the reversed pattern positions.

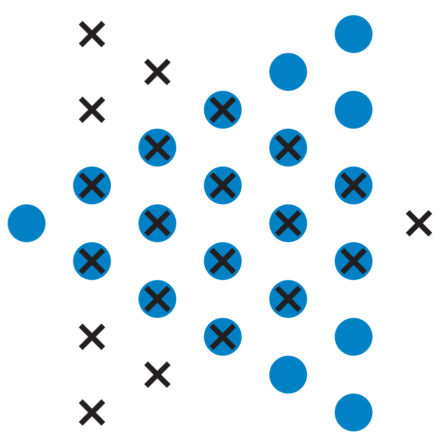

Figure 3

So why is the pattern not Fibonacci? Well, if you insert the results for the simplest degenerative problem involving only one fish in one column, you will have C=1, F=1, M=0 in the first row of the table. Now you can see that the pattern is not Fibonacci. So let us present the extended table discovered so far.

Columns (C)	Fish (F)	Minimum moved (M)
1	1	0
2	3	1
3	6	2
4	10	3
5	15	5
6	21	7

Figure 3 sheds other light on the solution to the general problem. Instead of concentrating on the minimum number to be moved, we could look at the maximum number not moved. It is observed that the number not moved has to be symmetrical about a column.

Obtaining the solution for the maximum number not moved is called the dual problem, since its solution solves the original problem immediately. The dual problem is one method for obtaining the proofs of earlier statements. For example, in the six-fish problem we can consider each of the three columns as the central column of the symmetric pattern of fish not moved. The results are shown in Figure 4 with ⊗ denoting the positions of fish not moved. Clearly the numbers not moved are 1, 4, 3 respectively, and hence the number moved are 5, 2, 3. Thus 2 is the minimum number needed to be moved, and 1 is impossible.

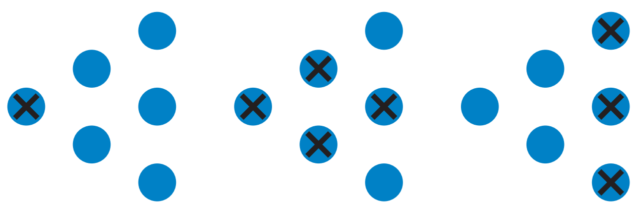

Figure 4

Your students should extend this idea to prove that three is the minimum number moved for the original ten-fish problem.

The stage is now set to continue with the experiment by adding more columns and extending the table. The pattern for C is the natural counting numbers, while that for F is the triangular numbers. But what is the pattern for M? By experiment this pattern is

0, 1, 2, 3, 5, 7, 9, 12, 15, 18, 22, 26, 30, 35, …

Look at the differences and note the pattern. Alternatively divide the value for *F* by 3 and note the results. Can your students find a general formula for any given integer value of *C*? It is

$$\lfloor C(C+1)/6$$

where $\lfloor$ denotes the floor function or integer just below or equal to the quotient. You could see if your students could discover that the number of fish for any value of *C* is $C(C+1)/2$, a well-known formula from arithmetic progressions which your students will meet later on in their mathematics courses, and is the formula for triangular numbers.

New experiments can be undertaken with different starting patterns. For example, you could start with an increasing odd number of fish in each column. (Having an increasing even number is easily seen to be a trivial problem.) Next your students could put the triangular numbers in successive columns and ascertain the minimum number needed to be moved to reverse the pattern. Extensions to square numbers, pentagonal numbers, etc., means that many experiments can be undertaken, but lots of buttons will be needed.

Finally, the counters could be replaced by matchsticks forming inter-connected equilateral triangles, and the minimum number sought to reverse the pattern. Some surprises await those who attempt this. It is much harder. Happy discoveries!

Reference

Polya, George (1945). *How to Solve It.* Princeton University Press.

From Helen Prochazka's

Scrapbook

CLASSIFYING MATHEMATICS

The Mathematics Subject Classification (MSC) was developed by the American Mathematical Society and Zentralblatt für Mathematik in Germany. It is used to categorize items covered by the two reviewing databases, Mathematical Reviews (MR) and Zentralblatt Math (Zbl). The current classification system, the 2000 Mathematics Subject Classification is a revision of the 1991 system. The MSC is broken down into more than 5000 classifications, each corresponding to a discipline of mathematics. It does not include classifications for elementary material. The first level is divided into just a few broad overlapping categories.

- **Foundations** considers questions in logic or set theory — the framework in which mathematics itself is carried out.
- **Algebra** is concerned with symmetry, patterns, discrete sets, and the rules for manipulating arithmetic operations.
- **Number theory** includes the study of rings and fields of numbers
- **Geometry** is concerned with shapes and sets, and the properties of them which are preserved under various kinds of motions.
- **Analysis** studies functions, the real number line, and the ideas of continuity and limit.