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# Abstracting Processes, from Individuals' Constructing of Knowledge to a Group's "Shared Knowledge"

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A model for processes of abstraction, based on epistemic actions, has been proposed elsewhere. Here we apply this model to processes in which groups of individual students construct shared knowledge and consolidate it. The data emphasise the interactive flow of knowledge from one student to the others in the group, until they reach a shared knowledge — a common basis of knowledge which allows them to continue the construction of further knowledge in the same topic together.

The relationships between the construction of knowledge by individuals and the "shared knowledge" of an ensemble is a fascinating issue, both from cognitive and socio-cultural points of views. An *ensemble* is defined as "the smallest group of individuals who directly interact with one another during developmental processes related to a specific context" (Granot, 1998, p. 42). The existence of an ensemble emerges from situations in which individuals contribute to the same activity.

Typical ensembles in classrooms include dyads, small groups, and entire classes. In each case the teacher may or may not be included in the ensemble. Understanding the relations between the construction of knowledge by individuals and what we will call "the constructing of the ensemble's shared knowledge" is crucial in research concerning learning processes in the classroom, and evolves from the cognitive as well as the social domain. Several scholars (e.g., Simon, 1995) have invested effort into integrating these domains. Rommetveit (1985) defined the term *intersubjectivity* to designate the set of common beliefs necessary to sustain an interaction, and Tomasello, Kruger, and Ratner (1993) elaborated the idea of *shared understanding* to explain why people attend to each other and reason together.

However, researchers who plan to observe and analyse, in detail, processes of constructing knowledge in an ensemble, within a context such as a classroom and over a period that may range from minutes to weeks, face great difficulties: the observation and documentation processes are complicated, and the data may be massive and messy. On the other hand, data on the behaviour of specific individuals may be sparse and there are no systematic clear-cut methodologies for analysing them (Schoenfeld, 1992). Some members of an ensemble might be silent and seem passive even when they are attentive — a situation that Kuhn (1972) designates "tacit interaction." The difficulties become particularly acute as the size of the ensemble increases.

Many researchers have been aware of the above difficulties. In their efforts to analyse the collective learning of a mathematics classroom community, Cobb and colleagues (2001) focused on the evolution of mathematical practices. For this purpose, they combined "a social perspective on communal practices with a psychological perspective on individual students' diverse ways of reasoning as they participate in those practices" (p. 113). They discussed the notion of *taken-as-shared activities* of the students in the same classroom:

> We speak of normative activities being taken as shared rather than shared, to leave room for the diversity in individual students' ways of participating in these activities. The assertion that a particular activity is taken as shared makes no deterministic claims about the reasoning of the participating students, least of all that their reasoning is identical. (p. 119)

Voigt (1995) emphasises the role of interaction in arriving at *taken-as-shared knowledge*:

> Through their discussion, the students and teacher constituted an explanation that perhaps neither would produce individually. They arrived at knowledge taken as shared. (p. 183)

Voigt analyses interactions between students and their teacher as well as interactions among students. According to him, interaction mediates both psychological and social processes: Individual cognition and social constructs are both promoted through the mediation of negotiation for meaning.

In this paper, we focus on processes of constructing knowledge in an ensemble of three interacting students, where personal diversity, the unique nature of each individual, is observed and analysed. We will emphasise the flow of knowledge from one student to the others, until they have a common basis of knowledge. The research focuses on the *constructing* processes and on the *constructs* at a given point of time, as well as on their *consolidation*. If their common base of knowledge allows the students in the ensemble to continue constructing knowledge collaboratively and actualising it in further activities, we identify this as *shared knowledge* — a common basis of knowledge which allows the students in the group to continue together the construction of further knowledge in the same topic.

Our research is standing on the shoulders of others' research, such as that of Voigt (1995), Cobb et al. (2001), and many others; but it goes beyond theirs in several perspectives:

1. The *micro*-perspective: We provide detailed evidence of the group's shared basis of knowledge, the manner in which it emerges from the individuals' knowledge-constructing processes, and the way in which it constitutes a shared basis that allows the students to continue constructing further knowledge together.
2. The *continuity* perspective of the micro-analyses: We tie the data and

their analysis together along a time span of several lessons in order to observe and analyse students constructing new mathematical knowledge in one activity, and to observe and analyse in detail if and how they use this knowledge in further activities, that is, if and how they consolidate the constructed knowledge.

3. The *theoretical* perspective: We consider how students go through an abstracting process in the sense of constructing a new piece of knowledge by vertical mathematisation, and how they interact with other students in the group to follow parallel processes of abstraction.
4. The *methodological* perspective: For the analyses in 1, 2 and 3 above, we use the RBC+C model of abstraction. We show how this model enables a deep analysis of a group's construction of shared knowledge.

The data that are presented in this paper form part of the corpus of data collected in a long-term research project whose goal is to investigate the construction and consolidation of knowledge in elementary probability. The learning of probability has been studied among individual students, small groups and whole class communities.

The data are presented in three "stories" taken from the activities of two groups from different schools that investigated problems which were extracted from the same elementary probability unit. Each story will be described and analysed in a few episodes. These stories exemplify different flows which describe how shared knowledge was constructed out of the individual knowledge in different interaction patterns. The stories also show different characteristics shared knowledge might have.

As mentioned, the RBC+C model (Hershkowitz, Schwarz, & Dreyfus, 2001 [HSD]; Dreyfus, Hershkowitz, & Schwarz, 2001 [DHS]) will be used as the main methodological tool for describing and analyzing the construction of shared knowledge and its consolidation in the three stories.

## The RBC Model

Researchers interested in studying knowledge construction by individuals and/or by ensembles have adopted various methodologies. For example, Resnick, Salmon, Zeitz, Wathen, and Holowchak (1993) used a double coding of argumentative moves on the one hand, and speech acts (claim, opposition, elaboration, explanation, etc.) on the other hand, in order to analyse reasoning in conversation. Although this methodology of double coding influenced several studies of knowledge construction, the focus of these authors was exclusively on the process of co-construction without relating to the specific content and knowledge concerned by the construction process. Therefore, Resnick and her associates did not provide criteria for analysing cognitive change. Pontecorvo and Girardet (1993), on the other hand, studied the construction of historical knowledge by observing historical epistemic actions (such as appealing to a source) during a conversation and by looking at the inferences collectively agreed among students.

Hershkowitz, Schwarz, and Dreyfus (HSD, DHS) presented a theoretical and practical model for the cognitive analysis of abstracting in mathematics learning. Abstracting in context was taken as a human activity of "vertical mathematisation" (Treffers & Goffree, 1985). Vertical mathematisation represents the process of constructing new mathematical knowledge within the mathematics itself and by mathematical means. Vertical mathematisation typically proceeds by reorganisation of previous mathematical constructs, interweaving them into a single process of mathematical thinking and leading to a new mathematical construct. Processes of knowledge construction are expressed in the model through three observable and identifiable epistemic actions: Recognising, building-with, and constructing (RBC). *Recognising* takes place when the learner recognises that a specific previous knowledge construct is relevant to the problem he or she is dealing with. *Building-with* is an action comprising the combination of recognised constructs in order to achieve a localised goal, such as the actualisation of a strategy or a justification or the solution of a problem. The model suggests *constructing* as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematisation to produce a new construct. It refers to the first time the new construct is expressed by the learner. It does not refer to the construct becoming freely and flexibly available to the learner, which is part of the construct's consolidation, as we will discuss later on.

Recognising and building-with are often nested within constructing actions. Moreover, constructing actions are at times nested within more complex or holistic constructing actions. Therefore the model is called the "nested epistemic actions model of abstraction in context", or simply the "RBC-model". The model has emerged and was first described by means of illustrative examples in a variety of contexts which differ by their mathematical content, their social setting, and their research setting (problem solving by individuals, and dyads' actions in problem solving). The researchers proposed and elaborated the RBC model on the basis of two case studies in which students were observed in laboratory settings: an interview with a single student (HSD) and the observation of dyads working in collaboration (DHS). In the second study, the construction of a new construct by each individual in the dyad was investigated, as well as the interactions between the two students. Interaction was investigated in detail as a main contextual factor determining the process of the dyad's shared abstraction.

Since then, the RBC model has been validated and its usefulness for describing and analysing processes of abstraction of other contents, in other social settings and other learning environments has been established by a considerable number of research studies by our group as well as by others (Bikner-Ahsbahs, 2004; Dreyfus & Kidron, 2006; Ozmantar & Roper, 2004; Ron, Dreyfus, & Hershkowitz, 2006; Stehlíková, 2003; Tabach & Hershkowitz, 2002; Tabach, Hershkowitz, & Schwarz, 2001, 2006; Tsamir & Dreyfus, 2002; Williams, 2002, 2003, 2004, 2005; Wood & McNeal, 2003; Wood, Williams, & McNeal, 2006). The settings considered by these researchers include classrooms, group interviews, individual tutoring situations, and even introspective self-reports of single learners. Some of the studies were carried out with computer tools available to the learners and others without. The age range of the learners extends from elementary school students to adult experts, and the content includes topics in fractions, algebra, functions, geometry, probability, and chaos theory.

The need for investigating *consolidation* of the constructed knowledge was mentioned in HSD, where consolidation is a process by which the construct becomes progressively more readily available to the learner. Consolidation is expected to be evidenced in learning activities which follow the one in which the new knowledge construct first emerges. Consolidation of construct X is a long-term process that starts after constructing X and continues indefinitely; every recognising of X and building with X contributes to this consolidation, in particular if the recognising and building-with occur as part of a later constructing action. Thus, investigation of consolidation should have a longitudinal dimension. The evidence we can obtain about the degree of consolidation is the ease, immediacy, confidence, self-evidence, and flexibility with which X is recognised and used, as well as the degree to which the learner is aware of recognising and using X (Dreyfus & Tsamir, 2004).

Research has made it clear that the RBC model can be extended to processes of abstraction and consolidation on a medium term time-scale (Dreyfus & Tsamir, 2004; Dreyfus et al., 2006; Monaghan & Ozmantar, 2004; Tabach & Hershkowitz, 2002; Tabach et al., 2001, 2006; Tsamir & Dreyfus, 2005). As a result, the RBC model is now called the RBC+C model.

In the present study, we will emphasise the common (shared) epistemic actions of constructing and consolidating and therefore, in order not to disturb the flow of ideas, will not identify explicitly recognising and building-with actions. We will use the characteristics mentioned above to identify and investigate consolidation processes and to define their place in the knowledge construction process.

## The Probability Research Project

A number of years ago, we began a research project to investigate students' learning during sequences of activities with a high potential for constructing and consolidating. We decided to focus on the basic concepts of probability, for several reasons:

- Probability is part of the Grade 8 curriculum.
- Research shows that this is an appropriate age group for learning structured probability concepts (Falk & Wilkening, 1998; Shaughnessy, 1992).
- The topic of probability has relatively little interaction with other topics. This minimises the influence of students' existing knowledge constructs at the start of the research program.
- Intuition plays meaningful roles in probabilistic thinking, because probability offers many exciting connections with daily life. Such intuition may serve as a basis for constructing abstract probability

concepts and thinking. Research shows that even young students often have intuitive knowledge about probability and that these informal conceptions are based to some extent on perception (Falk, Falk, & Levin, 1980; Konold, 1989). Moreover intuition might lead to wrong conclusions and hence to surprises and conflicts (Kahneman & Tversky, 1972). Students' initial knowledge is thus undifferentiated, as described by Davydov (1972/1990), and may become articulated and abstracted through appropriate learning activities.

- The hierarchal structure of probability is appropriate for our purposes: There are basic concepts as well as compound concepts and processes that may be constructed on the basis of the construction and consolidation of the more basic ones; it is thus possible to design a sequence of tasks that offers opportunities for constructing and consolidating a set of concepts and processes.

Based on this reasoning, a unit consisting of a sequence of activities embedded in a rich learning environment was designed, developed and used with student pairs and with entire classes of students. The sequence was carefully designed to offer opportunities for constructing and consolidating. The unit included:

1. A written pretest, to be answered individually.
2. Five activities, requiring approximately ten lessons, organised as sequences of problem-situations for whole class discussions, for investigative group work, and for homework which was mostly done on an individual basis.
3. Three post-tests to investigate students' knowledge: a written post-test to be answered individually, an individual interview, and an individual game-interview. It is worth noting that, because the aim of this work was to follow the process of constructing shared knowledge via the interaction among the individuals in the group, each individual's knowledge is considered as part of the group's shared knowledge. Therefore, at the end of the interactive learning processes, it is very important to investigate the knowledge of students as individuals and there is no value in a group posttest.

The research project had two cycles. The research goals of each cycle include the analysis of students' constructing and consolidating processes using the RBC+C model.

The *first cycle* was carried out in laboratory conditions. Six dyads of eighth graders worked through the unit in laboratory conditions. This cycle was used for research as well as for trialling the learning unit, and some minor revisions were accordingly made.

The *second cycle* was carried out in classrooms. Five Grade 8 teachers in four different schools taught the unit. In each lesson, one or two researchers were present and documented the lesson by means of two video cameras. One camera focused on a group of students (the same group in each lesson) and the second camera focused on the teacher and the whole class activity. The researchers also collected completed worksheets and written tests and took field notes.

In this second cycle, particular attention was paid to ensembles (whole class, small groups, and dyads) and the social interactions within them. In this paper, we analyse processes of abstraction within two groups of three students at time when all the students in the class were working in small groups. The group work formed part of the regular class work, which also included whole class discussions led by the teacher, teacher demonstrations, homework, and tests.

This paper is a continuation of the DHS paper, in which construction of knowledge of dyads in laboratory conditions was described. The dyads in DHS were investigated while they worked on a single activity. In the present paper, we extend the analysis carried out in DHS in two essential ways: First, the data stem from classrooms rather than from a learning laboratory; and second, the activity took place over an extended time scale, allowing us to focus on consolidating processes in addition to constructing processes.

## Constructing a 2D Sample Space: Principles and Tasks

The probability unit deals with concepts and problem-solving aspects of empirical versus theoretical probability, and one- and two-dimensional sample spaces.

The overall construct of sample space, which was investigated in the project, includes three hierarchical stages:

A. Sample spaces in one dimension (1D SS); for example the probability of obtaining 3 when rolling a die.
B. Sample spaces in two dimensions (2D SS) where the possible simple events are equiprobable and can be counted and organised into a table and the probabilities of the complex events found by inspection, for example, the probability of obtaining 3 on two dice rolled simultaneously.
C. Sample spaces in two dimensions where the probabilities of the possible simple events are given but are not necessarily equal. The different possibilities can be represented in an area diagram, from which the probabilities of complex events can be calculated.

The topics considered in Stage A (Activities 1 and 2) concern 1D SS and include theoretical probability as the ratio of the number of relevant outcomes to the number of all possible outcomes, as well as some experience with the fact that empirical probability values tend to the theoretical value as the number of trials becomes large. The data in this paper will be taken from students' activity at the beginning of Stage B of the learning unit. In order to be able to understand and analyse this activity better, we used an epistemic analysis in terms of the mathematical principles which are at the basis of this stage. The students' construction of the following principles will be investigated:

P1 The meaning of an event in 2D SS: A simple event in 2D consists of a simple event in each dimension (e.g., the possible outcomes of two dice rolled simultaneously are pairs of numbers). Thus, the possible simple events in each dimension have to be taken into consideration.

P2 All events: Identification of all possible simple events in a given 2D situation (e.g., identification of the 36 simple events when rolling two dice).
P3 All relevant events: Identification of all simple events that are relevant to a given complex event (e.g., the six events of obtaining equal numbers when rolling two dice).
P4 Calculating the probability of a complex event in a 2D SS.
P5 The importance of order in 2D simple events: The two simple events (a, b) and (b, a) are different, and each of them has the same probability as (a, a).

It is worth noting that:

- In 2D SS, P1 is a prerequisite for P2, P3 and P4.
- P2 and P3 are needed for P4.
- Complete versions of P2 and P3 necessarily include P5. But partial versions of P2 and P3, which do not include P5, might be quite relevant in various learning stages.

The probability tasks shown in Figure 1 were used in this study.

**Activity 3, Question 1**

1a: Yossi and Ruti roll 2 white dice. They decide that Ruti wins if the numbers of points on the 2 dice are equal, and Yossi wins if the numbers are different.

Do you think that the game is fair? Explain!

1b: The rule of the game is changed. Yossi wins if the dice show consecutive numbers: Do you think the game is fair?

1c: How many possible outcomes are there when rolling 2 dice?

1d: Suppose Yossi and Ruti play with one red die and one white die; does this change the answers to 1a, 1b and 1c?

**Activity 3, Question 2**

We again roll 2 regular dice. This time we observe the difference between the larger number of dots and the smaller number of dots on the 2 dice. (If the numbers on the two dice are equal, the difference is 0.)

2a: Make an hypothesis whether all differences have equal probability. Explain!

2b: How many different differences are there?

*Figure 1.* Activity 3, Questions 1 & 2.

Activity 3 is the first task in the unit that deals with 2D SS. Dealing with Questions 1a & 1b may lead to the constructing of P1 and at least partially constructing P2 and P3. Question 1c is explicitly related to P2 and its goal is to give students an additional opportunity to construct it. The goal of Question 1d is to motivate students to deal with the dilemma of principle P5. Question 2 presents a more sophisticated situation since there are six different differences, each of which is a complex event with a different probability.

## Data and Data Analysis

Three stories will be described and analysed in this section. Two of these stories are taken from the activity of the same triple of girls. The third story is taken from the activity of a triple in a different class in a different school. All three stories belong to Cycle 2 of the project and to Stage B of the learning trajectory. The three stories illustrate three different examples of constructing shared knowledge.

### *Story 1: Yael, Rachel, and Noam construct P1*

In this story the discourse among the three girls shows the process of constructing shared knowledge of P1. The knowledge flow starts from Yael, who has constructed this principle before or at the very beginning of this activity. (We have no information about how she constructed it.) The second student to construct P1 is Rachel. Later on, during Noam's interaction with Yael and Rachel, it becomes Noam's construct as well.

The three girls discuss Question 1a. Yael starts to enumerate winning number pairs for Yossi and Ruti, and finds 21 pairs for both of them together:

| | |
|---|---|
| [21] Noam: | What are you doing? Eh? |
| [22] Yael: | I did: What are the chances? One, one; two, two; three, three; four, four; five, five; six, six; and one, one; one, two; one, three; one, four; and two, one; two, two... |
| [23] Rachel: | I don't understand what you are doing. |
| [24] Yael: | It is because I have to know what is our whole, what are all the possible outcomes that might be, and all these outcomes are either (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6) or (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) and then 2 [meaning that now she has to count the combinations of 2 with the other numbers)]. |
| [25] Noam: | Yael, I don't understand anything of what you are doing. |
| [26] Rachel: | Yes, nor do I. |

It seems that Yael is already busy enumerating all the relevant pairs for calculating Ruti's and Yossi's chances to win (P3), and wants to know what all possible simple events are (P2). Yael [22] clearly refers to pairs of numbers, so we have used the notation (m, n) for the remainder of this story. In any case, she seems to recognise P1 right from the beginning and tries to use it to construct P3 and P2.

Rachel and Noam have not yet constructed the meaning of events as pairs of numbers (P1), and they do not appear to understand what, how, and why Yael is counting [21, 23, 25, 26]. When Yael realises that her friends are not aware that events in 2D SS are pairs, she explains again:

| | |
|---|---|
| [27] Yael: | Listen, there are some possibilities that 1 will appear: (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) and we finished with 1, now 2: (2, 3) (2, 4) (2, 5) (2, 6). |
| [28] Rachel: | O.K., O.K., we understood that, but why are you adding? I don't understand. |

Although it is not evident from Rachel's utterances that she has already constructed P1, we may assume that she already shares with Yael that one has to count pairs (P1), because now she only asks about Yael's counting system [28] and not about the nature of simple events in 2D SS. Between Rachel [28] and Noam [58], Rachel and Yael continue to discuss the number of all possible simple events and the number of relevant simple events (P2, P3). Throughout this discourse, Noam tries to understand the situation; she is still in the process of constructing P1. She confronts her friends with her misunderstanding:

| | |
|---|---|
| [58] Noam: | Look you don't... you did as if one side of the die is 3 and the second side is 4 and you did 3 plus 4 and it is as if... |
| [59] Yael: | I didn't do 3 plus 4. I will tell you exactly what I did... |
| [60] Noam: | No, one second, second. That's what I understood from what you did. |
| [61] Yael: | I will explain... |

But Noam wants to explain herself:

| | |
|---|---|
| [62] Noam: | One minute! No! You have to do 3 and 4; it is one possibility, and 4 and 5 is a second possibility, so it is two [possibilities]. |
| [63] Rachel: | That is what she did; (3, 4) is one possibility and (5, 4) is one possibility. |

We can conclude now that Noam [62] has constructed P1. Rachel [63] explains that this is exactly what Yael did, by repeating Noam's explanation. Rachel uses P1, which she had constructed earlier, to build-with in her explanation. She thus provides evidence that she is already consolidating P1.

More evidence for the fact that Noam has constructed P1 is that later, while the three girls work on Question 2 (Figure 1), she explains what is the meaning of the "differences of outcomes" on the two dice:

| | |
|---|---|
| [109] Noam: | If we look on the difference of one die and the other die, (2, 2), then the difference is 0. |

We may see that now Noam recognises P1, builds-with P1 and thus consolidates P1. She recognises the pairs representation for simple events of the 2D SS and uses it for building-with the explanation for the differences.

At this point in time, it seems that principle P1 has become a shared common basis of knowledge for the group.

*Summary*. The individual knowledge of P1 for each girl was constructed and, in some cases, consolidated on the basis of Yael's intuitive previous construct. But the questions of Noam and Rachel, the explanations of Yael, and the self-explanations of Noam and Rachel while they constructed this knowledge, played a crucial role. Examples:

1. The repeated questions of Noam and Rachel forced Yael to repeat and count pairs. While organising the counting, Yael consolidated the P1 principle [21-27].
2. Similarly, Rachel's explanation to Noam forced her to express things explicitly and by doing so, she did not just agree with Yael but also consolidated her knowledge [62-64].
3. Noam put the blame for her mistakes on Yael [58] and then corrected herself [62]. She not only realised that she had to relate to pairs but she could even explain her mistakes [58-62]. Thus she had constructed P1, and later started to consolidate it [109]. Here, consolidation of a construct is evidenced by the ability of the student to use the construct for building further explanations with it.

The group's shared knowledge in this story was constructed first through interaction of questions and explanations from one individual to the others, and then through the explanations of the two individuals who already shared P1 to the questions posed by the third student. The interaction in this story includes misunderstandings, a need to understand one another, and explanations. From observations of later activities, we have further evidence that P1 was consolidated here and remained applicable in the group's shared knowledge.

## *Story 2: Yael, Rachel, and Noam construct P2 and P3*

Chronologically, this story was happening in parallel to story 1, at least partially, because the three girls' discussions concerning P1, P2, and P3 were interwoven. We differentiate between the two stories in order to clearly separate the constructing of P1, which is a prerequisite for the constructing of P2 and P3, and because the constructing processes of P1 are quite different in nature from those of P2 and P3.

In this story we will exemplify how partial constructs of P2 and P3 ("All simple events", and "All relevant simple events" in 2D SS) become part of the shared knowledge of Yael and Rachel, and how they recognise them soon after constructing them. We also show that Noam finds a way to answer the task questions without a need to recognise or build-with the principles P2 and P3 (whether complete or partial), and thus seems to share the work with her friends without sharing the knowledge.

As described in Story 1, while working on Question 1 (see Figure 1) Yael is counting events in the 2D SS [22, 24] and explains that she is trying to find all the possible simple events in the sample space (in her words, "the whole" [24]) in order to calculate the required probabilities. Rachel, who has already constructed P1, still does not quite follow why and how Yael is counting the pairs [28], and Yael explains again:

[32] Yael: What is the whole? O.K., look! There are 6 out of ... [6 events with equal results on the two dice]. 5 + 4 is 9, and 3 is 12, and 2 and 1 is 15, and the 6 is ...

Yael counts the pairs with doubles, then pairs with a 1, 2, 3, 4 and 5, without taking principle P5 (the importance of the order of the numbers in a pair) into account.

[33] Rachel: 21!

It seems that Rachel just adds the numbers. (As we will see in [51], she still wonders about the meaning of the number 21.) The discourse continues as follows:

[41] Yael: I know that I am right. His frequency is 15 out of 21 ...

[42] Rachel: What?

And later:

[51] Rachel: What is 21? I don't understand what 21 is! [She laughs with embarrassment].

[52] Yael: O.K., there are 6 possibilities that we will get the same number on the dice ...

[53] Rachel: True!

[54] Yael: Then 15 plus 6 is 21.

[55] Rachel: Ah, O.K.

Then Noam joins in with the same clarification questions:

[69] Noam: Why 21?

[70] Yael: Ufff...!!!

[71] Rachel: Because there are 15 possibilities for him [different numbers] and for her 6 possibilities [equal numbers] and 15 plus 6 is 21.

[72] Yael: It is the whole, all the possibilities that can occur on the dice, two dice.

[73] Noam: Ah, you roll...

[74] Yael: Then, 6 divided by 21, it's her frequency [probability], and 15 divided by 21 it's his frequency [probability].

[75] Rachel: [summarising and dictating the answer] No [meaning not fair], because the chance of two equal numbers is 6 divided by 21 and the chance of different numbers is 15 divided by 21.

We may conclude now that Yael thinks that for checking the fairness of the various games, one needs to calculate the probabilities of the relevant events. This has motivated Yael to construct P2 and P3, but she is most probably not aware that she has a partial construct only. In any case, Yael recognises her (partial) constructs for P2 and P3 in order to build the needed probabilities with them (P4, which is basically the same for 1D and 2D SS).

Rachel is the one who tries to explain the issue to Noam, and she is also the one who sums it up for the reports in their notebooks. Hence we can conclude that Rachel, too, had constructed partial constructs for P2 and P3 and built P4 for 2D SS with them. Both Yael and Rachel share the partial constructs of P2 and P3, and even consolidate them, as they both build the calculation of the probabilities (P4) with them.

From the above discourse, we cannot tell whether Noam constructed some constructs for P2 and P3 or not. In Question 2, Noam becomes very active. It seems that the fact that she is now sharing knowledge of P1 with her friends (see Story 1) gives her some self-confidence and motivation. She is the first to build with P1, constructing the difference between a pair of numbers on the two dice:

[109] Noam: If we look on the difference of one die from the other, (2, 2), then the difference is 0.

The three girls then discuss the meaning of "different differences" (see Question 2 in Figure 1). They agree that a same difference might be obtained from different number pairs. For example, Yael says:

[115] Yael: If you get (1, 2), the difference is 1 and if you get (2, 3) the difference is also 1.

They start to discuss the question whether the probabilities of different differences are the same. In this discussion, Noam exposes her way of facing question 2a:

[119] Yael: Are the probabilities of [the differences] 1 and 4 equal?

[120] Noam: I understand. No! [meaning not equal]

[121] Yael: [repeating] Are the probabilities of 1, 2, 3, 4 equal?

[122] Noam: We understand! NO!

[123] Yael: How do you know? Did you calculate it?

[124] Noam: No! Because they are not equal.

[125] Yael: How do you know?

[126] Noam: If we say difference of 4 ... or we say difference of 5 there is only one [meaning only one giving a difference of 5 and more giving a difference of 4].

Noam compares the frequencies of the differences, and concludes that different frequencies mean different probabilities. For Yael, this way of reasoning does not seem to be sufficiently detailed:

[127] Yael: One moment! Let's take 1, then we have (1, 2) (2, 3) (3, 4) (4, 5) (5, 6) and the same number (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6).

Later Rachel joins Yael in counting, and both of them try to count how many pairs fit each difference (P3). When they try to summarise and write down their conclusion, Noam interrupts them and clarifies that, for her, examples are sufficient to conclude that different differences have different probabilities. Rachel does not agree; she thinks that examples are not sufficient:

[143] Noam: No, let's give examples.

[144] Rachel: It decreases, you understand?

[145] Noam: Come on, let's give simple examples, O.K.? Let's give examples!

[146] Rachel: No! Because the difference between 0 and 1 ...

[147] Noam: It is an example, you idiot!

Rachel is not satisfied with the minimal answer provided by Noam's counter example and ends the discussion by describing a functional dependence:

[148] Rachel: As the difference grows, the probability becomes smaller.

*Summary*. As in Story 1, each girl constructed her individual knowledge in her own way and time. This knowledge was constructed at least partially by answering the questions they asked each other, by dealing with their disagreements, and by their demand for explanations. At this stage of the activity, we may conclude the following:

1. The constructs for P2, P3 shared by Yael and Rachel, were partial (Ron et al., 2006). They explained their conclusions by trying to count all the relevant pairs (P3), and all the possible pairs in the sample space (P2), and they recognised that these numbers are relevant to build the calculation of the probabilities with them (P4). This provides evidence that they consolidated their partial constructs for P2 and P3.
2. Noam relates to relevant pairs (events) without trying to count them all and uses decisive (counter-)examples. She finds examples of the numbers of pairs of two differences, compares them, and by this explains that probabilities of the differences are not equal. Thus, she uses P3 and builds a correct answer with it, ignoring P2 and P4.

3. The three girls shared a lack of knowledge concerning principle P5. Thus their constructing of the P-principles was not complete. All of them counted two pairs such as (1, 2) and (2, 1) as a single simple event that has the same probability as the pair (1, 1).

The girls' knowledge concerning P2 and P3 was quite different. Yael and Rachel shared a systematic knowledge which was still partial. On the other hand, Noam did not feel the need for counting the events of P2 and P3 as a whole, but using P3 to find some examples enabled her to share the answer to Question 2 and go on learning with her friends. The methods to answer the questions were different and led to different levels of response. The interaction among the three students included objections, debates and finally agreement — but only on the answers to the given questions.

## *Story 3: Yevgenia, Chen, and Shany construct principle P5*

This story relates to the construction of principle P5: in 2D SS the events (*a*, *b*) and (*b*, *a*) are two different simple events, each of them having the same probability as the simple event (*a*, *a*).

In this story we describe how three girls, Yevgenia, Chen, and Shany (working as a group in a different class from the group in Stories 1 and 2) become aware of the issue of principle P5 and seem to agree on an incorrect relation between the pairs (*a*, *b*), (*b*, *a*) and (*a*, *a*). We will also show that this agreement is fragile, because one of the girls does not trust it and repeatedly raises a debate about it. Finally, we will show how the three eventually reject the agreement and construct the correct principle P5, thanks to a new tool presented by the teacher in a class discussion. At least one of the girls is even able to explain why and how their initial construct for P5 was wrong.

The first time that the question of principle P5 is raised in this group, is when they discuss Question 1b (see Figure 1). The three girls agree that there are 12 pairs in the sample space (thinking additively that 6 on one die and 6 on the second die gives 12 events), that Yossi's chance is 5 out of 12, and that Ruti's chance is 6 out of 12. Then Yevgenia raises the question of principle P5:

[56] Yevgenia: One moment... something does not seem logical to me...

And she explains:

[59] Yevgenia: No, no, no ... it can be different, every die is 10 ... he [Yossi] still remains with the same 10, (1, 2) and (2, 1).

Chen does not understand and Shany helps to explain the problem:

[60] Chen: So ... what have we done? (1, 2) is one and (3, 2) is one ...

[61] Shany: Or the opposite, on the second die 2 dots. This is one die and this is a second die [pointing at (2, 1) and (1, 2)].

Then they discuss the issue of winning the game without relating to principle P5, until Yevgenia connects the two issues:

[71] Yevgenia: It is clear that the game is not fair. It can be that on one die you get 1 and on the second 2, or the opposite 2 and 1.

Chen joins her, calculating the probability, and Shany asks for the help of the researcher:

[72] Chen: And then it will be 10 to 12? [meaning 10 out of 12]

[73] Shany: [to the researcher] We also have to reverse the dice?

At the end of a debate about whether there are 10 or only 5 simple events of successive numbers, Yevgenia decides:

[88] Yevgenia: Yes, there are a total of 5 pairs of successive numbers.

After Shany repeats that Yossi has 5 possibilities and Yevgenia adds that Ruti has 6, Chen again raises the issue of principle P5 and Yevgenia, in an assertive way, rejects it:

[92] Chen: But it is possible to reverse it!

[93] Yevgenia: No! It is not possible to reverse it!

[94] Researcher: What do you say, Chen?

[95] Chen: [clarifies] It is possible to reverse the dice: 1 and 2, and 2 and 1.

[96] Yevgenia: It will not change anything! It will not change anything!

It is worth noting that all three girls are aware of the dilemma, and Yevgenia clarifies that she, who had first raised the issue [56, 59], is now sure that the order is not important.

[98] Yevgenia: I totally changed my mind, it will not change anything. (2, 1) or (1, 2) is the same successive numbers! It will not change anything. It is the same successive numbers! The same successive numbers! I was wrong before and I totally changed my mind!

After Chen raises the issue a few more times, Yevgenia starts to summarise, but Chen tries to object. Shany joins Yevgenia (the strongest one):

[109] Yevgenia: It does not make any difference.

[111] Chen: If Yossi gets 2, and she gets 1...

[112] Yevgenia: Even if it will come out the opposite it will not make any difference.

[113] Shany: She [Yevgenia] is right, it does not make any difference if it comes out (1,2) or (2,1).

[114] Chen: I don’t know, maybe he [Yossi] has more chances.

[115] Yevgenia: He has not! He has not! He has only the successive numbers from 1 to 6.

[117] Chen: [to the researcher] What shall I do? 10 out of 12 or 5 out of 12?

[118] Yevgenia: It is finished ... thanks Chen ... we have still not read the [next] question.

Chen, who seems to be in a process of constructing principle P5, tries to get help from the researcher and Yevgenia, who shows strong confidence in denying the importance of order in a pair, closes the discussion and moves on to the next question. Shany does not hesitate in supporting Yevgenia.

Thus we can say that the construct of "wrong P5" is shared by Yevgenia and Shany, while Chen, as we will show, does not agree with them.

As they work on Question 1c (see Figure 1), Yevgenia counts all the possible pairs (principle P2 without P5) and finds 21, showing consistency in denying the importance of order. We observe that she has now abandoned the additive thinking which had led them to a 2D SS with only 12 elements.

Chen is still bothered about principle P5:

[153] Chen: [to the researcher] Are (1,2) and (2,1) different outcomes?

[154] Researcher: Ask your friends, you have to decide if they are different outcomes.

[156] Chen: [to her friends] Are (1,2) and (2,1) different outcomes?

[157] Yevgenia: Yes, no! I counted it as one outcome.

Chen gives in and writes down the shared agreement that there are 21 pairs:

[174] Chen: Ah! Yes. It will be 21.

But Chen's doubts are reinforced by the next question, 1d (see Figure 1), where dice of different colours are considered:

[177] Shany: It does not make any difference, right? It does not matter what colours the dice are?

[178] Chen: For sure there is a difference.

[179] Yevgenia: How come?

Chen remains uncertain:

[189] Chen: I still don't understand what it will be: 5 [out of] 12 or 10 [out of] 12?

The girls move on to Question 2 (see Figure 1). While working on this question, Chen gives up her idea of principle P5 and generalises further: She claims that if the order is not important, then there is no need to count more than once pairs in which the difference between the numbers is the same. (For example, pairs like (6,4) and (3,5) should not be counted as two different pairs because the difference between the numbers is the same.)

[202] Chen: 0, 1, 2, 3, 4, 5 — six differences.

[203] Yevgenia: How come?

[204] Chen: Zero is 6 minus 6, right? 5 and 6 is 1, right?

[205] Shany: But if one gets 3 and 5?

[206] Chen: Two, I counted 2. If you get 6 and 4 it is the same. Why count it again if it is the same?

The lesson comes to an end without the girls reaching a shared conclusion.

In the next lesson, the class carries out a computer simulation of the differences task designed to confront the students with their wrong results to Question 2 (see Figure 1). Unfortunately, we have no data showing whether and how the conflict is resolved in our focus group.

After the simulation, the teacher gives a demonstration in which she presents the idea of a 2-dimensional table as a tool for representing all possible pairs in a 2D SS (Figure 2). She then conducts a whole class discussion concerning principle P5. For this purpose she focuses on Question 2, colours her two dice yellow and green, and asks the students to colour yellow (green) all the cells in their tables that represent a difference in which the number on the yellow (green) die is larger.

The three girls start to work together on their table. Shany is the first to discover the symmetry in the table.

[114] Shany: It gives two halves and a diagonal in the middle [see Figure 2].

[115] Chen: This makes the differences table clear.

Then the class discusses the issue of principle P5 and concludes, with the teacher's help, that each cell in the table represents a different simple event.

From now on, all three girls count pairs with the same numbers but a

| Yellow | | | | | | Green |
|---|---|---|---|---|---|---|
| 6 | 5 | 4 | 3 | 2 | 1 | |
| 5 | 4 | 3 | 2 | 1 | 0 | 1 |
| 4 | 3 | 2 | 1 | 0 | 1 | 2 |
| 3 | 2 | 1 | 0 | 1 | 2 | 3 |
| 2 | 1 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

*Figure 2*. The teacher's table for Question 2. In the original, the lighter shading was yellow and the darker shading was green.

different order as different pairs. Chen is now dominant in the group and seems to be happy to provide an explanation. The discussion is between Chen and Shany, who are now working together. Yevgenia withdraws and works by herself.

As requested in the worksheet, they return to find answers to Questions 1a and 1b (Figure 1). The following is their discussion concerning Question 1b:

[169, 171] Chen: Look, in this game Ruti wins if the number of dots is equal and Yossi wins if the numbers on the dice are successive. You have to complete the table.

Chen marks R in table cells corresponding to Ruti's wins and Y in cells corresponding to Yossi's wins. Later she summarises for Shany:

[180] Chen: The probability for Yossi to win is 10/36 and for Ruti 6/36.

The researcher confronts the girls with the answers they gave earlier, when they agreed that the sample space contains only 21 pairs of numbers. Chen is the one who explains their incorrect thinking, using the P5 principle, and she seems to be happy that at the end she was right concerning the importance of the order of numbers in the pair:

[188] Researcher: You wrote 21 [before], now what is the number of all the possible outcomes?

[189] Chen: Now 36!

[190] Researcher: Then why did you think 21?

[191] Shany: Because we thought (1,1) (1, 2) (1, 3) ...

[192] Researcher: What was the problem there? What understanding do you have now?

[193] Chen: That (1, 2) and (2, 1) are not the same! It is a different event!

The researcher wants Yevgenia to participate and check what she is thinking now. It seems that she understands her mistake, but she stays silent about it and smoothly adapts to the idea of 36 events in the sample space:

[194] Researcher: [asking Yevgenia] what is the probability in game 1a and what is the probability in game 1b?

[195] Yevgenia: In any case Ruti is losing!

[196] Researcher: What is the probability in the second game?

[197] Yevgenia: Ruti 6/36 and Yossi 10/36.

[198] Researcher: Where is his chance to win bigger, in the first or in the second?

[199] Yevgenia: In the first. Come on, let's go ...

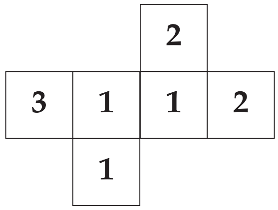
Here we have evidence for the construction of principle P5 and hence for completing the construction of the P-principles in this question, first by Chen and later by Yevgenia [197]. But, although Shany seems to agree with her friends, we cannot conclude if it is a construct of her individual knowledge.

Later, in one of the posttests, each girl was interviewed individually. One of the interview questions was the following (Figure 3):

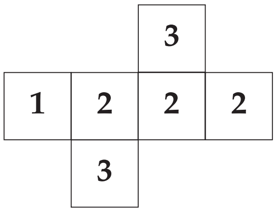
**Posttest question**

Ronen and Rina play with 2 dice.

On one of them the following numbers are written:




On the second:




Ronen wins if the numbers on the 2 dice are the same. Rina wins if the numbers are different.

Do you think that the game is fair? Explain!

*Figure 3.* The posttest question

Each girl represented all 36 pairs in a table. However, their tables were all different. We will first present part of Shany's answer in order to show that she had also constructed principle P5.

[20] Shany: We have to do a table.

She drew a 2-dimensional table (see Figure 4) and said:

[22] Shany: Ronen 11 and Rina 28.

After some questions from the researcher, she wrote *Ronen 11/30* and concluded:

[23] Shany: Rina 19/30 [she erased the 28 and wrote 19 instead].

(She wrongly considered the number of possible events to be 30, despite the fact that she had filled 36 squares in her table.) Finally, she summarised:

[28] Shany: According to the chances, the whole sample space is 30 and Ronen's chances are 11 times and Rina's 19. Rina has more chances to win.

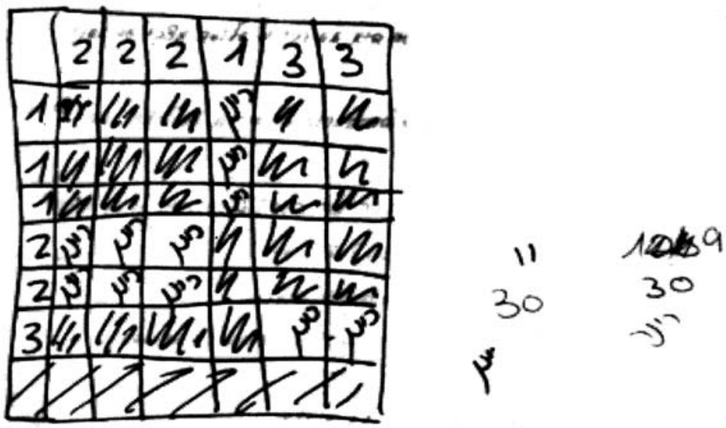


*Figure 4.* Shani's table for the posttest question.

Shany did not count or calculate correctly the numbers of cells of the table. But, as far as principle P5 is concerned, she did take account of the order of the numbers in a pair when she built the table using principle P5 and when she calculated Ronen's and Rina's probabilities of winning. Thus we can conclude that she had constructed principle P5.

Each of the other two girls drew their own unique table (see Figures 5a & 5b) and used them to calculate and explain the required probabilities.

**(a)** **(b)**

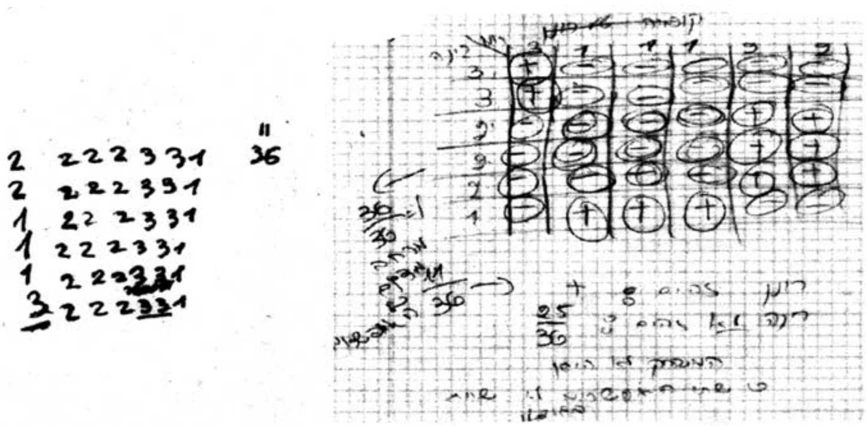


*Figure 5.* Yevgenia's table (a) and Chen's table (b) for the posttest question.

If we examine the three tables together we can see that each girl drew her own table, and that each of the three tables is different from the teacher's table. In this item, the frequency of the numbers on each die is different; thus the learner has to draw a row (column) in the table for the number on each side of the first (second) die, and then identify and count the pairs of numbers which fulfil the given conditions. Thus principle P5 becomes very important and indeed Chen and Yevgenia counted all the relevant simple events and calculated the probability correctly (see Figure 5). We may conclude that the three girls showed evidence for consolidating the P5 principle.

*Summary*. In Story 3 we showed the construction and consolidation of a principle (P5) with the help of the teacher, who demonstrated an appropriate tool and orchestrated a class discussion concerning the issue.

The three girls had been aware of the issue of order from early on, but they did not arrive at an agreement until much later, in a dialectic process. There was a set of three interaction cycles in this story:

1. In the first cycle, the three girls engaged in an argumentative process to elaborate hypotheses concerning the importance of the order in a given pair (principle P5). A shared agreement about the need to discuss this problem is evidenced.
2. In the second cycle, Yevgenia changed her mind about P5 and aggressively tried to force her opinion on the other two. Shany followed her, but Chen opposed her whenever an opportunity occurred. The rejection of P5 was shared by Yevgenia and Shany only.
3. In the third cycle, after the table was presented by the teacher and immediately adopted by the three girls, it became clear that Chen was right. Then Yevgenia became quite silent and worked alone, leaving Chen to lead the work of the group. The posttest a few weeks later shows that principle P5 had by then been constructed and even consolidated by all three of them.

The group reached *shared knowledge*. They reached this shared knowledge in a dialectic flow of opposing arguments and illustrations from one girl (Chen [95, 99, 153]), and forceful claims by another (Yevgenia [93, 98, 118]).

## Discussion

Abstracting new knowledge in mathematics involves high-level mental functions and as such is triggered by the need for a new construct. In earlier papers, we showed that the epistemic actions of the RBC model allows the description and analysis of the emergence of such abstracted constructs in their specific social and cultural context. We also showed that the epistemic action of consolidation can be observed within a larger continuum of activities, thus expanding the model to RBC+C and considerably strengthening its descriptive and analytical power.

The power of this model has been achieved by tracing, at a fine-grained level, the repeated recognising and building-with actions referring to a newly abstracted construct in further activities. The present study, which spans a number of lessons, emphasises the expanded role of the epistemic actions. On one hand, recognising and building-with are bricks in constructing a new construct; they are nested in the constructing process. On the other hand, when these actions occur within the continuum of a learning process, they can serve as evidence for consolidating a previous construct. These two roles can happen simultaneously. For example, the teacher's drawing of the table in Story 3 might be considered as a brick in constructing the principles P1, P2, P3 and P5. But the building of the different tables in the posttest serves as evidence for the consolidation of the same constructs.

This dynamic relation between abstracted constructs and epistemic actions is special to the RBC+C model. It differs from Mitchelmore and White's (2004) theory of mathematical abstraction, labeled as the theory of Empirical Abstraction. Following Skemp (1986), Mitchelmore and White distinguish between two kinds of abstraction: The first is abstraction in mathematics in which mathematical concepts are defined within the mathematical system; these concepts are considered as existing apart from any external reference and are thus called *abstract apart*. The second kind of abstraction, empirical abstraction in mathematics learning, pertains to mathematical ideas that arise through the investigation of real world situations. Empirical abstraction is based on similarity recognition, the identification of key common features within real world situations. The embodiment of a similarity, recognised empirically in a new idea, is an *abstract general* concept.

Together with the earlier RBC model based studies, the present study considerably departs from the theory of empirical abstraction. It focuses on the description of processes at a fine-grained level of analysis of students' epistemic actions during the construction and consolidation of new knowledge. This analysis differentiates recognising from constructing and building-with actions. As we explained above, recognising can be nested within a building-with action or the process of construction. It can also serve as an evidence for consolidating a previous construct. Hence the identification of recognising is never the main goal of the analysis of abstraction processes using the RBC+C model.

The present research has traced successive activities during which three students constructed knowledge in their natural classroom framework. They worked as a group, or as individuals, or participated in whole class discussions orchestrated by their teacher. The traces and effects of these various learning modes were expressed in the group's construction processes as we have described them.

We list here some conclusions we draw from the present study:

1. It was found that the shared knowledge of the group was characterised by its great diversity — many times each partner constructed a piece of knowledge in her own way. Yet all three students in the group may have benefited from this multifaceted shared knowledge in their common further work, and may have continued to construct or consolidate

constructs in follow-up and assessment situations, where the construct continued to have an individual flavour. The process was also varied from one group to the other. The variability evolved from the different needs and interaction patterns at various points in time, and shows the uniqueness of the construction for each individual and each group.

2. The stories that serve as our data sources have enabled the researchers to observe the development of probability knowledge. The episodes are not isolated — they allow the investigation of the effect of past learning and intuition (e.g., Yael in Stories 1 & 2) and the traces of the stories on future performance (e.g., on the posttest in Story 3). Each story tells us the way the group constructed a different principle: Story 1 shows the construction of P1, and how it started in Yael's need to construct P2 and P3. The P1 construct then appears in Story 2, evidencing its consolidation. In Story 2 at least two members of the group were constructing P2 and P3, but because they did not yet construct principle P5, their constructs of P2 and P3 remained partial. In Story 3, a different group constructed principle P5, which again stemmed from a need to complete the construction of P2, P3, and P4 but led to a very different manner of sharing the process of construction.
3. Several patterns of interactive constructing of a group's shared knowledge could be identified.
4. In Story 1, Yael was the knowledge source. In a very intensive series of questions and requests for clarification by Rachel and Noam, Rachel constructed P1 as well. In a new interactive phase, both Rachel and Yael helped Noam to construct P1 through the presentation of 2D simple events as pairs of numbers, and the three students in the ensemble shared the constructed knowledge.
5. In Story 2, the interaction was mostly between Yael and Rachel. Through Rachel's demands to be very explicit and Yael's responses, knowledge was constructed and shared by both of them. The interaction with Noam was then characterised by her objecting to the constructs of her friends' shared knowledge and by constructing her own unique strategy to solve the problems.
6. In Story 3 the shared knowledge of the three girls was constructed in a 3-cycle process: from a shared awareness, to denying the importance of the right construct, to constructing shared knowledge by all three students.
7. We observed some general features of interaction among the three girls. In most cases only two students were active; but the third nevertheless seemed to be involved and, on some occasions, intervened. Being active or passive in the interaction seems to be dependent on many factors such as psychological traits or a specific history of immediate successes or failures.

In conclusion, our empirical work has strengthened the evidence for the high diversity in individual students' ways of participating in a group's construction of knowledge, as claimed by Cobb and colleagues (Cobb et al., 2001) when defining taken-as-shared activities of students in the same group or class. But in addition, it exemplifies a number of paths by which individuals really did construct a piece of knowledge. We also showed, as did Voigt (1995), that interaction — even between students in a small group — has many faces.

We are interested in the construction of shared knowledge as it emerges through interaction during the individuals' construction of knowledge, providing a common basis that enables the group to continue to construct further knowledge in the given subject. We investigated the cognitive and interactive processes of constructing shared knowledge as a single process. The RBC+C model has served as a powerful and especially suitable methodological tool to carry out the necessary micro-level analyses of processes of abstraction over several successive activities. This has provided us insight and understanding of the ways by which knowledge is abstracted by a group.

Although we showed that knowledge was highly diverse and subjective, we could define the shared knowledge of the ensemble in an analytical and objective way. The fact that students in the ensemble kept on working collaboratively to construct knowledge and to build with it in further activities, allowed us to identify this shared knowledge not only *ad hoc* but also *post hoc*, that is, through the observation of consolidation of the shared knowledge. It may have been the existence of this shared knowledge that enabled the students to continue to interact productively in a succession of learning activities.

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