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Repeating Patterns and Multiplicative Thinking: Analysis of Classroom Interactions with 9 -Year- Old Students that Support the Transition from the Known to the Novel

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ABSTRACT

In early years' (primary grade) classrooms in Australia repeated patterns are commonly explored as an early introductory activity to mathematics. Most young students have an extensive knowledge of and exhibit success in copying, continuing, creating and transferring patterns into other media. By contrast, research indicates one of the most difficult concepts with which students grapple in their later years of elementary school is the notion of ratio. This paper reports on a design (teaching) experiment conducted over a four-lesson period in two classrooms comprising 51 students whose average age was 9 years and 6 months. The focus of these lessons was using students' knowledge of repeating patterns, an understanding that traditionally remains in the precinct of early years, to scaffold the introduction of ratio. The theoretical frameworks that underpinned the classroom interactions and learning were the socio-constructivist theory of learning, inquiry-based discourse and the simultaneous use of multi-representations to build new knowledge. The results show that after a short intervention period, repeating patterns can act as effective bridges for introducing the ratio concept. They also show that particular representations and teacher actions assisted students to identify ratio, recognize equivalence between particular ratios, and begin to represent these ideas in abstract notation systems.

INTRODUCTION

In response to the difficulties many adolescent students continue to experience with algebraic thinking (e.g., Warren, 1996) and their unwillingness to participate in higher levels of mathematics (Australian Council for Education Research, 1998), recent research has turned to examining how young children learn to embed algebraic reasoning *in* arithmetic reasoning. This movement to examining young children's learning of mathematics, in order to understand and inform the processes of learning at higher levels of schooling, indicates a shift *from* the traditional approach that views algebraic reasoning as occurring after the development of arithmetic reasoning *to* an approach that views algebraic reasoning as occurring in conjunction with arithmetic reasoning. Carpenter and Levi (2000) argued that the artificial separation of arithmetic and algebra "deprives children of powerful schemes of thinking in the early grades and makes it more difficult to learn algebra in the later years" (p. 1). The introduction of the Patterns and Algebra strand in the new Queensland Syllabus is a direct response to this concern. This is the first time that the state-wide syllabus includes a strand focused on combining arithmetic and algebraic reasoning as part of the outcomes for Year 1 through to Year 10 students.[1] It contains many changes that require teachers to embrace new content and pedagogy and to reconceptualise arithmetic as procedures rather than products, that is, a refocus on the underlying *structure of arithmetic* rather than *arithmetic as a computational tool*.

Exploring the Teaching and Learning of New Content

This article reports on a design (teaching) experiment conducted on one aspect of this strand, namely patterns and functions. The participants were 51 Year 5 students (average age 9 years and 6 months) and their teachers from two classes in two middle class State primary schools from two inner city suburbs in Brisbane, Australia. The teaching experiments undertaken for this study built on the conjecture-driven approach of Confrey and Lachance (2000). The conjecture consists of two dimensions: mathematical content and pedagogy linked to the content. The design aimed to produce both theoretical analyses and instructional innovations (Cobb, Yackel, & McClain 2000) with one variation:

[1] Compulsory schooling in Australia start in Year 1, when students reach 5 years of age and conclude with Year 10, when students are typically 15 years of age. Years 11 and 12 are not compulsory and are referred to as college. Those who attend universities are "university" students, not college students as is the case in other countries, such as the US.

one of the researchers acted as teacher (teacher/researcher), similar to Carraher, Schliemann and Brizuela (2001). The teacher/researcher entered the classroom at pertinent times throughout the school year to conduct small teaching episodes focusing on new content knowledge, with the aim of ascertaining the types of concrete materials, teacher actions and classroom conversations that promoted engagement. During these episodes the regular classroom teacher and the second researcher acted as participant observers. In this type of research, instructional design and research are interdependent (Cobb, Dean & Zhao, 2005). The design involves attempting to support the development of students' learning while at the same time investigating the processes and actions that assist the learning. Thus a hypothetical learning trajectory is postulated and conjectures are formulated about envisaged learning processes and specific means that might support these processes.

THE FOCUS QUESTIONS

The specific design (teaching) experiment that is examined in this article occurred over a four-lesson sequence. In line with this approach, during and in between each lesson, hypotheses were conceived 'on the fly' (Steffe & Thompson, 2000) and modifications in the design were responsive to observed actions and understandings of the teacher/researcher and the students. For example, although instructional tasks were generated prior to the commencement of each lesson, during the lessons some tasks were modified according to the classroom discourse and interactions, with new representations being introduced in order to challenge students' thinking and encourage them to justify their understandings.

The research activity and the tasks were closely aligned with one giving direction to the other, an evolutionary process. The tasks were intended to assist students in reconceptualizing repeating patterns as representations of ratios and equivalent ratios. We identified four interdependent classroom lessons in the design (teaching) experiment (Schoenfeld, 2006) that (a) explored repeating patterns to expose their mathematical structure, and (b) used this structure to explore how teachers supported the development of early understandings of the notion of ratio, that is, how they attempted to bridge the transition from the known to the novel. The particular questions that we attempted to address in this research episode were:

1. How can repeating patterns act as a bridge to the ratio concept?
2. What teacher actions assist in making these connections?
3. What roles do external representations play in scaffolding to the ratio concept?

By examining classroom interactions involved in teaching and learning mathematical patterns and functions, we demonstrate how the merging of arithmetical and mathematical thinking can support and constrain student learning of ratio and proportional reasoning.

Design of the Experiment and Theoretical Framework for Analysis

In the classroom episode examined in this study, students were given geometric tiles in order to construct repeating patterns. Duval (2002) provided the theoretical perspective we used to examine how external representations assist elementary students to negotiate meaning about visual patterns. Duval (2002) categorized mathematical knowledge as consisting of four registers (1) natural language (multi-functional) (2) tables/ figures/ diagrams (multi-functional), (3) notation systems/algebraic symbols (mono-functional), and (4) graphical representations (mono-functional). Mono-functional registers are characterized by processes that are algorithmic. In most instances, Duval argues, mathematical comprehension results from the coordination of at least two of these registers. Duval (2002) believes that such coordination of registers does not come naturally.

He further classifies mathematical transformation as being characterized as staying within one register (e.g., carrying out calculations while remaining strictly in the one notation system) or changing the register without changing the objects being used (e.g., passing from natural language of a relationship to using letters to represent it). These are respectively referred to as treatments and conversions. Duval suggests that conversions are the 'real' mathematical activity as they lead to the mechanisms underlying understanding. It is also the more difficult of the two and hence is commonly avoided by many teachers. Thus, the requirements for learning mathematics involve comparing similar representations within the same register, converting representations from one register to another, and understanding the mathematical processing that is performed in each register (Duval, 1999).

Within the longitudinal study, a part of which is analyzed in this article, teaching tasks focus on conversions between registers and domains (Duval, 1999) and linking, integrating and moving between representations to show mechanisms underlying understanding (particularly in terms of visual, table and symbolic generalizations of repeating patterns as ratios). Central to this approach and Duval's definition of mathematical reasoning and classroom interaction were the socio-constructivist theory of learning, inquiry based discourse and the simultaneous use of multi-representations to build new knowledge. Throughout the design (teaching) experiment the science of semiotics (e.g., Peirce, 1960; Radford, 2001; Saenz-Ludlow, 2001; Warren, 2003) reminded us that social discourse (interpretation) of differing representations (signs) assist in reaching an understanding of the ratio concept (the object). Neither the cognitive domain of the individual nor the social interaction is primary. Learning is an evolving process moving beyond particular signs to more and more complex representations, each giving deeper understanding to the object itself. In the teaching experiment reported here, students were encouraged to share their ideas in an investigative climate and explain these using appropriate representations and contexts.

METHODS

All lessons were videotaped using two video cameras, one on the teacher/researcher and one on the students, particularly focussing on students who actively participated in the discussion, thus capturing both teacher-student and student-student dialogical interactions. Both the teacher and teacher/researcher continually endeavoured to view students' responses in terms of strategies and sign choice from their own perspectives. Consequently the teacher had to interpret simultaneously her own mathematical actions and those of the students, with an understanding that this assisted in maintaining a meaningful dialogue.

The basis of rigour in participant observation is "the careful and conscious linking of the social process of engagement in the field with the technical aspects of data collection and decisions which that linking involves" (Ball, 1997, p. 311). Thus both observers (the second researcher and regular classroom teacher) acknowledged the interplay between them as classroom participants and their role in the research process. At the completion of the teaching phase, the researcher and teacher reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection, and arrive at some common perspective of the instruction that occurred and the thinking exhibited by the students participating in the classroom discussions.

The videotapes were transcribed and worksheets collected. The videos and participant observation scripts served to provide insights to the learning of the community and to identify specific actions, specific use of representations and specific conversations that supported this learning. The worksheets assisted in ascertaining how the individual student was progressing along the learning trajectory proposed for the whole community of learners. Thus the data was two tiered, the first tier relating to the classroom learning and the second tier focussing on the individual students within this classroom.

Five main dimensions were developed across the four lessons: (1) introducing language and notations to describe repeating patterns, (2) separating repeating patterns into their repeating components and discussing the number of different colored tiles in different number of repeats, (3) recording this information in tables of values and from the tables of values generalizing relationships for repeating patterns, (4) introducing ratio as the comparison between the components of repeating patterns, and (5) creating repeating patterns for various ratios and discussing the notion of equivalent ratios. The teaching episodes occurred over a four lesson sequence, with each lesson of approximately one hour's duration. Each lesson predominantly focused on one of the above dimensions per lesson. The language chosen to represent repeating patterns was *repeat*, *term*, *repeating part*, *number of repeats*, *component*, *ratio* and *equivalence*. Some typical activities conducted in the teaching phase are illustrated in Figures 1 and 2.

The first set of activities focused on physically reconfiguring repeating patterns to represent the concept of a ratio and recording this data in a table of values. The activities delineated within the lesson sequence proceeded from type (a) activities to type (d) activities (Figure 1).

The second set of activities were considered to be higher order thinking tasks, namely generalizing the patterns in the tables of values, creating repeating patterns when given the ratio of its components, and exploring the notion of equivalent ratios.The sequencing of activities proceeded from type (a) activities to type (c) activities (Figure 2).

Processes were established to try to ensure that the worksheets were truly representative of student's own thinking. First, students were not permitted to use erasers at any time throughout the lessons. Instead they were asked to 'cross out' what they had written if they wished to change their responses. This in itself required some discussion about how we were interested in their thinking rather than whether the answer was correct or not. This approach was at odds with the type of activity that commonly occurs within most Queensland mathematics classrooms, where student responses are either ticked as correct or incorrect and their ability to do mathematics is gauged on how many correct responses they were given. These students, however, had been working with us for a two-year period and had developed a response to our entry to their classrooms -- they automatically remove their erasers. This approach was also important for encouraging them to take risks and openly posture explanations about the activities. Second, students were not permitted to have a pencil in hand while classroom discussion occurred. Third, all worksheets were collected at the end of each discussion phase of the lessons. The responses of the worksheets collected during the four lessons were also analyzed. In this instance, responses were either marked correct or incorrect. In the

FIGURE 1

Activities that transform the repeating patterns to ratios.

a) Use tiles to create the repeating pattern

b) Separate the pattern into components (repeats)

c) Compare different number of repeats

1 repeat 

2 repeats 

3 repeats 

d) Record your data in a table of values

No of Repeats	*No of* □	*No of* ■	*Total Number*	*Ratio of* □ *to* ■

FIGURE 2

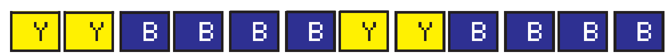
Higher order thinking activities.

a) Generalizing patterns in the table

No of Repeats	*No of* □	*No of* ■	*Total Number*	*Ratio of* □ *to* ■
				23 to 46
			240	
?				

b) Creating repeating patterns from ratios
Using the tiles ■, □, *and* ▨, *create the repeating pattern.*
Ratio of □ *to* ■ *is 2 to 3 and ratio of* ▨ *to* ■ *is 2 to 2*

c) Equivalent ratios
Suppose I had the ratios of yellow to blue tiles of 2:4 and 4:8 are they the same ratios or are they different? How are they the same? How are they different?



case where multiple responses were possible, grounded methods were used to identify categories of responses.

The videos were viewed by the researcher and the research assistants who were present during the teaching phases. This served as a member check on all stages of the analysis. From each video, significant episodes were selected. These episodes exemplified key points during the teaching phase and exhibited classroom dialogue that evidenced students' understanding of the stages of the lessons. Thus, the focus in the first stage of the analysis was on student learning and in particular on the students' responses to questions that encouraged them to explain and justify their answers and their interpretations of the signs (e.g., How? Why? What pattern can you see? Who agrees/disagrees?). The videos were reanalyzed to identify significant teacher actions that were believed to support this learning. The focus of this second stage was to identify questions, actions, or particular instruction that encouraged students to reinterpret the representations with the aim of supporting a deeper understanding of the ratio concept. While the transcripts provided the dialogue, the videos provided examples of the interactions between the dialogue, materials and gestures. The field notes served as a point of triangulation. The episodes and field notes were combined into rich descriptions of positive and negative relationships between the teaching and learning. The findings were used to develop hypotheses concerning effective teaching, including effective activities and purposeful questions.

RESULTS

The results are organized chronologically exemplifying significant episodes that occurred across the four lessons. Across the analyses, we discuss how the interactions within the episodes contribute to answering the three questions posed at the beginning of the article.

In the first lesson, students were asked initially to create a repeating pattern using their tiles and matchsticks, and then to physically separate the repeating pattern into its repeating parts. Under each part, they were to place the words 1st, 2nd, 3rd and so on. They were then asked to rearrange their repeats so that the same elements in each *repeat* were grouped together and questions such as: *How many blues in 1 repeat, 2 repeats, 10 repeats, 20 repeats? How many yellows? How many matchsticks?* were discussed. The data analysis indicated that students had a good understanding of what repeating patterns were. They could copy and continue repeating patterns, break the pattern into repeats and talk about the patterns that they saw across the repeats including relating the elements in each repeat to the number of repeats.

The focus of the second lesson was to introduce *the table of values representation* to assist in summarizing the data for the repeating pattern (see Figure 1, d). It was conjectured that the table of values would support conversations about the general structure of the pattern. For most students, filling in the tables of values was very mechanical. In many cases after they had completed the first couple of rows in the table, they abandoned the physical materials and counting the number of tiles, and simply completed the table by continuing counting patterns that appeared in each column. For example, if one repeat had 2 blue tiles, then 2 repeats had 4, and 3 repeats had 6. Thus, the pattern for the blue tiles was 2, 4, 6, 8, 10, and so on.

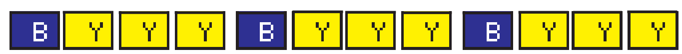
This strategy, while assisting students to quickly complete the table initially impeded the teacher/researcher's conversations with the students about the relationship between different elements of the repeating pattern, that is, relationships between the columns in the table. Specific language and actions were introduced to assist students to focus their attention on relating the columns. The relationships that they could see in the table of values were classified as 'down' rules or 'across' rules. Examples of down rules offered by the students were: *the number of repeats goes up by 1, the number of yellows is going up by 3, the total number increases by 4.* These rules focus on one data set, the data in a particular column and finding the relationship within this set. Examples of across rules were: *there are 2 times as many yellows as blues; the number of yellows is three times the number of repeats*. These rules focus on examining two data sets and finding the relationship between the two sets.

The students exhibited great difficulty in identifying across rules and even when they did, their explanations tended to focus on one line of the data rather than the patterns that existed across all lines of data. For example, Matthew commented, "The number of repeats and number of blues and total numbers are all multiples of 5," referring to the row 5, 5, 10 and 15, instead of commenting on the more general relationship that the number of repeats is the same as the number of blue tiles and so on. They also exhibited some difficulty in verbally expressing the generalizations that they saw. Specific questions and the introduction of large numbers appeared to assist the students in refining the descriptions that they offered. The following transcript provides insights into the types of conversations that occurred in this phase of the lesson.

T: If you were going to give me an instruction to make the pattern what would you say?
C1: You multiply the number by 4.
T: You multiply the number of?
C1: I am not sure.
T: Tell me, what have I got to do and I will try and act it out to see if it works.
C1: You would times the number of 3's, so the yellows you would 3 times.
T: If I said I've got 363 repeats how would I work out the number of yellows?
C: You would times 363 x 3.
T: And what's 363? What's another name for 363?
C2: The number of repeats.
C1: You times the number of repeats by 3.
C3: Take the number of repeats and multiply it by 3 and you get the number of yellow squares.
C4: Yeah um 1 times 3 is 3, 2 x 3 is 6, 3 x 3 is 9, 4 x 3 is 12.

At the conclusions of extensive conversations about different repeating patterns and the relationships between the columns in the table of values, the conversations moved on to expressing these relationships in both language and symbols.

For the repeating pattern



T: Suppose I had an unknown number of repeats. How would I work it out?
C5: You add this unknown [the number of blue tile] and that unknown together and this unknown equals unknown times 3 [the number of yellow tiles]. So unknown plus unknown x 3 [the total number of tiles].
T: Unknown but what symbol did we use, we used one didn't we? Yes?
C6: A question mark
T: Can you write yours up because it is a little bit different?
C5: Well that one, there's one blue and three yellow so its (*child writes on board*): 1 x ? + 3 x ?

The task that the students were asked to complete at the conclusion of this phase of the teaching required them to create the pattern □□■▣▣ for 5 repeats and then complete the table. Students were asked to complete the table consisting of 5 columns, namely: (1) number of repeats, (2) number of □, (3) number of ■, (4) number of ▣, and (5) total number of tiles for 1, 2, 3, 4, 5, 12, 27, 88 and an unknown number of repeats represented as [?]. The responses fell into four broad categories, created to represent differing levels of thinking.

Category 1 Incorrect pattern

This category consisted of responses where students only completed the first three rows of the table but their responses in the first three rows were incorrect.

Category 2 Small countable number of repeats

This category consisted of responses where students correctly completed the pattern for the repeats 1, 2, 3, 4, 5, 12. Classroom observations indicated that these students tended to focus on single data sets (the columns of values) and identify patterns within these sets (e.g., the number of □ is 2, 4, 6, 8, 10, 24).

Category 3 Large uncountable number of repeats

This category consisted of responses where students correctly completed the pattern up to and including 88 repeats. These responses were considered different from Category 2 responses. From the classroom observations it was conjectured that in order to arrive at a correct answer for 88 repeats students had to identify the relationship between the columns.

Category 4 Unknown number of repeats.

This category consisted of responses where students not only correctly completed the table of values for the given steps but also expressed the relationship between the number of repeats and the number of tiles as an abstract expression. Figure 3 presents some typical responses for this category.

By the completion of Lesson 2 three quarters of the students could create the generalization for large uncountable numbers of repeats. Fourteen of these students could also write the relationship between the columns as a series of abstract expressions representing the unknown as [?].

Lesson 3 began with a discussion on the concept of fractions and representing the relationship between the columns as fractions. Students were encouraged to create one repeat of the repeating pattern and use gestures, such as, placing their hand over the pattern or running the finger around the edge of the pattern to identify the whole. The conversations then turned to a discussion about the parts in the repeat: *How many parts in the whole? What is each part called? Are they equal?* Then, questions turned to naming the fraction that each component represented. The next phase consisted of examining two repeats and reiterating the above dialogue. The students were then directed to record the fractions in a table of values. Students struggled with this concept. It is conjectured from the field notes and analysis of the videos that this was due to (a) difficulties in distinguishing the changing size of the whole as the number of repeats increased (e.g., for the pattern byyybyyybyyybyyy the whole for 1st repeat consists of four tiles but for 2 repeats the whole is eight tiles) (b) the lack of specific fraction language used to identify fractions, for example, many struggled with fourths, eighths, sixteenths and then thirty-seconds, and (c) the representation of the fraction using the set model, a representation that past research has indicated is the most difficult representation of the fraction concept. As was evidenced in Lesson 4, this lesson served to initiate a focus for students' attention on repeats as consisting of parts and wholes.

In lesson 4 the concept of the ratio was introduced. The lesson began with a discussion about what ratio was and continued to the utilization of the ratio to represent the relationship between the number of elements generated for a differing number of repeats. The activities in this lesson were similar to lesson 2, but in this instance the table of values was extended to include columns where students recorded the relationships between the different parts of the repeating pattern as ratios (see Figure 1d). The table of values on an accompanying worksheet mirrored this format. Table 1 presents a summary of the responses to Question 1 and Question 2 on the worksheet. A correct response indicates that the student correctly recorded the data for up to 5 repeats in the table of values, including the ratios between the elements of the repeats.

Overall, most students experienced little difficulty with the concept of a ratio and successfully related this concept

FIGURE 3

Expressing the generalization for an unknown number of repeats.

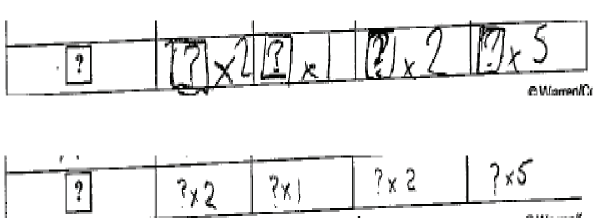


TABLE 1

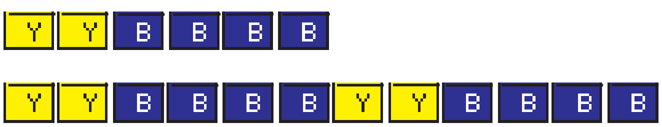
Frequency of correct responses for 5 repeats of each pattern.

Task	Correct	Incorrect
Record your data in the table of values.		
1(a) ■■■■□□ 1(b)□□□■■	46 (90%)	5 (10%)
2(a) □□□■▨▨ 2(b)□□■■■▨	33 (65%)	18 (35%)

to their repeating patterns. They certainly found this lesson much easier than lesson 3 where the focus was on using fractions to discuss the relationship between the elements. We conjectured that this was due to: (a) comparisons between parts are simpler than comparisons between wholes and parts, and (b) the simplistic language used to express these comparisons. Ratio language closely mirrors the language used to compare numbers. For example, the ratio 8 to 32 is simply read as *eight to thirty-two* (cardinal language) whereas 8/32 is *eight thirty-seconds* (ordinal language), language with which students experienced difficulty in Lesson 3. The one difficulty that they experienced was in understanding that the order in which we say the ratio is the order that the comparison occurs, however this appeared initially and predominantly only when the concept was first introduced.

The second phase in the lesson required the students to create repeating patterns for different ratios. Students found this phase to be more difficult. Thirty three students correctly created the ratio for □ white to ■ black is 6:2. The results from their responses suggested that as the complexity of the task increased (e.g., ratios involving three different colored tiles or the inclusion of distracters in the task), the number of correct responses decreased.

Finally, students were challenged to compare equivalent ratios and discuss whether they believed they were the same or different. A typical question was: *Suppose I had the ratios of yellow to blue tiles of 2:4 and 4:8; are they the same ratios or are they different*? Students modeled these two ratios with yellow and blue tiles.



They were asked to indicate by a show of hands if they thought they were the same or different. Most thought they were different as one had more tiles than the other, indicating that they were focusing on the comparing the total number of tiles rather than comparing the number of blue tiles for each yellow tile for each ratio.

To further students' understanding of ratio concepts, we introduced different conditions in which ratios were potentially useful. Two jugs were drawn and each ratio was placed in each jug. The students were then asked: *If the ratio of cordial to water in jug 1 is 2:4 and the ratio of cordial (fruit juice concentrate) to water in jug 2 was 4:8, which jug has the stronger cordial*. Many students thought that they were the same strengths. A typical response from students who thought they were different was:

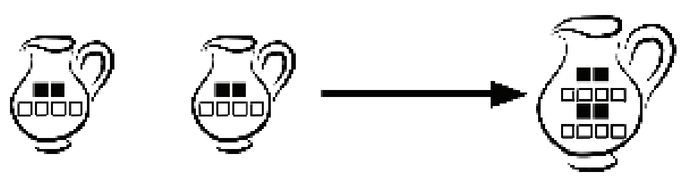
T: What makes you think that this is different? Is it because there is more cordial?
Sarah: No, because of more water. It is more watery.
T: What's your favorite drink? What do you like drinking?
Sarah: Lemonade.
T: What if I had two cans of lemonade and mixed them together, would it be stronger?
Jill: Same.
T: Would it? Why?
Sarah: Um... because it's the same thing.
T: Yes, but - Do you think it would be stronger, would it? Who thinks it would be stronger? If you had 2 cans of lemonade into a jug, would it be stronger?
Sarah: But there would be 2!
T: Does that make it stronger?
Jill: It would make it more.

These students were then asked to explain to the class why they thought they were the same. John's explanation exemplified one possible solution. John drew three jugs on the board, 2 small and 1 large (Figure 4). In each of the 2 small jugs he placed 2 yellow tiles and 4 blue tiles and said, pointing to the 2 small jugs:

John: The cordial in these jugs is the same strength. If we take 2 of these small jugs and pour them into the large jug we have 4 yellow and 8 blue, but it isn't stronger it is the same because it was the same in the smaller jugs.

FIGURE 4

John's proof that 2 to 4 is the same ratio as 4 to 8.



By contrast, Amy simply took the tiles in each jug and re-arranged them so that each yellow tile was 'attached' to 2 blue tiles. An excerpt of her explanation was:

Amy: For every yellow tile there are 2 blue tiles so it must be the same. If it was stronger there would be more than one yellow tile for every 2 blue tiles in the larger jug.

Cam utilized the concept of 'balance' to justify that the ratios were the same.

Cam: It's just the same because it balances it out because if there were 4 on these ones it would be balanced out [pointing to the tiles in the small jugs]. If there were still only 4 blues and the yellows have gone up to say 10 - so it wouldn't balance out. So what you have to make sure that what you are doing all stays the same.

When comparing 2:4 and 1:2 Emily rearranged the tiles on the board as shown in Figure 5.

Her comment was: "They now look the same, they are the same ratio. They are all the same ratio. It's pretty much what we were doing yesterday. Now we've add two together (pointing to the first two repeats). It's become a whole. It is just a bigger amount, but it is still a whole." While it is possible to have a discussion about equivalent ratios from the table of values, we are suggesting that the justification phase involves the use of visual materials, as it is in the reformulation of these materials that meaning started to occur. Thus, the process of constructing understanding in the classroom setting involved the manipulation of concrete materials in a real life context. Finally returning to Sarah:

T: You convinced or not? (asking Sarah)

Sarah: I am convinced with the cordial, but lemonade I am not.

DISCUSSION AND CONCLUSIONS

Three areas of findings were identified, each relating to one of the questions posed for this study. The first focuses on the role of repeating patterns in bridging between arithmetic reasoning and algebraic reasoning, the second on the role of the teacher in supporting such bridges, and the third on the role of representations in learning to reason algebraically. Finally, the role and value of the methodology, design (teaching) experiment, is discussed.

Repeating patterns as a bridge to introducing the ratio concept

The results suggest that repeating patterns can act as effective bridges for introducing the ratio concept to young students. These students found it easier to discuss the concept of a ratio than they did the concept of a fraction. Past research suggests that this occurs because ratio involves a comparison between parts whereas fractions involve a comparison between parts and wholes, a comparison many students find more difficult (e.g., Shwartz and Moore, 1998). The results from this study add to this research, providing evidence that language also plays an important role. The language used for describing ratio more closely mirrors the language used for numbers (i.e., cardinal language rather than ordinal language), and students found it easier to articulate ratio relationships than to give the fraction name for different numbers of repeats. Thus, in learning fractions, they are not only struggling with identifying the whole but also with using complex language such as *eight thirty-seconds* compared to *eight to thirty-two.*

Many young adolescents in the Queensland context find the concepts of ratio and proportion difficult. One must ask: *Is this because its introduction occurs after the introduction of fractions? Does the complexity of the fraction concept and the difficulties they experience impact the later introduction of the ratio concept?* The results of this research indicate that very productive conversations about ratio and proportion can occur earlier than suggested by the current syllabus, which mandates that the concept should only begin to be introduced to students age 13 years. In addition, as evidenced by the conversations presented here, the concept of a ratio also encourages students to (re)visit their understanding of multiplication and division. Thus its earlier introduction could also provide realistic contexts and empirical problems to situate these discussions (Lo & Watanabe, 1997).

Teacher actions that assisted in making the connections

Particular teacher actions also helped to forge the relationship between repeating patterns and the concept of a ratio.Two categories of teacher actions were identified in this study: breaking into parts (a physical action) and using tables of value (a graphic organizer) for recording data. Both actions utilized special language and discourse.

Breaking into repeats

Students initially experienced difficulty in breaking the repeating pattern into its discrete repeats. The teacher's emphasis of breaking patterns into parts not only allowed students to identify the repeats, but also to begin to discuss the structure of one repeat, two repeats and so on, and the similarities and differences between these differing repeats. This involved the development of common class words (e.g., 'repeating part') with which to describe what was happening. The students were also encouraged to rearrange the elements in the repeats so that the same elements were clustered together. The introduction of cards to place under the differing number of repeats also enabled students to explicitly see the data that was under consideration in ensuing conversations, namely, the number of repeats and the number of different elements in these repeats.

Introduction of the table of values for recording data

This representation was introduced to assist students to summarize their data for differing numbers of repeats. Tables of values have long been recognized for their role in assisting students' understanding of mathematics (e.g., Warren, 1996). This research suggests that specific strategies need to be incorporated in the discussion to ensure that this understanding is maximized. The first strategy entails the introduction of explicit categories for differentiating generalizations identified in the table of values. As indicated in the classroom conversation, the table of values assisted many students in searching for generalizations between data sets. Initially, students appeared to search for patterns in only one element of the repeats (i.e., finding generalizations in one of the columns); for example, they described how the number of yellow tiles changed by 2 as the number of repeats increased. Specific strategies assisted in categorizing generalizations as either searching for generalizations in one data set (down rules) or searching for generalizations by linking two data sets (across rules), thus assisting students to change their thinking from single variation thinking to co-variation thinking, a richer form of understanding. These strategies were enabled by the introduction of terms 'down' and 'across' (e.g., "that is a down pattern").

The second strategy involved generalizing from a small number of repeats to a larger number of repeats, thus making it almost impossible to pattern down the table of values. For example, if there are 88 repeats, how many yellow tiles are there? How many blue tiles are there? The third strategy entailed acting out students' verbal descriptions of the generalizations that they identified. This assisted in ensuring that these descriptions were very precise and directly related to the data set under scrutiny. These strategies, while all being situated in Duval's (2002) mono-functional register, significantly contributed to the conversion from mono-functional to multi-functional register; that is, they both supported the discussion about generalizations in one register and scaffolded the conversion of these discussions to algebraic systems.

The development of the discourse around jugs and the language of 'stronger' and 'weaker' cordial were important to the development of equivalent ratio. Student, John, used this discourse explicitly in his argument in support of equivalence. As well, the development of ratio was assisted by the continual use of inquiry discourses (e.g., asking students to defend their positions).

The Role of Representations

External representations like the use of tiles and the value table played differing roles in assisting students to reach an understanding of the ratio concept. Summarizing the data in tables of values assisted students in identifying the relationships between the various data sets (the number of repeats and the number of tiles in each repeat), whereas, the rearrangements of the physical tiles representing related ratios (e.g., 2:4, and 6:12) assisted students in identifying that these ratios were equivalent. Even the fact that the tiles were magnetic and flexible assisted student learning (e.g., Emily's demonstration of equivalence was assisted because the tiles could be partially placed on top of other tiles and still hold to the steel whiteboard).

We argue that it was in the synergetic interplay amongst a variety of representations that deep understanding began to occur. When the table of values was introduced, it assisted students in summarizing their data and supported them in searching for generalizations within the data set. It also resulted in the translation of visual representations to

FIGURE 5

Emily's explanation of equivalent ratio.

number representations and hence students began to operate in a number world. This certainly assisted them in searching for numeric patterns and generating a variety of ratios for differing numbers of repeats. However when it came to a discussion about the relationship that existed between the ratios in the table of values it was the return to the visual representations of the ratios that assisted students' conversations about equivalence. As exemplified from the results of this study, in this world, they were able to generate three different ways to justify that the ratios were indeed the same, each dependent on manipulating the tiles themselves.

Interpreting mathematical signs is a personal process. In some instances it appears that students were unable to go beyond the written mark, the literal interpretation. The inherent triadic nature of sign relations (object, representations and interpretation) are exhibited in this research. The tasks presented in this research induce an interaction between these three dimensions and also exhibit how the interplay between different signs and their interpretations bring deeper meaning to the object itself. Our role as teachers is to respond to these interactions, ensuring that both the classroom discourse and mathematical contexts are rich and representative of the full range of understandings, so that the original intended meanings are reached by students participating in the dialogue.

Saenz-Ludlow (2001) refers to this as engaging in interpreting games, continually presenting different representations and conversations about the object to support the development of deeper and richer understanding of the object (e.g., the use of tables of values, gestures, and manipulations), and using language and questioning to unpack the impact of these representations on cognitive development.

The Role and Value of the Design (Teaching) Experiments

The four imperatives of the design (teaching) experiment methodology: interventionism, teacher-student interactions, simultaneous exploration of teaching and learning, and contingency, all played their role in this study. The methodology gave us the license to trial our own lesson creations for using repeating patterns as a basis for ratio. It directed our analysis onto what was happening in the lessons and what might have caused it, and so we are able to evaluate the effectiveness of individual tasks as well as map out learning sequences and explore students' potential for learning. It is even partially responsible for the study in the first place; the repeating pattern lessons on which this article reports extended and complemented other studies of growing and repeating patterns within a large scale longitudinal design experiment.

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