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Abstraction as a Natural Process of Mental Compression

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This paper considers mathematical abstraction as arising through a natural mechanism of the biological brain in which complicated phenomena are compressed into thinkable concepts. The neurons in the brain continually fire in parallel and the brain copes with the saturation of information by the simple expedient of suppressing irrelevant data and focusing only on a few important aspects at any given time. Language enables important phenomena to be named as thinkable concepts that can then be refined in meaning and connected together into coherent frameworks. Gray and Tall (1994) noted how this happened with the symbols of arithmetic, yielding a spectrum of performance between the more successful who used the symbols as thinkable concepts operating dually as process and concept (procept) and those who focused more on the step-by-step procedures and could perform simple arithmetic but failed to cope with more sophisticated problems. In this paper, we broaden the discussion to the full range of mathematics from the young child to the mature mathematician, and we support our analysis by reviewing a range of recent research studies carried out internationally by research students at the University of Warwick.

The term "abstract" has its origins in the Latin *ab* (from) *trahere* (to drag) as:

- a verb: to abstract (a process),
- an adjective: to be abstract (a property),
- noun: an abstract, for instance, an image in painting (a concept).

The corresponding word "abstraction" is dually a process of drawing from a situation and also the concept (the abstraction) output by that process. It has a multi-modal meaning as process, property, or concept.

Gray and Tall (2001) envisaged at least three distinct types of mathematical concept: one based on the perception of objects, a second based on processes that are symbolised and conceived dually as process or object (procept) and a third based on a list of properties that acts as a concept definition for the construction of axiomatic systems in advanced mathematical thinking. Each of these is an abstraction: a mental image of a perceived object (such as a triangle), a mental process becoming a concept (such as counting becoming number), and a formal system (such as a permutation group) based on its properties with concepts constructed by logical deduction.

Our purpose in this paper is to unite these various different ways of abstracting concepts in mathematics into a single construct by seeking an underlying mechanism in human thinking that gives rise to them all. We then review how this mechanism works successfully in some cases but not in others.

How do Humans do Mathematics?

We begin with a much more fundamental question: How does a biological creature like *Homo Sapiens* do mathematics? First, many individuals develop and build on each other's work to construct a body of mathematical knowledge that is recorded in books and other products of human culture and shared with the community. Every individual develops from being a child who knows no explicit mathematics into an adult who may learn to share the mathematical culture. Even mathematicians who created that culture also went through such a development — from being a baby dependent on mother's milk to becoming a sophisticated adult, such as Plato, Newton or Einstein. This has profound implications when we analyse mathematical thinking in general and abstraction in particular. By gaining insight into the way that mathematical thinking develops from child to adult, we also gain insight into the mathematical thinking of adults and into the nature of mathematics itself.

Homo Sapiens think using the biological structure of the human brain. In common with other species, actions can be repeated until they can be performed automatically without conscious thought. Such learning by rote often occurs in mathematics and can be used to respond to standard problems presented in a familiar context. But to be able to solve novel problems, connections need to be made to develop a more flexible form of knowledge.

The evidence shows that human brains, though exceedingly complex, are only able to concentrate consciously on a few things at once, requiring a mechanism to cope with the complication:

> The basic idea is that early processing is largely parallel — a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) "object(s)" can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once). (Crick, 1994, p. 61)

In addition to filtering out information, there must also be a mechanism to enable the essential information to be held in the brain in an economical manner.

Compression

The mechanism by which information is held in an economical manner relies on a phenomenon that we term *compression* (after Thurston 1990):

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics (Thurston, 1990, p. 847).

Compression involves taking complicated phenomena, focusing on essential aspects of interest to conceive of them as whole to make them available as an entity to think about.

Although other species have such mechanisms to function in their own context, *Homo sapiens* has a tool that enables it to grasp a complex situation, to reflect upon it at various different levels of sophistication, and to communicate with others: language. The essential feature of this tool is to name a phenomenon as a word or phrase, to allow the name to be spoken when referring to that phenomenon, and then to use language to discuss its various aspects and to focus on its properties and its relationships with other phenomena. We use the term *thinkable concept* to refer to some phenomenon that has been named so that we can talk and think about it. This can be any part of speech, and may refer to any phenomenon, such as number, food, warmth, rain, mountain, triangle, brother, fear, black, love, mathematics, category theory. The phrase "thinkable concept" is, of course, a tautology, for a named phenomenon is a concept and is therefore thinkable. However, given the many meanings of the term "concept", we choose to use the term "thinkable concept" here to emphasise its particular use in our theoretical framework.

Thinkable concepts are noticed before they are named. First various properties and connections are perceived in a given phenomenon, but it is only when these are verbalised and the phenomenon is named that we can begin to acquire power over it to talk about it and refine its meaning in a more serious analytic way.

As an example of the development of a thinkable concept, consider the notion of *procept* itself (Gray & Tall, 1994). As we sat looking at data from children solving arithmetic problems, we saw how some children seemed to use number symbols both for counting procedures and also as thinkable "things" to operate upon. Suddenly we realised that this phenomenon needed a name to talk about it and the term "procept" was born. At this point it was just a word linked to a complicated phenomenon. But by naming that phenomenon, we acquired the power to think about it and talk about it to each other and to our colleagues.

Whereas others had talked about a process becoming encapsulated, or reified, as an object (Dubinsky, 1991; Sfard, 1991), they did not have a name to talk about the elusive underlying concept that was both process and concept. Though our invention built heavily on their work, it moved it to a more sophisticated level. We can now talk about different kinds of procepts, including *operational* procepts such as 2 + 3 in arithmetic, which always have a procedure to produce a result; *potential* procepts in algebra such as $2 + 3x$ which represent both a general process of evaluation that cannot be carried out until x is known and also a concept of an algebraic expression that can be manipulated; and *potentially infinite* procepts, including the concept of limit (Tall et al., 2001). We can go on to discuss how different kinds of procepts involve different kinds of cognitive advantages and difficulties.

This typifies the way in which a complicated phenomenon (here, operating with symbols as process and concept) can be compressed into a thinkable concept (here, procept) to allow us to think about the phenomenon in a more sophisticated way. We suggest that this is the underlying mechanism of abstraction — to compress phenomena into thinkable concepts — that enables human thought in general and mathematical thinking in particular to operate at successively higher levels of sophistication.

Making Connections between Thinkable Concepts

Having considered the compression of knowledge into (thinkable) concepts, we now address how the brain connects them together. This is through a biological process called *long-term potentiation*, which is an electro-chemical modification of the links between neurons to strengthen those that are useful (Hebb, 1949). All neurons have multiple inputs from other neurons and a single output (the axon) that passes electrochemical messages down its length and branches out to connect to other neurons. A particular neuron receives charges from other neurons and when it reaches a threshold, it fires down its axon. This occurs typically several times a second. Links that fire more often are changed chemically and are more easily fired for a time. This leads to the recency effect, in which we continue to be conscious of more recent events and can sustain a continuous train of thought. If a link is repeated and put on a "high," it may then be strengthened to such a level that it fires automatically, making the link essentially permanent. This process is long-term potentiation that builds connections between thinkable concepts. It operates by a process akin to Darwinian selection in which successful links are enhanced and dominate others in the long-term (Edelman, 1992).

The necessary corollary of long-term potentiation is that the brain can only think using either built-in structures, such as vision, taste, smell and their respective connections, or mental constructions based on previous experience. The successive experiences that we have therefore deeply affect the ways in which we are able to think at later stages.

If a child compresses ideas into thinkable concepts, this will build the tools to work at a more sophisticated level. If not, more diffuse ideas may simply be too complicated for the individual to cope. Krutetskii (1976) studied the success of four groups (gifted, capable, average, incapable) in terms of their compression of solutions procedures and found that the gifted were likely to curtail solutions to solve them in a small number of powerful steps, whilst the capable and average were more likely to learn to curtail solutions only after considerable practice and the incapable were likely to fail altogether. Gray and Tall (1994) report a spectrum of different performances in arithmetic that they described as *the proceptual divide* between those who cling to the comfort of counting procedures that, at best, enable them to solve simple problems by counting and those who develop a more flexible form of arithmetic in which the symbols can be used dually as processes or as concepts to manipulate mentally. *Proceptual thinking* occurs when counting procedures are compressed into number concepts with rich connections — for example, knowing things like "4 and 2 makes 6, so

6 take away 4 must be 2" and using these "things" to derive new knowledge, such as "26 take away 4 must be 22 because 26 is just 20 and 6."

The compression of complex phenomena into thinkable concepts is a natural biological development. However, to trigger the compression requires a specific focus on relevant aspects of a situation to name and compress into a thinkable concept. Abstraction is the process of drawing from the situation the thinkable concept (the abstraction) under construction. This thinkable concept is then available for use at more sophisticated levels of thinking.

Abstraction occurs in various ways. We concentrate on three that are widespread in mathematics. One way arises from a focus on the *properties* of perceived objects, giving them names that are compressed by categorisation into different hierarchical levels (Rosch, 1978). A child may say, "that is a dog, that is a cat, they are both animals," and a whole tree of classification becomes possible. It includes individuals such as Rover or Tiddles through generic cat and dog, which are both mammals, as are also elephants and rats but not frogs, however all are animals, and so on. This kind of hierarchy occurs in geometry, studying categories of figures such as triangle, square, rectangle, circle and classifying, for example, different kinds of triangle (scalene, isosceles, equilateral, acute-angled, right-angled, obtuse), seeing that a square is a special kind of rectangle and both are quadrilaterals which, along with triangles, pentagons, hexagons, and so on, are all polygons. Each category is a thinkable concept. The study of the properties of these objects and the actions upon them (geometric constructions) builds eventually into a coherent theory of Euclidean geometry.

A second form of abstraction focuses on *actions* on objects, which leads through compression to the computable symbols in arithmetic, the manipulable symbols in algebra and symbolic calculus. Numerical symbols are thinkable concepts with properties such as even, odd, prime. They provide the basis for and extension to wider concepts such as fraction, decimal, rational, irrational, real, complex and algebraic concepts as expressions that we can factorise or simplify and equations we can solve.

Eventually the focus may turn to the properties of mental objects, compressed through several stages leading to a third form of abstraction by formulating a set-theoretic *concept definition* to construct a thinkable concept from the definition using mathematical proof.

All of these cases show abstraction in action: focusing on relevant aspects and naming or symbolising them to become thinkable concepts, be they mental images of objects (e.g., triangle), symbolism for a process compressed into a concept (eg 3 + 2 as a process of addition and the concept of sum) or structures defined by a list of set-theoretic axioms (such as the complete ordered field $\mathbb{R}$ or the infinite cardinal $\aleph_0$.

Gray and Tall (2001) hypothesised that these three forms of abstraction lead to the construction of three distinct forms of mathematical object which were later formulated as occurring in distinct *worlds of mathematics*:

> the *conceptual-embodied*, based on perception of and reflection on properties of objects;

the *proceptual-symbolic* that grows out of the embodied world through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (procepts);

the *axiomatic-formal* (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions. (Tall, 2004, quoted from Mejia & Tall, 2000).

Abstraction is a natural process in the sense that the brain focuses on some aspects of a situation and not on others to reduce the cognitive strain for decision making. However, progress to greater sophistication requires the focus of attention on the generalisable aspects that can be compressed into forms suitable for the brain to operate with them at a higher level, rather than on inflexible procedures or incidental detail. Such a phenomenon is related to the distinction between *abstract-apart* and *abstract-general* made by White and Mitchelmore (1996):

> Using mathematical symbols may be an abstract operation if the symbols have no concrete reference: they are 'apart'. The only context for the symbols is the symbols themselves ... An example of 'abstract-apart' is knowing how to manipulate algebraic symbols without having any sense of what the letters stand for. On the other hand, 'abstract-general' indicates that the mathematical objects involved are seen as generalisations of a variety of situations and so can be used appropriately in different looking situations. (White & Mitchelmore, 1996, pp. 574-575)

Earlier Theories

Most of the major theories of mathematical development in recent years have thematic frameworks that are consonant with a theory of embodiment, symbolism and formalism using various forms of abstraction to compress knowledge into forms that can be used at successive levels of sophistication. While all these theories have individual aspects that are distinct, there is a common thread of development in many of them that deals with the long-term development from new-born child to adult, whereby physical action and perception are interrelated through increasingly sophisticated mental abstractions.

Piaget's seminal work has two major aspects: his conceptions of abstraction and his stage theory. His conceptions of abstraction distinguish construction of meaning through *empirical abstraction* (focusing on objects and their properties) and *pseudo-empirical abstraction* (focusing on actions on objects and the properties of the actions). Later *reflective abstraction* occurs through mental actions on mental concepts in which the mental operations themselves become new objects of thought (Piaget, 1972, p. 70).

His distinction between empirical and pseudo-empirical relates directly to our focus on the properties of objects in the embodied world and the compression of action into concepts in the symbolic world. However, we see these distinctions continuing throughout cognitive development — one using imaginative thought experiments, the other using numerical computation and algebraic manipulation. Essentially, our distinction between embodied and symbolic ways of thinking relates to the contrast between abstraction from objects on the one hand and abstraction from actions on the other, beginning with the real world and developing through further abstractions within the mind. The long-term development of conceptual embodiment and proceptual symbolism therefore includes the shift to Piaget's stage of formal operations. Our axiomatic-formal world relates to the more sophisticated axiomatic approach of Hilbert, based on set-theoretic definitions with further properties deduced by mathematical proof.

The work of Skemp (1971) focused on the fundamental human activities of perception, action and reflection. Perception involves input from the senses, action involves output through interaction with perceived phenomena, and reflection is the process whereby we think about relationships between perception and action. Skemp (1979) talked about two distinct systems: *delta-one* which involves perception of and action on the actual world we live in, and a second internal system, *delta-two*, whereby our brains imagine internal perceptions and actions and reflect on them. Underlying this theory is a structure in which perceptions of and actions on objects are reflected upon, producing an increasingly sophisticated mental framework.

Fischbein (19878) focused on three distinct aspects of mathematical thinking: fundamental *intuitions* that he saw as being widely shared, *algorithms* that give us power in computation and symbolic manipulation, and the *formal* aspect of axioms, definitions and formal proof. There are significant differences in detail, but the underlying tripartite categorisation of embodiment, symbolic manipulation, and formal proof is clear.

Bruner (1966) focused on three modes of operation: enactive, iconic and symbolic, which inhabit a similar theoretical discourse. We see the conceptual embodied world as a combination of enactive and iconic and the symbolic and formal worlds corresponding to Bruner's subdivision of symbolism into "the two artificial languages of number and logic" (Bruner, 1966, pp. 18-19).

Biggs and Collis (1982) built on the stage theory of Piaget and the modes of Bruner to construct the theory of Structure of Observed Learning Outcomes (SOLO) to assess the progress of students through successive modes: sensori-motor, ikonic, concrete-symbolic, formal, and post-formal. The SOLO taxonomy took a step further by formulating a cycle of construction within each mode (unistructural, multi-structural, relational, extended abstract). Pegg (1992) revealed each mode could contain more than one cycle and Pegg and Tall (2002) refocused these cycles to apply to the construction of individual concepts in each mode, dealing first with a single aspect, then several separate aspects, related aspects, then the whole idea. They referred to this as the "fundamental cycle of concept construction" in general and noted, in particular, how process-object encapsulation could be seen to involve successively a single procedure, several procedures (with the same effect), a single process producing the required effect (perhaps by several procedures) and a procept.

Overall, this gives a general process of abstraction to compress knowledge into thinkable concepts as children build on their experiences over time. Development is formulated in three different styles of thought that are claimed to constitute different worlds of mathematics, beginning with categorisation of concepts in conceptual embodiment, branching out in parallel to the symbols of proceptual symbolism, and on to the axiomatic-formal construction of abstract structures through concept definition and formal proof.

Supporting Empirical Data from Around the World

The framework we have formulated is consistent with the research performed over recent years in different countries to further our studies of the development of the individual from the young child to the sophisticated adult. The focus of these studies includes the way in which more successful children focus on the subtle compression of knowledge of arithmetic while less successful children remain fixated on the more visible complication of the physical detail (Gray & Pitta, 1996). The development through the proceptual-symbolic world and the transition to the axiomatic formal world reveals a bifurcation between conceptual and procedural thinking arising from different levels of success in compressing procedures into thinkable concepts (Tall et al., 2001). There are distinct forms of proof available as the child develops cognitively from physical interaction and thought experiments in the conceptual-embodied world, to proof of properties of procepts using calculation and manipulation in the symbolic world, to the formal proof available to the mathematical expert (Tall, 1999). For example, the calculus has three distinct approaches in terms of conceptual-embodied manipulation of graphs using physical drawing and interactive computer software, symbolic-proceptual calculations and symbol manipulations, and formal proof in mathematical analysis (Tall, 2002).

More recent work with our own research students in Britain, Malaysia, Turkey and Brazil has revealed many cases where the desired abstraction of thinkable concepts often does not occur as required, with many students remaining at an inflexible procedural level of operation. This has long-term consequences for the successful teaching of mathematics at all levels around the world.

The Development of Early Arithmetic

Early arithmetic evolves from actions of manipulating and counting collections of objects. It involves repeating and refining the action-schema of counting until it becomes apparent that the effect of a counting procedure on a given set always gives the same number word, leading to the thinkable concept of number. The addition of two numbers begins by putting two sets together and counting the combination by various counting procedures, with a symbol such as 3 + 2 evoking either a process of addition or the concept of a sum.

In any context that involves an action on objects, the individual has the possibility of attending to different aspects of the situation — a theme that Cobb, Yackel, and Wood (1992) see as one of the great problems in learning mathematics. In the terms of our framework, the essential question is whether the child focuses on the actions of counting leading to a procedural interpretation or is able to contemplate the effect of those actions in terms of thinkable number concepts.

The greater majority of young children learn to count. For a young child, counting can be seen as part of a stage in concept development. However, an older child's extensive reliance on counting may be the result of necessity. Counting procedures that work with small numbers, such as calculating 8 + 3 by counting-on three after 8, are no longer practicable in dealing with larger numbers such as 855 + 379. Learned routines may be used without meaning and — when they lead to error — confusion and alienation may ensue. In a long-term study, Pitta (1998) found that students who succeed use compressed number concepts to perform arithmetic tasks, while children who fail often focus on other aspects. For instance, there is a tendency for those who are successful in arithmetic to see the picture  as representing the fraction one half (comparing the black and white parts of the square) while the less successful child may see it as the doors of a lift or even an open window at night with a white curtain. This study found that the more successful children shifted from embodiment working with physical materials to flexible use of number as compressed number symbols, while the less successful remained within an increasingly complicated world where their counting procedures were not powerful enough to solve more sophisticated problems.

Is it possible to help those who are struggling with counting procedures and unable to cope with more sophisticated problems? Gray and Pitta (1997) worked with an eight-year-old child who had difficulty counting on her fingers and used mental images of counters in specific arrays to perform arithmetic calculations with small numbers. They provided her with a graphical calculator and tasks to perform such as "find a sum whose answer is 9". Trying 5 + 3 gave 8, but adding another 1 gave 5 + 3 + 1 is 9. The essential facility of the graphic calculator is that it shows both the arithmetic expression and the result without the need for counting, allowing the focus of attention to be shifted from long procedures of counting which are no longer required to the visible relationships in arithmetic displayed on the screen. She found combinations such as 4 + 5, 3 + 6, 4 + 4 + 1, 3 + 4 + 2 and as she did so, she began to see number patterns. Slowly her activities no longer depended totally on counting. As she worked through the program set for her, she became more adventurous, building sums such as 2 + 9 + 1 – 6 = 6, 90 – 80 – 4 = 6, 30 – 15 – 9 = 6, 5 + 20 – 19 = 6, 40 – 30 – 5 = 5, and 10 + 30 – 30 – 2 = 8. Over ten weeks of exploration she became comfortable with larger numbers and using number patterns. At the end of the course she was asked what "4" meant to her, and she replied "a hundred take away ninety-six." Not only had she become familiar with number relations in a way highly unusual for a slow learner, she had developed a quality that is usually only shown by much more able children: a sense of humour with numbers.

This shows that the use of a graphic calculator representing both the arithmetic expression to be computed and the result of the computation can enable a child to focus on relationships rather than on the time-consuming procedures of counting, leading to a powerful abstraction of number symbols as thinkable concepts.

However, such remediation has not proved successful in all cases. Howat (2005) found that there were failing students in arithmetic with a median age of 8.5 who were unable to cope with place value because they had not constructed the concept of "ten" as a thinkable concept that could be both ten ones and one ten. Without this concept, they were overwhelmed by the arithmetic of two digit numbers — since their response to any problem was to attempt counting on in ones. Even when working closely with them, Howat found that some had cognitive difficulties that started far back in their development that were deeply ingrained and seemed no longer subject to her remedial action.

The Ambiguous Number Line

The shift to number as measurement is embodied using a number line. In the English National Curriculum, this is intended to give learners an overall picture of numbers in order on a line as part of a long-term development of successive abstractions: from a line drawn with pencil and ruler, to a mental image of an arbitrarily long line with no thickness that is subdivided to imagine fractions, finite decimals, then infinite decimals and eventually to the formal thinkable concept of the real numbers as a complete ordered field. It is a journey that is made by those who become mathematicians, but the initial path proves stony for many young children in school.

In English primary schools, the number line is introduced as a key classroom resource within the Primary National Strategy (PNS) in the *Framework for Teaching Mathematics* (DfEE, 1999). It begins with a "number track" consisting of blocks placed one after another in order, then moves on to a number-line in several different guises including a "washing line" of numbers, table-top number lines, some marked with specific numbers, others left open to place the numbers in an appropriate place. The overall aim is to use these representations to promote the understanding of the number sequence and the order of the whole numbers marked on the line, introducing addition and subtraction in terms of operations on lengths, then later expanding the children's knowledge to include fractions, decimals and negative numbers.

Within the documentation there is no reference to the conceptual differences between a discrete number track and a continuous number line, or to the subtle shifts in meaning involved in the introduction of broader number concepts.

This ambiguity is reflected in schools by the way in which teachers interchangeably use the terms number line and number track as if they are the same idea, when they are not. The number track consists of discrete numbers 1, 2, 3, ..., with each number followed by a next number and no numbers in between. The number line is a continuous line on which we may mark numbers as points, with fractions between whole numbers and the possibility to extend the line in either direction to include positive and negative numbers, rationals, decimals, and irrational numbers.

When Doritou (2006) interviewed a range of primary school children, most of them simply described the number line in terms of some perceptual features of a particular line or explained a particular line in the context of an action. Overall, the quality of the children's responses did not change significantly between children in Year 3 (median age 7.5 years) and those in Year 6 (median age 10.5). There was an over-riding preference to label calibrated lines with whole numbers and a limited acknowledgement that an interval could be subdivided, linking back to their experiences with the number track rather than the intended number line.

When the teachers came to use what they perceived as a number line to demonstrate the subdivision of intervals for fractions and decimals, most children carried an embodiment pre-loaded with prior active, linguistic and relational experience with whole number. There was no sense of the conceptual structure that underpins the number line as an ideal representation connecting whole number and fraction (Baturo & Cooper, 1999).

Procedural Conceptions of Fraction

After the initial challenges of handling whole numbers, the shift to handling fractions is a problematic one for many children. Instead of a single name for a single number, like "3," a fraction has many names: "two-thirds," "four-sixths", "sixteen twenty-fourths" and so on. A fraction begins as an embodied activity of breaking an object or collection into equal parts and assembling the required number of parts. Whereas two-thirds, four-sixths and sixteen-twenty-fourths involve quite different procedures and produce parts of different sizes, the quantity in each case is the same. The major act of abstraction shifts attention from the sharing procedure as a sequence of steps to the effect of that procedure, namely the quantity produced in the result. Focusing on the effect, one-third is the same as three-sixths. If fractions are seen as procedures, then addition is almost too complicated to contemplate. But if fractions with the same effect are seen as the same, then adding one-third and one-half is the same as adding two-sixths and three-sixths — which is a simple case of the more general addition of two things and three things to get five things. The child with proceptual flexibility of number is more likely to see the essential simplicity of adding fractions as things, where the things are the same size (in this case, sixths), than the child with a procedural view who may attempt to learn complicated rules such as "put the fractions over a least common denominator" that may have little meaning.

As part of the *Malaysian Vision for 2020* to develop the country's economy to the highest standards by that date, the Malaysian curriculum is designed to teach fractions in a caring and helpful way to include seeing that multiplication can be done in different ways to give the same result. For instance, "two-fifths of twenty-five" can be performed either by working out a fifth of twenty-five and then multiplying the result by two, or by multiplying two by twenty-five and then dividing by five.

To ensure that all children can accomplish these tasks, the teacher encourages the pupils to remember the procedures, reciting successive parts of the procedure and inviting the children to fill in missing words. For instance, the teacher might say (in Bahasa Malay), "How do we work out two-fifths of twenty five?" and draw three circles on the board one above the other for numerator and denominator of the fraction, the other for the whole number. "What do we put in the top circle? The nu..." to which the class gleefully says "the numerator." "What do we put in the bottom circle? The de...", "denominator". "Of means mul...", "multiply". And so the lesson continues, building up the ritual of the procedure of multiplication by a fraction.

In a study observing lessons and interviewing students, Md Ali (2006) found that children's achievement in fractions was improving. But although the teachers subscribed to the aspirations of Vision 2020 to help children "really understand mathematics," the general consensus was that they felt constrained by the teaching schedule and the need for success in the National UPSR Examination. Interviews with the students revealed that success in examinations was achieved procedurally with some degree of flexibility in choosing which procedure to use — but the compression of knowledge into flexible thinkable concepts to solve unfamiliar problems proved elusive.

Magic Embodiments in Algebra

The shift from arithmetic to algebra involves an abstraction from the computable operations of arithmetic to the use of expressions representing generalised arithmetical operations. Such a transition proves relatively easy for some who have a flexible proceptual approach to arithmetic, but it is far more difficult for those who continue to think of expressions purely in procedural terms. Rosana Nogueira de Lima (de Lima & Tall, 2006) worked with a group of committed teachers in Brazil to understand what students were actually learning in their regular algebra classes. Here the solution of linear equations was being taught by the principle of "doing the same thing to both sides" to maintain the equality and manipulate the equation to give the solution. In practice, the students focused not on the general principle but on the actions they performed. Subtracting 2 from both sides of the equation $3x + 2 = 8$ and simplifying to $3x = 8 - 2$ soon became "change sides, change signs" while the final simplification of $3x = 6$ by dividing both sides by three to get $x = \frac{6}{3}$ became "move the 3 over the other side and put it underneath." Interviews with the students revealed that they interpreted these moves in an embodied way, picking up the terms and putting them somewhere else with additional actions such as "change signs" or "put it underneath." This procedural embodiment, carried out by mentally moving the terms with an additional meaningless piece of magic to get the right answer, was used successfully by some but proved fragile for others who made errors mixing up the rules such as shifting the 3 in $3x = 6$ with the additional magic of changing signs to get $x = \frac{6}{-3}$. Such errors may increase the confusion of students who may then try alternatives to get the correct answer, producing what appear to be random errors.

In solving quadratics, the problems became worse: The teachers, knowing the difficulties with linear equations, focused only on teaching the formula because it gives a solution for any quadratic. However, this single procedure first requires the equation be manipulated into the form $ax^2 + bx + c = 0$ and so proves to be more complicated in cases where an alternative approach may be more insightful. For instance, a problem arose when the students were asked to show that the equation $(x - 2)(x - 3) = 0$ has roots 2, 3. Most students failed to respond at all and those that did attempted to multiply out the brackets to solve the equation using the formula. Few succeeded.

Here the focus on limited procedures and lack of compression into thinkable concepts has a cumulative effect. Some students see the essential simplicity of an algebraic expression as a potential calculation that can be manipulated in its own right and develop an effortless mastery of algebra. Others focus only on procedures and become involved in more and more complicated activities that increase the cognitive strain and become unmanageable.

Complications in the Function Concept

As we move through into the secondary curriculum, we come to concepts like the notion of function which the NCTM standards see as being an essential underpinning of a wide range of mathematics. In some countries, such as Turkey, the function concept is taught from its set-theoretic definition and seen as a foundational idea. It is quite simple: There are two sets A and B and for each element x in A, there is precisely one corresponding element y in B which is called $f(x)$ (read as "f of x"). That's all there is to it!

However, this definition is used in the curriculum to weave a huge web of knowledge: linear functions, quadratic functions, trigonometric functions, exponentials and logarithms, formulae, graphs, set diagrams, and so on. How does one help the student make sense of this complicated array of ideas? Two routes are possible. One is to focus on the simplicity of the definition and continually link back to it to make powerful connections. Another is to look at the difficulties that students are seen to experience and teach the students how to cope with them.

Bayazit and Gray (2006) report a study of the teaching of two teachers with very different approaches. Ahmet saw his duty to mentor the students and help them make sense of the function concept. At every opportunity, he emphasised the simple property that a function $f: A \to B$ mapped each element of the domain A to a specific element in the domain B. For example, in considering when a graph could be a function, he looked at the definition and related it to the fact that each x corresponded to only one y, and linked this to the vertical line test. When he considered the constant function, he considered the definition and revealed the constant function $f(x) = c$ as the simplest of functions which maps every value of x in A onto the element c in B. Likewise, when he studied inverse functions, such as the square root, the inverse trigonometric functions and the relationship between logarithm and exponential, he patiently referred everything to the definition. With piece-wise functions, which were new to the students, he used the definition to confirm that these too satisfied the simple requirement that for every x there was a unique y.

The other teacher, Burak, was well aware of his students' potential difficulties and misconceptions. He considered that students rejected the constant function because "it did not vary with x." He interpreted the students' difficulties with the inverse function as an indicator of their inability to move back and forth between the elements of domain and co-domain. He knew that students had problems with the discontinuities of the graphs of piecewise-defined functions, predicting that they would draw lines to fill in any gaps.

However, he made no effort to eliminate these obstacles during his teaching. Instead he gave the students the details he considered that they needed to answer the examination questions. He taught the vertical line test as a specific test for functions, practising examples to get it right. He introduced the inverse function with a simple case, finding the inverse of $y = 2x + 3$, by seeking to express x in terms of y, subtracting 3 from both sides and dividing by 2 to get $x = \frac{y-3}{2}$ then interchanging x and y to get $y = \frac{x-3}{2}$. He dealt with the problem of the constant function by affirming that a function does not need to involve x and that its graph is a horizontal line parallel to the x-axis. He dealt with piece-wise functions by showing students how to cope with particular examples.

He would often indicate that an examination or test required particular tactics:

> If you want to succeed in those exams, you have to learn how to cope. Do not forget simplification. It is crucial, especially [in] a multiple-choice test.

Even though he was aware of student difficulties he did not attempt to address them meaningfully as teachers had done in other studies (Escudero & Sanchez, 2002; Tirosh, Even, & Robinson, 1998). His students scored significantly lower than those of Ahmet.

Here we see that a gifted teacher focusing on the essential ideas in the function definition can encourage the abstraction of the function concept as a powerful thinkable concept, while the emphasis on a host of specific instructions to deal with known difficulties may be counter-productive.

Drawing Together the Threads

What does the available evidence tell us? The overwhelming message is that powerful learning arises from compression of sophisticated knowledge into thinkable concepts that enable the learner to make links that lead to powerful ways of thinking in more complex situations. On the other hand, limited focus on the steps needed to perform standard procedures such as column subtraction, long division or factorisation of quadratics, without the focus on the underlying conceptual ideas, can lead to actions that work in time without necessarily creating thinkable concepts that can be used to reflect on the concepts and relationships involved. For instance, the procedure of factorisation turns $x^2 + x - 6$ into $(x - 2)(x + 3)$, which can be seen as converting one thing into something different. These two expressions are different as procedures of evaluation, but they are the same in effect. Being able to see 8 + 6 as 8 + 2 + 4 and then as 14, or to see $\frac{1}{2}+\frac{1}{3}$ as $\frac{3}{6}+\frac{2}{6}$ and then to hear this as "three-sixths plus two-sixths" to get $\frac{5}{6}$, transforms arithmetic procedures into flexible thinkable concepts. In the same way, seeing $x^2 + x - 6$ and $(x - 2)(x + 3)$ as different ways of expressing the same thing is a significant simplification that turns algebraic expressions into thinkable concepts that can be handled fluently.

If at one stage a learner fails to focus on relevant aspects to produce subtle thinkable concepts and instead learns the steps of the procedure to carry out a specific task, then the human brain lacks the thinkable concepts to build on the sophistication required at the next stage and is more likely to resort to the primitive strategy of learning by rote. The effects are cumulative. As all of us go through the long-term development of learning mathematics, if compression of knowledge required for the next stage does not occur, then procedural learning becomes more likely, not only in the children we are teaching but in those who are adults and have already been through their mathematics education. Thus, despite the widespread call for more meaningful conceptual learning, the perception that the way to learn mathematics is to learn procedures proliferates and is held, not only by children, but also by many teachers, administrators and politicians.

To improve long-term conceptual learning, the framework formulated here suggests that the whole curriculum must be framed with an awareness of the abstraction process to produce thinkable concepts at every stage. This requires the teacher to become a mentor who encourages children to focus on the appropriate essential ideas in a way that enables them to compress the phenomena into thinkable concepts. This in turn requires mathematics educators to aid the development of such a vision by working to formulate how this transformation can be attempted in ways that make sense both to the teacher and the many different learners.

This journey will not be easy. In our international studies, we have found that a sincere desire to improve children's capacity to perform mathematical algorithms more accurately and efficiently may lead to such improvement without improving the ability to solve more sophisticated problems.

In the UK, the National Numeracy Strategy (DfEE, 1999) was initiated as a response to low attainment in mathematics in many schools. It informed teachers, through its annual objectives that built new ideas on those previously taught, what should be taught in mathematics during each school year. It also explained how the mathematics specified should be taught by presenting a recommended three-part lesson format: mental/oral phase, main phase and plenary phase — and advocated that the first and last phase, together with part of the main phase, should be taught with the whole class. While this strategy was not enforced by law, the majority of schools responded to the initiative, partly due to pressure from government agencies such as OfSTED (Denvir & Askew, 2001).

In an evaluation of mathematics provision for 14 to 19 year-olds, OfSTED (2006) reported that the majority of teachers were preparing students for examinations by "teaching to the test," which might ensure that students passed examinations but would not ensure mathematical flexibility. The government report intimated that the problems arose from the inadequacies of teaching.

Our theoretical framework suggests differently. We believe that the natural process of abstraction through compression of knowledge into more sophisticated thinkable concepts is the key to developing increasingly powerful thinking. It occurs naturally with the most able and others can be helped by using techniques that encourage a focus on the essential elements for compression into thinkable concepts. But there is no evidence that it can work for all children. Until we grasp the nature of the required sophistication to compress complicated phenomena into thinkable concepts and are able to express it in a way that makes sense to teachers, students and, if possible, to politicians, mathematics will remain for many a world of overbearing difficulty relieved only partially by limited rote-learning.

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