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Abstracting in the Context of Spontaneous Learning

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There is evidence that spontaneous learning leads to relational understanding and high positive affect. To study spontaneous abstracting, a model was constructed by combining the RBC model of abstraction with Krutetskii's mental activities. Using video-stimulated interviews, the model was then used to analyse the behaviour of two Year 8 students who had demonstrated spontaneous abstracting. The analysis highlighted the crucial role of synthetic and evaluative analysis, two processes that seem unlikely to occur under guided construction.

I became interested in spontaneous learning as a secondary mathematics teacher. I wondered why students sometimes became so engaged in tasks that they lost all sense of time, self, and the world around them, were completely focused on the task at hand, reported developing a deep understanding of the mathematics they explored, and displayed indicators of pleasure during the process. I wanted to increase the likelihood, and number, of students experiencing this phenomenon. To do this, I needed to explore the learning process in more detail, in particular looking for examples of abstraction (Dreyfus, Hershkowitz, & Schwarz, 2001).

Spontaneous abstracting may be more easily identified than abstraction resulting from guided construction due to body language indicators of the high positive affect I have just described. For this reason, study of spontaneous abstracting could provide a useful way to learn more about other forms of abstraction.

Spontaneous Learning

The term *spontaneous* has previously been used to denote student learning that is not caused by the teacher:

> We do not use spontaneous in the context of learning to indicate the absence of elements with which the student interacts. Rather we use the term to refer to the non-causality of teaching actions, to the self regulation of the students when interacting ... we regard learning as a spontaneous process in the student's frame of reference. (Steffe & Thompson, 2000, p. 291)

Steffe and Thompson's expression "in the student's frame of reference" is crucial to understanding the nature of spontaneous learning, as I will show below.

Creative mathematical activity during the spontaneous development of new mathematical concepts has been identified in secondary and elementary classrooms (Barnes, 2000; Cobb, Wood, Yackel, & McNeal, 1992) and linked to positive affect (Barnes, 2000; Liljedahl, 2006; Williams, 2002a). In these studies, students were working above their present conceptual level on a self-set intellectual mathematical challenge that was almost out of reach (Csikszentmihalyi & Csikszentmihalyi, 1992; Williams, 2002a).

Spontaneous abstracting has been described as a process of progressively discovering complexities (Williams, 2000a). A student (or group of students) discovers a mathematical complexity that was not evident at the commencement of the task and decides to explore it. To facilitate this exploration, students spontaneously formulate a question about this complexity and engage in complex mathematical thinking in order to answer it. Students use the "smaller" concepts and ideas they have developed during exploration of earlier complexities to assist in unravelling later ones.

This type of thinking during the development of new conceptual knowledge is described in different but consistent ways by other researchers (Krutetskii, 1968/76, p. 292; Chick, 1998, p. 17; Csikszentmihalyi, 1997, p. 65). It is described as "not only choosing the cues and concepts — and often unexpected cues and concepts — but even the very question" (Chick), and "not so much direct attempts at solving the problem as a means of thoroughly investigating it, with auxiliary information being extracted from each trial" (Krutetskii). Csikszentmihalyi (1997) described the cumulative effect of the small discoveries:

> You may have only one big insight, but as you try to elaborate, as you try to explain what the insight is, you have small insights coming up all the time too.

The process of spontaneous abstracting thus involves posing questions to explore, in order to gain increased knowledge about a mathematical complexity, and synthesising aspects of these new understandings during the process of developing insight.

Making connections between mathematical concepts in the process of such thinking develops relational understanding (Skemp, 1976) — a connected form of understanding where students know why mathematics is relevant and are able to select and use it in unfamiliar situations. Relational understanding enables flexible thinking and opportunities for future learning of associated topics (Sfard, 2002) and for making sense of the world (mathematical literacy) (Kilpatrick, 2002). Local and international concerns about low mathematical literacy highlight the need to develop relational understanding (Kilpatrick, 2002; Skemp, 1976).

The RBC Model of Abstraction

According to Dreyfus, Hershkowitz, and Schwarz (2001), abstraction is an activity of "vertically reorganising previously constructed mathematical knowledge into a new structure" (p. 377). The term "vertical" refers to forming a new mathematical structure as opposed to strengthening connections between a mathematical structure and a context ("horizontal"), following a distinction initially made by Treffers and Goffree (1985).

The genesis of an abstraction passes through (a) a need for a new structure; (b) the construction of a new abstract entity; and (c) the consolidation of the abstract entity by using it in further activities with increasing ease (Dreyfus, Hershkowitz, & Schwarz, 2001). Through analysis of the dialogue of participants as they undertake the social process of critical inquiry, the cognitive elements of the process of abstraction (recognising, building-with, and constructing) are made visible. *Recognising* involves identifying a context in which a previously abstracted mathematical entity applies, or identifying mathematics relevant to a context (Hershkowitz, Schwarz, & Dreyfus, 2001). To recognise the usefulness of mathematics in a new context, the student must understand the relationship between the mathematics and the context. *Building-with* involves using a mathematical procedure the student has recognised in a context in which it has previously been used or in a new context. It can involve using mathematics that has been recognised in a new sequence or combination. *Constructing* involves integrating abstracted entities to develop new insight. Recognising and building-with are often *nested* within constructing. In other words, during constructing, other previously abstracted mathematical entities may be progressively recognised and built-with to support constructing.

Branching is a variation of constructing that occurs when constructing separates into two different directions to study two different aspects of the mathematics involved and then later rejoins unexpectedly (Dreyfus & Kidron, 2006). *Consolidating* can occur when students work with familiar mathematics, and also when students use a newly abstracted entity as part of further abstracting (Dreyfus & Tsamir, 2004).

Krutetskii's Mental Activities

Krutetskii (1968/76) studied the problem-solving activity of students who thought out loud as they solved unfamiliar problems individually. He identified various mental activities (cognitive activities) that were initiated and controlled by the high ability students and not the interviewer. From least complex to most complex, these activities included analysis, analytic-synthesis, synthesis, and evaluation. The hierarchical nature of these thought processes is implicit in Krutetskii's descriptions and supported by his empirical data (Williams, 2000b, p. 18). These four activities are important components of spontaneous learning.

Krutetskii described analysis as an initial process of examining a problem element by element, commenting that "to generalise mathematical relations one must first dismember them" (p. 228). Since, as I will show, types of analysis also entered into other activities, I will call this process *element-analysis*.

Krutetskii called the simultaneous analysis of several elements analytic-synthesis, but since this is a more complex type of analysis, I prefer to call it *synthetic-analysis* (Williams, 2002b). This process is "more or less drawn out in time" (p. 231) for different students or the same student at different times. Sometimes, it occurs almost instantaneously; Krutetskii called this flash of inspiration *analytic-synthetic vision*. At other times, it progresses from perceiving an unfamiliar problem in terms of "its separate mathematical elements [where] 'going outside' the limits of the perception of one element often means 'losing it'" (p. 229) to "connecting the mathematical elements of the problem" (p. 229). Krutetskii called this process *analytic-synthetic* orientation.

Students in Krutetskii's study who encountered difficulties were refocused on elements of the problem by the interviewer who asked, for example, "Well, how do you solve this group of examples you have selected in contrast to the other group? Compare the course of solution" (p. 236) and "Look carefully at the previous example ... doesn't it suggest anything to you?" (p. 241). These students were not undertaking synthetic-analysis because the processes was externally directed rather than self-initiated. The interviewer told the students what mathematics to focus on (external control) and what to do with it (external elaboration), whereas other students in the study spontaneously pursued their own pathways and evaluated the reasonableness of their mathematics as they explored. I call this externally directed process *guided element-analysis*.

Krutetskii described *synthesis* as the identification of "generality hidden behind various particular details" where students "'grasp' what was main, basic, and general in the externally different and distinctive [and find] elements of the familiar in the new" (p. 240). In other words, synthesis involves recognising something that is already known integrated within something new.

Evaluation was identified by Krutetskii as a continual checking of consistency of the mathematics developed during the abstracting process, or the recognition of a mathematical entity just abstracted for another purpose. Evaluation included progressively reflecting on the situation as a whole for the purpose of recognising inconsistent information, or reflecting upon the process of problem solution for the purpose of identifying its limitations or applications to other contexts. It also involved reflecting upon the solution pathway developed and its possible contribution to generic mathematical processes for future use. This activity was illustrated by Dreyfus, Hershkowitz, and Schwarz (2001), where the student-pair recognised algebra as a tool for justifying.

Krutetskii briefly discussed the case of students who did not undertake evaluation but who considered whether their mathematics was correct after the solution had been obtained. This process can involve considering or checking ideas after a pattern had been identified, or after an answer had been found, or deciding about a prediction that had been made, or deciding on the relative elegance of solution pathways. I call this activity *evaluative-analysis* (Williams, 2002a). It is a more complex process than synthetic-analysis (Williams, 2002a) because it involves synthetic-analysis for the purpose of making a judgment. But it is also less complex than synthesis.

To summarise: Spontaneous problem solving includes the following five mental activities. These are (from least to most complex):

- Analysis:
  - Element-analysis: isolating parts and examining them one by one
  - Synthetic-analysis: simultaneously examining several elements
  - Evaluative-analysis: synthetic-analysis for purposes of judgement
- Synthesis: identification of generality
- Evaluation: reflection on mathematics progressively developed and results obtained

The Spontaneous Abstracting Model

In an attempt to describe the abstracting that takes place in spontaneous abstracting, I have integrated the RBC model of abstraction with my extension of Krutetskii's mental activities to create the Spontaneous Abstracting Model illustrated in Figure 1 (Williams, 2002b). The three ellipses representing recognising, building-with and constructing are to be seen as lying in parallel planes, and the vertical arrows represent the nesting of the cognitive elements within each other.

Notice that all subcategories of analysis are contained within building-with and constructing contains synthesis and evaluation. Only processes associated with spontaneous learning (so not guided element-analysis) are included in the Spontaneous Abstracting Model.

The study to be described below was designed to answer the research question:

- Is the Spontaneous Abstracting Model sufficient to describe thought processes that occur during spontaneous abstracting?

Answers to this question could assist teachers to identify thought processes and make informed decisions about how to promote spontaneous abstraction during problem solving.

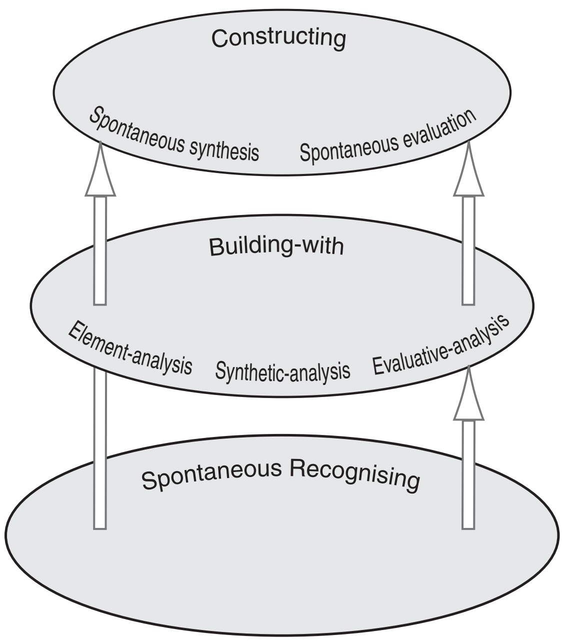

Figure 1. The Spontaneous Abstracting Model.

Research Design

This paper reports on the spontaneous abstracting of two students selected from a broader study (Williams, 2005). This study was itself embedded in the Learner's Perspective Study (Clarke, 2006), which investigated students' perspectives on their learning in a variety of Year 8 classrooms in nine different countries including Australia and the USA. In order to capture the creative development of new mathematical knowledge by students, and to examine the social elements in more detail, three cameras simultaneously captured the activity of the teacher, a different pair of focus students each lesson, and the whole class. A mixed video image produced during the lesson (with two focus students at centre screen and the teacher as an insert in the corner) was used to stimulate student reconstruction of their thinking in individual post-lesson interviews. The students used the video remote to identify parts of the lesson that were important to them, and discuss what was happening, and what they were thinking and what they were feeling. Interview probes were informed by findings from Ericsson and Simons (1980) on how to generate high quality verbal data about cognitive activity. The interviewer did not ask questions that included constructs the subject has not previously reported, so the subject was not likely to "generate answers without consulting memory traces" (p. 217). Instead, the subject spontaneously "described one or more specific sub-goals, and these were both relevant to the problem and consistent with other evidence of the solution process" (p. 217). A sketchpad and pen were provided to facilitate communication for students who were unable to express their newly developed knowledge in verbal form (Ericsson & Simons, 1980).

Although this design allowed a detailed study of the focus students' thinking, there was a high likelihood that spontaneous abstracting would take place among students who were not being videotaped by the student camera. Therefore, non-focus students who displayed evidence of high positive affect in one lesson were frequently focus students the following lesson. The subsequent interview then focused on the lesson in which the abstracting might have occurred as well as the lesson in which the student was a focus student. Both students in this article were interviewed in the lesson following their spontaneous abstracting.

The classroom videos and the subsequent interviews were used to determine whether the focus students' thinking was or was not spontaneous. The crucial determinant was whether the students developed their ideas themselves or whether there was mathematical input from others. Of the 86 students studied, 5 were found to have been spontaneously abstracting on 8 separate occasions. On only 5 of these occasions were the students focus students. There was no evidence of non-spontaneous abstracting.

The two students whose spontaneous learning is described below, Kerri (USA) and Eden (Australia), came from similar situations. Both classes contained students with higher than average mathematical abilities, and student interactions were integral to the learning process. Both of the lessons where spontaneous abstracting took place involved learning about linear functions, and both teachers commenced the topic with a hands-on activity. Both students could read and plot Cartesian coordinates before the lesson. However, there were also a number of differences between the two situations.

Kerri's class was composed of students identified as gifted. Before the research period, they had found equations to linear graphs by plotting two points, generating the line on graph paper, drawing a "slope triangle", measuring the lengths of the two smaller sides of the triangle, finding the *y*-intercept by inspection, and substituting the gradient and the *y*-intercept into the general equation for a linear function. Eden's class was the top stream (track) of non-accelerated students in a school with an acceleration program. They had not studied linear functions, but Eden had met them when accelerated by his Year 7 teacher. Eden had forgotten most of this work and if he had encountered the term gradient and the associated concept, he did not remember it. The first time he recollected an awareness of the concept of gradient was in the lesson where he abstracted: He did recognise that he had used the concept of gradient without knowing the term: "I sort of ... used them [gradients] except I wasn't — didn't know what exactly they were." Neither student knew how to substitute values into an equation to find the values of the constants.

Although analysis of these two students' thinking was hindered because they were not captured continuously on the student camera while they were abstracting, each student contributed sufficiently to class discussions during subsequent lessons to demonstrate they had abstracted the relevant concepts before they were formally taught. Further, although Eden was not a focus student in the lesson in which his abstracting was inferred, he was seated beside a focus student (Darius). Darius' computer screen was visible and was the focus of joint attention by Eden and Darius several times, and the focus of Eden's individual attention just prior to his exploratory activity. During the seven minutes from when Eden formulated his spontaneous question and began to explore it, he was also visible on the whole class camera. He then rolled his chair back towards Darius and his exclamation was captured.

Each student's activity will now be reported through a narrative summary, an analysis of their relevant cognitive and social activity, and a diagrammatic representation of each student's spontaneous abstracting activity.

Kerri's Spontaneous Abstracting

Narrative Summary

The teacher reminded the class of the previous lesson before teaching her class how to find the equation to a line without plotting the graph: "[Remember you were] drawing a graph using the slope triangle and estimating where the graph crosses the *y*-axis." In her interview, Kerri confirmed that she had already learnt how to do this:

Interviewer:	Oh, okay. So now this, um, procedure that you showed me here — you already knew that before today?
Kerri:	Well I didn't like know it like formally introduced, but I was doing that. ... It was during a test. And it said graph, and I

> didn't have any graph paper. ... If you find the slope and the ... difference of the points and ... then we can substitute, *oh perfect*. So I just wrote the equation.

Kerri described how she recognised a slope triangle "'cuz you can picture a line in a little right triangle on it". She then used her knowledge of the Cartesian coordinate system to find the lengths of the horizontal and vertical sides in this triangle by subtracting the appropriate coordinates of the two points (a new idea for her), and then applied the rule she had previously learnt to find the gradient. Next she realised (another new idea) that the y-intercept (b) could be found by substituting the gradient and the x value and y value of a point on the line into the equation $y = mx + b$. The interviewer acknowledged the novelty of this mathematics for Kerri: "It's very impressive that you figured out the substitution thing on your own."

While doing her homework after the test (using the teacher's plotting method), Kerri generalised the spontaneous building-with she had undertaken in the test:

> I was doing my graph — and then I like realised like — *really solidly*, ... I got the same answer [as] if you do the subtraction.

Kerri's reflection led to a new insight: The graph was not needed, the ordered pairs could always be operated on instead.

Kerri also recognised another use for her discovery: She could find the length of a segment on the line by applying Pythagoras' theorem to the differences in the ordered pairs:

> We had to find the distance between the two plots, [we were] supposed to graph them too — I was using the Pythagorean theorem. ... You're really finding- ... like if you make it a right triangle, it's the hypotenuse, not just the distance.

Evidence of Kerri's previous constructing activity was found during the lesson. When the teacher used a slope triangle diagram to demonstrate finding the equation without graphing, saying "You would graph (3, 4) and (4, 6) and draw a little slope triangle", Kerri queried the teacher's process: "You still graphed it." After the teacher explained the new work, the other students in Kerri's group built-with previously known ideas by plotting graphs and measuring intervals. In her interview, Kerri articulated the differences between how she and the other students approached the exercise:

> [It] said graph and find the distance, and most people would graph the line, and then do the little thing [slope triangle]. But I would find what — see that'd be two and then one [subtracting y values and x values], so you do um, *a* squared plus *b* squared equals *c* squared.

The other students did not recognise the significance of the coordinates in eliminating the need to plot graphs, even though this had been discussed in the lesson and they were considered gifted students.

Kerri's Abstracting Process

Figure 2 represents Kerri's spontaneous abstracting process. Her inferred thought processes are numbered in the order in which they occurred. Within each constructing process, smaller numbers are generally associated with recognising and building-with and larger magnitude numbers with constructing. Nesting of recognising within building-with and recognising and building-with within constructing is represented through the dotted arrows.

She constructed three new insights, two of which (ß and γ) relied upon the constructing of the first (α). The lower half of Figure 2 is a version of Figure 1 that represents the abstraction of Insight α. The additional two versions of Figure 1 positioned above this and side by side represent branching into two abstracting processes. These two smaller RBC diagrams share the same recognising ellipse containing Insight α, and the final constructing processes for Insights ß and γ also draw from the original recognising ellipse that includes cognitive artefacts possessed before the test. Kerri's branching differs to that previously identified (Dreyfus & Kidron, 2006) because the two branches do not rejoin but result in two separate insights.

Figure 2 shows further features of the Spontaneous Abstracting Model. In Step 8, in the lower building-with ellipse, Kerri analysed her newly developed procedure and her teacher's method (synthetic-analysis) and then compared them (evaluative-analysis). Synthesis is represented within the constructing ellipse above it, when she recognised the equivalence of the numerical and graphical representations of length (Step 10). Kerri's abstracting led to curtailment (Krutetskii, 1968/76) because she no longer needed to consider a sketch first. Her realisation of the elegance and generality of her method led to spontaneous evaluation. Insight α led to further development above that.

Processes associated with Step 11 were not studied in detail, but there is sufficient data to suggest that it constitutes building-with other cognitive artifacts rather than analytic-synthetic vision (cf. Figure 1). It is difficult to know what cognitive artefacts Kerri relied upon in Step 11. [From her interview, it would appear that Kerri understood gradient as more than a rule because she discussed that rise over run was convention but that the convention could just as easily have been run over rise. She stated she would need to think about that a little further]. Further synthesis occurred when she combined Pythagoras' Theorem (Step 6) with her new mathematical structure (Step 10), to create Insight α. In developing Insights ß and γ, Kerri built-with Insight α (an example of consolidation during spontaneous building-with (Dreyfus & Tsamir, 2004).

Kerri's activity shows the significance of synthetic-analysis and evaluative-analysis in enabling constructing. Without the opportunity to compare her newly developed procedure with the teacher's procedure, she may not have recognised the equivalence of the numerical and graphical representations of length and made decisions about the value of her method. When she recalled her realisation of the crystallisation of her learning after she had worked with the homework problems, Kerri expressed high positive affect.

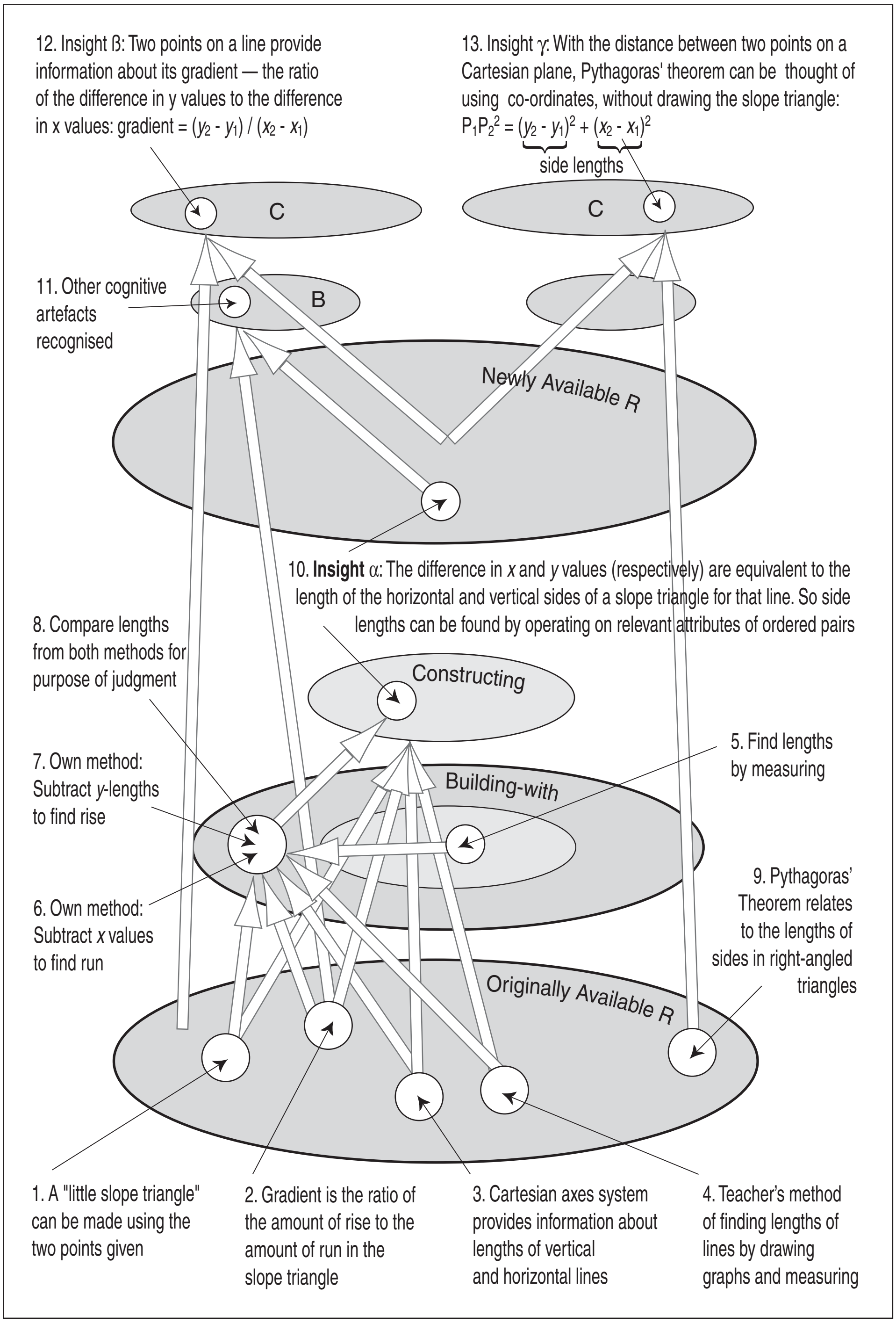

Figure 2. Kerri's spontaneous abstracting processes.

Eden's Spontaneous Abstracting

Narrative Summary: Eden

Eden's class experimented with Green Globs, a computer game that randomly displays globs on a Cartesian coordinate plane for students to "hit" with linear functions. In this game, the score depends on how many globs are hit in a turn.

Eden and his friend Darius, working side by side, had different goals in this lesson. Eden's main focus was on making sense of the generation and positioning of "angled" lines (Eden's term). Eden's interaction with Darius showed he was aware that Darius was only using trial and error. When Eden asked Darius, "What's the rule for that [sloping line on Darius' screen]? That's the sort of angle ...", he took Darius' answer "Two x plus three" literally and wondered why nothing happened when he entered the term $2x + 3$ into the computer. This supported his interview statement that he did not remember much about linear equations. When Eden entered $y = 2 + 2$ soon after (probably omitting the x by accident) and generated a line crossing the y-axis at 4, he exclaimed, "Oh I get it — if you do two plus two is four", suggesting that he was unaware that constant terms can be collected together in equations.

Some time later, Darius generated a family of parallel lines coming closer and closer to points he wanted to hit (Figure 3). Eden moved across to Darius, remaining motionless as he watched the display evolve on Darius's computer screen. He asked Darius, "I don't know how you get that" but Darius did not respond. Eden's interview response afterwards suggested that he was asking himself, "How are the x and the y co-ordinates of points on a line related, and why are they related this way?"

Eden suddenly returned to his own computer and worked intently for seven minutes before making an almost inaudible statement, "y is the cross with x". In his interview, he was initially unable to communicate his ideas, but then he used the sketchpad and explained as he sketched. He had found a relationship

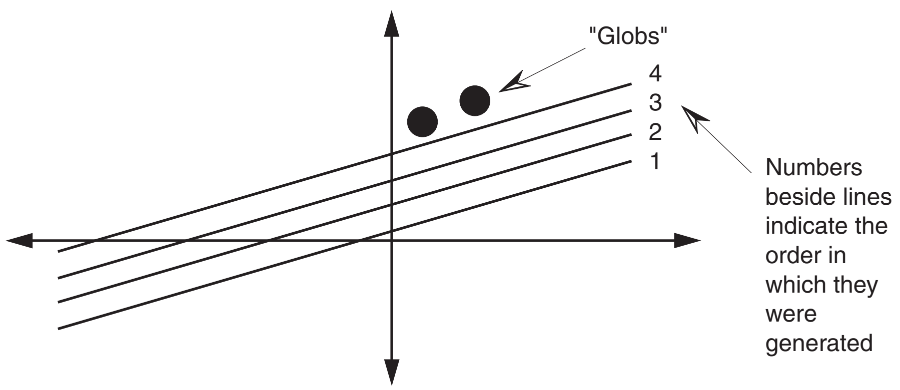

Figure 3. Lines that appeared progressively on Darius's computer screen.

between the x and y co-ordinates of points on linear graphs. He tried to explain why the line sloped as it did and how this was related to the presence of the x in the equation, in particular why some equations gave sloping lines and others horizontal lines:

Eden: ... if it's just y equals minus one (pause) it's gunna be (pause) horizontal — I am not quite — I have forgotten (pause) why it [the diagonal line] just is (pause). The x makes it — makes the rule more complete sort of ...

Interviewer: Okay

Eden: ... because it actually (pause) crosses over with x.

Exactly what Eden meant is unclear. He might, for example, have been trying to convey that the y value relied upon the value of x, or that the presence of an x in the equation moved the y value so the line crossed over the horizontal line formed without the x. But he thought his explanations were clear, and was surprised when the interviewer probed again. "Didn't I tell you that already?" he asked.

Green Globs provided a way for Eden to evaluate his ideas as he worked at his own computer without mathematical input from other students or the teacher. He did not discuss "angled lines" with others after this. Eden's comment "y is the cross with x" coupled with the intensity of his behaviour just prior to that and his actions in the subsequent lesson, were consistent with his developing new knowledge. In the next lesson, Eden was the first to explain that the equation could be found using the relationship between elements of ordered pairs for points on the line. When an exercise was set, Eden immediately recognised that all questions were of the same type. When the students seated on either side of Eden (Darius and Marius) experienced difficulties with the exercise later in the lesson, they asked and received Eden's assistance. Both of these students usually performed equally as well as Eden.

Eden's Abstracting Process

In his interview, Eden drew a graph of his own choosing to assist in communicating his ideas. After the interviewer indicated she needed to know more about what he was saying, he converted this graph into a table of values. Eden pointed to cells within the table as he explained the relationship he had found between the x and y values:

> You've got (pause) a little table like x and y ... y is minus — starts off on minus three (pause) minus two (pause) minus one (pause) and zero (pause). ...
>
> Then it's minus one to minus two, zero to minus one and then it keeps going like that (pause) so it [x] is always one ahead. ...
>
> And then (pause) one (pause) and then the rule (pause) is ah (pause) would be (pause) um (pause) y (pause) equals (pause) x (pause) minus one.

Eden was clearly attempting to express in words the algebraic relation $y = x - 1$. His hesitations suggested he was working out this equation as he spoke, and that his new knowledge was fragile.

When the interviewer asked him to summarise what he had found, he stated that the graphs on the Green Globs screen were all that was provided to assist in finding the equation to the line: "The graph's drawn up already (pause) for you to look at — that's the only help you get to answer". His comment suggests that during his intense focus on Darius's screen he came to realise that the pattern he could see within the coordinates was also expressed in the algebraic representations of the linear function at the bottom of the computer screen.

Eden's interview explanations, considered in conjunction with his lesson activity, suggest that his constructing process followed the sequence shown in Figure 4. As before, the smaller numbers represent recognising activities and the highest number constructing.

Eden's recognition of the pattern linking the x- and y-coordinates of points on a line (Step 7) constituted spontaneous element-analysis. In using the dynamic visual display on Darius' screen to simultaneously consider different representations of the pattern he had generated (Step 8), he was performing synthetic-analysis. During this action, he displayed intense interest. When he returned to his own computer, he undertook evaluative-analysis to judge the reasonableness of ideas he had developed (Step 9). His final exclamation about y crossing over with x indicated that he had constructed a mathematical insight (Step 10). Eden realised he no longer needed the graphical, tabular or verbal representation to describe linear functions because this information was "hidden inside" the algebraic representation and could be unpacked and explained as required. He was satisfied with what he had found and knew why it occurred.

Although the mathematical structure Eden developed was still primitive in that it provided information about slope but not yet position, the relational understanding he developed (Skemp, 1976) was a good foundation for learning further aspects of linear functions. The quality of his understanding was evident in the way he moved flexibly between representations when explaining his new knowledge.

Discussion

Each student developed a different "tightly structured and connected knowledge base" (Skemp, 1980, p. 535) associated with linear functions through their idiosyncratic foci. Each student's "frame of reference" related to what they had decided they needed to know (Steffe & Thompson, 2000, p. 291) and was influenced by the resources they had available at the time. Kerri wanted to answer the test question and Eden wanted to find mathematics that would assist him in gaining high scores in the computer game. Each student searched for a way to proceed and identified a mathematical complexity that they had not previously been aware of. Kerri identified a link between graphical and numerical representations, and Eden identified a pattern linking the x and y value for points on a line.

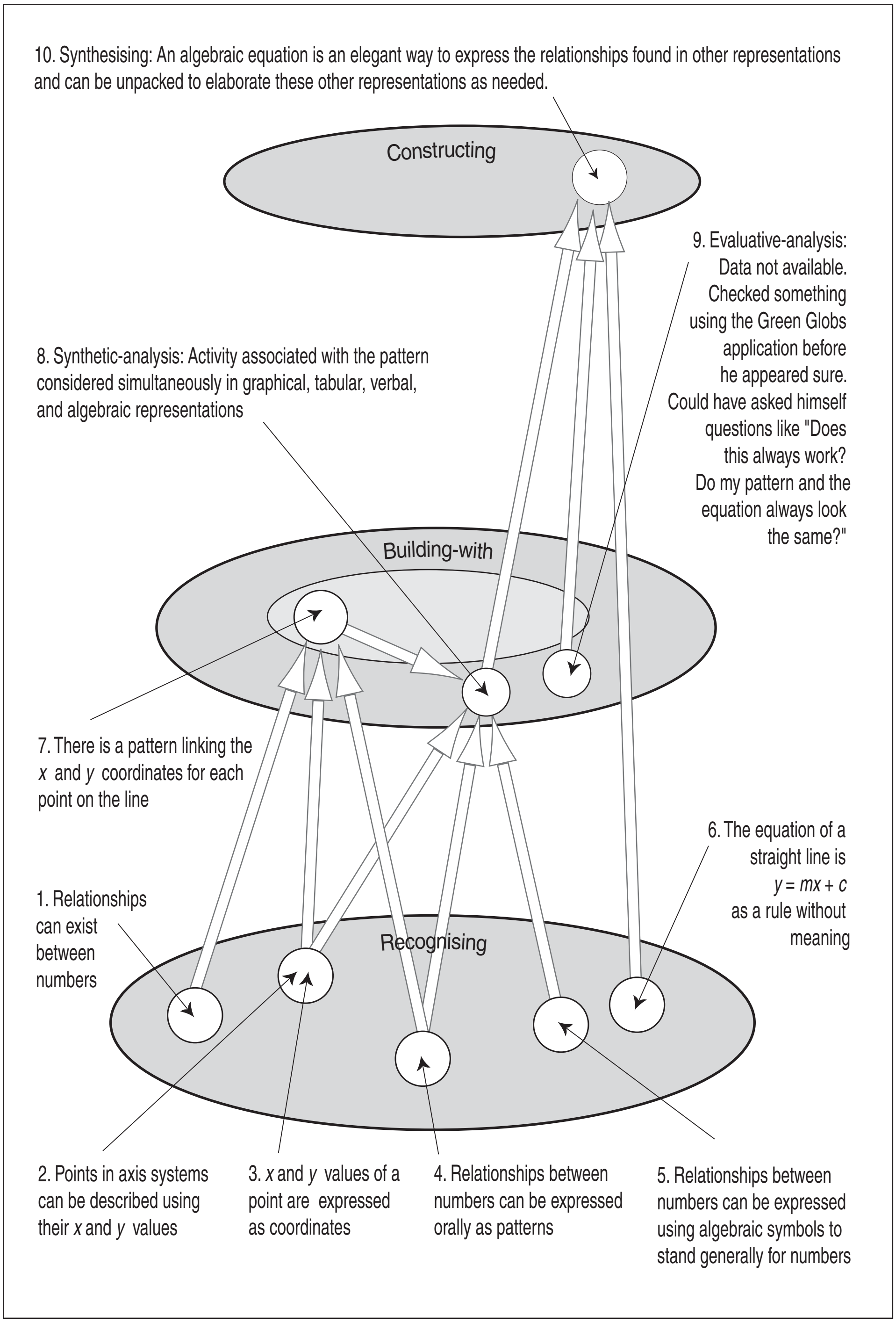

Figure 4. Eden's spontaneous abstracting process.

Kerri's activity when completing her homework would outwardly have looked like practice, but it was actually more complex. Sfard and Linchevski (1994) identified the potential of practice with exercises to contribute to the abstracting process. Kerri's activity raises questions about what might be happening during practice. Kerri practised her homework exercises as she simultaneously attended to her recently developed method. She undertook synthetic-analysis and evaluative-analysis as she compared the processes and found equivalent attributes. These more complex types of thinking within building-with finally resulted in synthesis. The type of interconnected conceptual understanding Kerri developed has previously been identified by Schoenfeld, Smith, and Arcavi (1993) in the way mathematicians think:

> [Mathematicians link] manipulations in the algebraic world, in which m is simply calculated by the formula $(y_2 - y_1)/(x_2 - x_1)$, and the graphical world, in which m has graphical entailments. (p. 58)

Kerri judged the numerical operations as an elegant way to find gradients and subsumed the other representations into her mathematical structure. She generalised these numerical operations into algebraic manipulations of coordinates. She did not just apply rules (empirical generalisations), she was aware of the meanings behind the procedures she applied (theoretical generalisation) (Davydov, 1972/1990). She realised that the Cartesian coordinate system could be used as a tool to find lengths. Other students in her group did not develop theoretical generalisations. Their activity supported Davydov's (1972/1990) research on the need for mental reorganisation to develop insight, and the inadequacy of teacher transmission in achieving this.

Eden considered his activity of generating and checking his pattern simultaneously with the algebraic symbolisation, and this process led to his insight. This "dual use of symbolism as process and concept" has been recognised previously (Gray, 2002, p. 205). Symbols that evoke these dual processes have been referred to as "procepts" (Gray & Tall, 1994). Proceptual thinking provides the student with "far greater power and flexibility" providing "a process to do mathematics and as a concept to think about it" (Gray, 2002, p. 206). Eden displayed this duality in his interview by encapsulating his concept of function in the activity of recognising and building-with the numerical relationship between the elements of the ordered pairs as he described the function. He displayed "economy of thought" by using the algebraic representation to encapsulate the relationship he recognised (Davis & Tall, 2002, p. 139), and an ability to recognise his new structure with increasing ease (consolidation) when he exclaimed that the questions in the exercise were all of the same type (Hershkowitz, 2004; Hershkowitz, Schwarz, & Dreyfus, 2001).

It is helpful to compare Eden's learning with that of IN, a bright, articulate, high achieving sixteen year old interviewed by Schoenfeld, Smith, and Arcavi (1993) as she used Black Blobs (from which Green Globs is derived). IN had difficulty commencing her exploration and the interviewer provided her with a great deal of help but generally did not allow her any control over what she focused upon. He often told IN what to use (control over recognising) and how to use it (control over selection of procedures to build with). She did not have opportunity to trial ideas and decide whether or not they were productive. For example, the interviewer quashed IN's attention to the x-intercept because he considered a graph to be determined by its y-intercept and gradient (p. 59). She did not have opportunity for evaluative-analysis because an external source queried the moves she made and provided his explanation and elaboration. IN did not develop relational understanding (Skemp, 1976) over the six lessons using Black Blobs:

> The simple conclusion that "she had it right when she left" is deceptive. IN's knowledge of slope was not tied in any deep way to an understanding of the mathematical structures that compels lines of positive slope to move up to the right. (Schoenfeld, Smith, & Arcavi, 1993, p. 99)

Eden also needed extra information to enable him to start his explorations, namely information about the general equation, what information the computer application required, and how another student had positioned his graphs. The answers were sufficient for Eden to begin to explore how and why linear equations positioned sloping lines. His exploratory activity and the mathematical structure he developed were consistent with what Schoenfeld, Smith, and Arcavi (1993) valued:

> Learning even simple knowledge in a complex domain means making connections, that is, a piece of knowledge is robust and stable to the extent that it is connected to other pieces of knowledge. (p. 99)

Consistent with Davydov's (1972/1990) findings, the high quality of the mathematical understanding developed by Eden contrasted to the fragmented understanding of rules displayed by IN. Had the mathematical structure of the equation been explained to Eden, it seems very likely that he would not have reached such a deep level of understanding.

The outcomes of interactions with Green Globs/Black Blobs by Eden and IN demonstrated that learning about linear functions through the attributes that are formally presented in many texts (gradient, y-intercept) does not necessarily lead to a rich understanding. IN (compared to Eden) was in a higher year level, studying more complex mathematics, had a wider mathematical vocabulary and interacted with the program over several sessions. Eden explored linear graphs using unconventional characteristics and developed relational understanding of linear function without knowing the standard terminology. His only mathematical advantage was his sound basic understanding of the Cartesian coordinate system.

The cognitive processing undertaken by Kerri and Eden included the same processes. Through simultaneous consideration of more than one representation (synthetic-analysis), these students checked whether the ideas they had developed were reasonable and considered their relative elegance (evaluative-analysis). Kerri undertook evaluative-analysis when she used the teacher's measuring method to simultaneously check her newly developed method of mentally calculating lengths. During this process she realised "really solidly" (Kerri's comment in her interview) what she could achieve without needing the graph. She developed an elegant method that did not necessitate plotting and measuring lengths on graphs. Eden undertook evaluative-analysis when he developed an algebraic representation of his pattern and checked its equivalence to the general form of a linear function that had previously had no meaning for him (synthetic-analysis nested in evaluative-analysis). He then judged the respective elegance of these representations for retaining knowledge. The crystallisation of his elegant way to conceptualise his new knowledge involved subsuming representations within each other so that thinking could be undertaken through the most elegant representation (synthesis during constructing). Transition from evaluative-analysis to constructing becomes more transparent when the activities of these two students are considered: Mathematical structures developed through subsuming of representations within others when equivalent attributes were found through evaluative-analysis. This raises questions about the relative quality of mathematical structures developed with and without evaluative-analysis.

Eden and Kerri used different cognitive artefacts and were situated in learning contexts that differed in some respects. One was technology assisted (Eden) and the other involved individual work (in a test and at home). Even so, both students simultaneously considered various elements of the mathematics involved (synthetic-analysis) and checked the developing ideas (evaluative-analysis). Using the same types of thought processes, they developed different insights about linear functions.

Conclusions

The Spontaneous Abstracting Model was found sufficient to describe and elaborate the spontaneous abstracting processes employed by Kerri and Eden, and to illuminate the commonalities between them. The diversity of these two cases supports the robustness of the model. This study confirms that deep understanding can result from spontaneous abstracting and suggests that this type of learning could lead to more connected understandings than guided learning.

The question arises of how to provide classroom environments in which opportunities for spontaneous learning occur. What we do know is that "heavy" guiding (e.g., in IN's case) inhibits the types of thinking associated with spontaneous learning (e.g., recognising, evaluative-analysis) and results in fragmented learning. But, what are the effects of "light" guiding where decisions made or answers gained are simply affirmed by an external source? This study suggests that affirming activity could reduce opportunities for developing connected understandings because affirming decreases the need for evaluative-analysis, and evaluative-analysis was integral to connecting representations.

This study suggests that increasing opportunities for looking-in activity could support spontaneous learning (Williams, 2006). *Looking-in* is an activity that can occur when students do not possess appropriate cognitive artefacts to progress their exploration. To compensate for this, they examine artefacts generated by others and extract mathematics that was not explicit within them (e.g., as Eden observed Darius's display).

Each of these cases indicates that spontaneous learning can increase student interest in mathematics. High positive affect accompanied self-initiated recognising and synthesising for Kerri, and Eden's synthetic-analysis and evaluative-analysis was accompanied by an intense focus on the mathematics.

Study of these two cases of spontaneous abstracting has assisted in elaborating the social elements associated with spontaneous abstracting, lightly guided constructing, and heavy guidance. The findings could inform teachers and teacher educators making decisions about interventions during student learning. Study of other cases of spontaneous abstracting would test the generalisability of the Spontaneous Abstracting Model and assist in elaborating it further. Further research also is required to find ways to promote spontaneous learning and to find whether tasks that provide opportunities for looking-in could provide a productive way forward.

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References

Barnes, M. (2000). "Magical" moments in mathematics: Insights into the process of coming to know. *For the Learning of Mathematics, 20*(1), 33-43.

Chick, H. (1998). Cognition in the formal modes: Research mathematics and the SOLO Taxonomy. *Mathematics Education Research Journal, 10,* 4-26.

Clarke, D. J. (2006). The LPS Research Design. In D. J. Clarke, C. Keitel, & Y. Shimizu, (Eds), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 15-36). Rotterdam, The Netherlands: Sense Publications.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interaction analysis. *American Educational Research Journal, 29,* 573-604.

Csikszentmihalyi, M. (1997). Flow and creativity. *The North American Montessori Teachers' Association Journal, 22,* 61-97.

Csikszentmihalyi, M., & Csikszentmihalyi, I. (Eds.). (1992). *Optimal Experience: Psychological Studies of Flow in Consciousness*. New York: Cambridge University Press.

Davis, G., & Tall, D. (2002). What is a scheme? In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics: A tribute to Richard Skemp* (pp. 131-150). Flaxton QLD: Post Pressed.

Davydov, V. (1990). Types of generalization in instruction: Logical and psychological problems in the structuring of the school curricula (J. Teller, Trans.). In J. Kilpatrick (Vol. Ed.), *Soviet Studies in Mathematics Education* (Vol. 2). Reston, VA: National Council of Teachers of Mathematics. (Original work published 1972).

Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). The construction of abstract knowledge in interaction. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 377-384). Utrecht, The Netherlands: PME.

Dreyfus, T., & Kidron, I. (2006). Interacting parallel constructions: A solitary learner and the bifurcation diagram. *Recherches en Didactique des Mathématiques, 26,* 295-336.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*, 271-300.

Ericsson, K., & Simons, H. (1980). Verbal reports of data. *Psychological Review, 87*, 215-251.

Gray, E. (2002). Processes and concepts as false friends. In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics: A tribute to Richard Skemp* (pp. 205-218). Flaxton QLD: Post Pressed.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*(2), 115-141.

Hershkowitz, R. (2004). From diversity to inclusion and back: Lenses on learning. In M. Høines & A. Berit Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (pp. 55-68), Bergen, Norway: PME.

Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*, 195-222.

Kilpatrick, J. (2002). Understanding mathematical literacy: The contribution of research. *Educational Studies in Mathematics, 47*, 101-116.

Krutetskii, V. (1976). *Psychology of mathematical abilities in schoolchildren* (J. Kilpatrick, & I. Wirzup, Eds., J. Teller, Trans.). Chicago: University of Chicago Press. (Original work published 1968)

Liljedahl, P. (2006). Pentominoes and AHA!'s. In R. Zazkis & S. Campbell (Eds.), *Number theory in mathematics education: Perspectives and prospects*. Mahwah, NJ: Lawrence Erlbaum.

Schoenfeld, A., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in Instructional Psychology* (Vol. IV, pp. 55-175). Hillsdale, NJ: Lawrence Erlbaum.

Sfard, A. (2002). Thinking in metaphors and metaphors for thinking. In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics: A tribute to Richard Skemp* (pp. 79-96). Flaxton QLD: Post Pressed.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification — The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Skemp, R. (1980). The functioning of intelligence and the understanding of mathematics. In M. Zweng (Ed.), *Proceedings of the Fourth International Congress on Mathematical Education* (pp. 533-537). Boston, MA: ICME.

Steffe, L., & Thompson, P. (2000). Teaching experiments methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education. In L. Streefland (Ed.), *Proceedings of the 9th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 97-123). Utrecht, The Netherlands: PME.

Williams, G. (2000a). Collaborative problem solving and discovered complexity. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle WA, Vol. 2, pp. 656-663). Sydney: MERGA.

Williams, G. (2000b). *Collaborative problem solving in Mathematics: The nature and function of task complexity*. Unpublished Masters thesis, University of Melbourne, Melbourne, Victoria, Australia. Accessed 23 July 2007 at http://eprints.infodiv.unimelb.edu.au/archive/00002770/

Williams, G. (2002a). Associations between mathematically insightful collaborative behaviour and positive affect. A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 402-409). Norwich, UK: PME.

Williams, G. (2002b). Identifying tasks that promote creative thinking in mathematics: A tool. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 698-705). Sydney: MERGA.

Williams, G. (2005). *Improving intellectual and affective quality in mathematics lessons: How autonomy and spontaneity enable creative and insightful thinking*. Unpublished doctoral dissertation, University of Melbourne, Melbourne, Australia. Accessed 23 July 2007 at http://eprints.infodiv.unimelb.edu.au/archive/ 00002533/

Williams, G. (2006). Autonomous looking-in to support creative mathematical thinking. In D. Clarke, J. Emanuelsson, E. Jablonka, & I. A. C. Mok (Eds.), *Making connections: Comparing mathematics classrooms around the world*. Taiwan: Sense Publishers.

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