Students’ conceptions of models of fractions and equivalence

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A solid understanding of equivalent fractions is considered a stepping-stone towards a better understanding of operations with fractions. In this article, 55 rural Australian students’ conceptions of equivalent fractions are presented. Data collected included students’ responses to a short written test and follow-up interviews with three students from each year. This exploratory study found most participating Years 4, 6 and 8 students were familiar with geometric area models, particularly circles, and able to explain equivalent fractions when presented geometrically as area models but had difficulties when equivalents were presented numerically as $\frac{a}{b}$.

Introduction

Many studies found middle primary and junior secondary students have difficulties understanding, and working with, fractions. According to Niemi (1996, p. 6), fractions, because of their importance, are conventionally introduced to children in kindergarten and continue to occupy a prominent place in school curricula from the second year of primary. The concept of fraction is very important in understanding equivalent fractions. Although learning equivalent fractions is repeated in subsequent years, Kamii (1994, p. 2) found the performances of middle primary years and junior secondary students are still disappointing. This paper reports how some Years 4, 6 and 8 students view some fraction models and simple equivalent fractions.

The study

Three classes, from an Independent school in regional Australia who agreed to participate in the study, were selected to represent the learning continuum from the formal introduction of simple fractions in Stage 2 (Years 3–4) through to operations with more complex fractions in Stage 4 (Years 7–8) (NSWBOS, 2002). Fifty-five students (21 Year 4, 12 Year 6, and

22 Year 8), who had signed consent forms, undertook the same paper-and-pencil test. Three students, each recommended by their teachers and representing different ability levels — high, medium and low — were selected for individual interviews following the test. The test (Kerslake, 1986), comprised six questions on fraction models and equivalent fractions. All students answered the same test and semi-structured interview questions. The latter (Kerslake 1986) built on students' test responses with four additional questions presented as placards to further explore student understanding. Tests were administered during their 20-minute mathematics periods. Individual interviews, conducted in their teachers' presence, were audio taped.

This paper presents students' responses to questions that focused on a general understanding of fractions and simple equivalent fractions. Data collected also provided information on students' conceptions of other, more complex, equivalent fractions and addition of simple fractions. This is not discussed further in this article.

Test results

Question 1: Choose and tick the correct ways of saying fraction $\frac{2}{5}$ from the following sentences.

(a) Two fifths, (b) Two over five, (c) Two by five, (d) Two upon five,
(e) None of the above.

Students were most familiar (Table 1) with "two fifths" though three Year 4 students omitted it. Description "two over five" was quite popular in Years 6 and 8 but less popular in Year 4. Some Year 6 and a few Year 8 students chose "two by five."

Table 1. Question 1: student responses.

Items	Year 8(n=22)		Year 6(n=12)		Year 4(n=21)	
	Chosen	Not Chosen	Chosen	Not Chosen	Chosen	Not Chosen
Two fifths	100%	0	100%	0	85.7%	14.3%
Two over five	63.6%	36.4%	66.7%	33.3%	14.3%	85.7%
Two by five	0	100%	16.7%	83.3%	0	100%
Two upon five	4.5%	95.5%	25%	75%	0	100%
None of the above	0	100%	0	100%	0	100%

Everyone who chose "two fifths" explained it was the way they were taught:

"[A]s if I was looking at a pizza cut into five and two fifths have pepperoni. That is the way I have always been told to say it." (Year 8)

"[B]ecause I was taught that way and also I read the top number first and the bottom number next." (Year 4)

Those who accepted "two over five" explained:

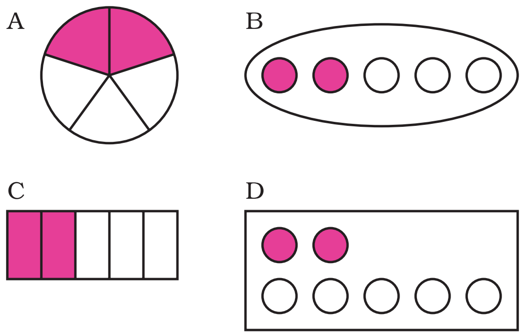
"Two is on a line above 5 so you say 2 over 5 or two fifths." (Year 8)

"There is a two over a five" (Year 6)

A Year 4 student said, "The teacher taught us that two over five is two fifths."

Overall, "two fifths" was popular with the majority of Years 6 and 8 students also choosing "two over five." Students seemed to reproduce the sanctioned, classroom language confirming the (a) pedagogical influence on students' developing fraction language, and (b) different capacities of students to explain their understanding depending on their year level.

Question 2: Which of these pictures would help you know what the fraction $\frac{2}{5}$ is?



Models A and C, in that order (Table 2), were accepted by most students in each year. Least selected by Year 4 is model D, while Years 6 and 8 students were certain it was incorrect.

Increased percentages, especially from Years 4 to 6 for C and B, and highly consolidated A[1] reflect the pedagogical mediation of student learning along a developmental continuum (from exploration and introduction to consolidation) of types of:

1. geometric area models (with increased exposure to, and consolidation of, rectangle-models to match established circle-model conceptions); and
2. fraction models (increased exposure to discrete models (i.e., part of a set or collection) to match established area-model conceptions).

However these trends gradually decreased (B and C) from Years 6 to 8 though percentages for A and C were still relatively high with just over half of Year 8 students selecting B. Students' conceptions, and by implication the teaching of fraction models, peak at Year 6 with a tapering off post-Year 6.

1. Highly consolidated meaning the increased percentage from 83.30% (Year 6) to 90.90% (Year 8) implies more consolidation of the circle-model whilst C (83.30% to 72.7%) and B (66.7% to 54.50) both slightly decrease. Latter are decreases instead of stabilising or improving.

Table 2. Question 2: Student Responses.

Items	Year 8(n=22)		Year 6(n=12)		Year 4(n=21)	
	Chosen	Not Chosen	Chosen	Not Chosen	Chosen	Not Chosen
Model A	90.9%	9.1%	83.3%	16.7%	90.5%	9.5%
Model B	54.5%	45.5%	66.7%	33.3%	47.6%	52.4%
Model C	72.7%	27.3%	83.3%	16.7%	57.1%	42.9%
Model D	0	100%	0	100%	14.3%	85.7%

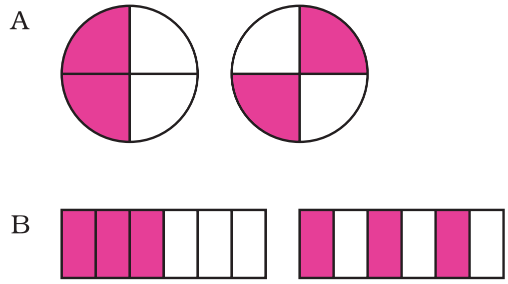
Example justifications (Figure 1) show that the first Year 8 student confidently reasoned that the unfamiliar B was also correct. This is in contrast to the Year 6 student's certainty that seemed to reflect a familiarity with the three different representations. Interestingly, the second Year 8 student (Figure 1) excluded B while the second Year 4 student chose only A. Students' reasoning, as inferred from these explanations, reflect the implied pedagogical trends discussed earlier.

Two Year 4 students who incorrectly chose D, said, "D helps me to know two over five because there are 2 balls over five," and, "'I think all of them help me, especially D because it has got 5 bottom and 2 above." Conceptually, these students interpreted $\frac{2}{5}$ as representing two unrelated whole numbers, not as a part–whole relationship, and appeared not to have developed any deeper understanding of pictorial and numerical representations beyond their visual spatial features.

Students	Responses
Year 8	"I chose A, B and C because I believe they represent $\frac{2}{5}$. A and C are the ones I have been shown before but B also makes sense." "A and C, because they are more clear, that two are only coloured out of five."
Year 6	"The three I ticked showed five equal portions, of which two portions are highlighted in someway."
Year 4	"There is five in each picture that I have chosen (A, B and C) and two of them are shaded so it is two out of five." "I chose this answer (A) because it is like a cake that has divided into 5 and someone ate two and it is also easy to know that it is $\frac{2}{5}$ to me."'

Figure 1. Question 1: Student justifications.

Question 3. What fraction is shaded in each diagram given in A and B? Are they equal? Explain.



Most students wrote correct fractions for all diagrams as representing $\frac{1}{2}$, by considering the partitioning of the regions into equal parts. For example, two Year 8 students explained: "because, in both, 2 parts out of 4 are shaded, indicating $\frac{2}{4}$. When simplified, $\frac{2}{4}$ can be expressed as $\frac{1}{2}$ " and "because there are 2 shaded shapes in each circle". In comparison, a Year 6 pointed out they are equal "because both have half of them shaded" while a Year 4 reasoned "because there are 2 parts shaded in each picture and they both equal a half." Overall, students found establishing equality of halves easy.

Question 4. a. Draw a model to represent $\frac{2}{3}$ and $\frac{4}{6}$.
b. What could you tell by comparing your models?

Almost all students (Figure 2) correctly represented each fraction with area models such as circles (most common) and rectangles, suggesting that representing thirds and sixths diagrammatically was easy. Describing equivalence between pictorial models was progressively easier, as expected, from Years 4 to Year 8 (e.g., 76.2% Year 4, 33.3% Year 6 and 27.3% Year 8 students could not describe equivalence).

According to the *K–10 NSW Mathematics Syllabus*, at Stage 3 (Years 5–6), children learn modelling, comparing and representing the new fractions (thirds, sixths and twelfths) and finding equivalence using pictorial representations (NSWBOS, 2002). Percentages suggest the development of students' conceptions (fraction models and equivalence), seem to occur more steeply between Years 4–6 than between Years 6–8. Some responses (Figure 2) indicated (a) the dual use of "amount" to describe both area and number of parts to justify equivalent fractions and equal areas (circle, Year 8 students), and (b) Year 4 students' (circle, first student; rectangle student)

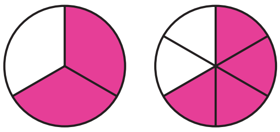
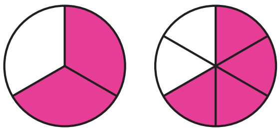
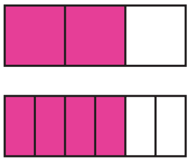
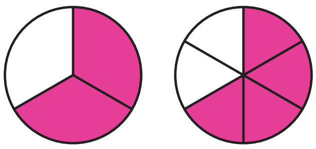
Diagrams drawn	Reasons
	Year 8: The amount shaded is the same. Year 8: That $\frac{2}{3}$ was half of the amount of $\frac{4}{6}$. Year 8: They both hold the same amount. $\frac{4}{6}$ simplified is $\frac{2}{3}$.
	Year 4: The whole of the first model [is] has doubled to make the second. The shaded parts of the first model has doubled to make the second. Year 4: They are both the same because if you put together they both equal the same fraction.
	Year 8: The $\frac{4}{6}$ is double the $\frac{2}{3}$. Year 6: There is the same quantity shaded in each diagram. Year 4: $\frac{2}{3}$ is shorter and $\frac{4}{6}$ is longer and has more pieces.

Figure 2. Question 4 student justifications.

contextually bound knowledge and difficulty abstracting meaningful relationships. The decline in percentages (Years 4–8) is expected, given curricular expectations for Stage 2 (Years 3–4), namely, thirds and sixths are yet to be introduced.

Question 5. Are the following shaded circles equal? Compare and explain your answer?



Every Year 6 and Year 8 student (Table 3) could represent the shaded parts as fractions but a few incorrectly explained equivalence. Some justified equivalence by simplifying fractions (procedural) and some by matching areas (visual spatial). Although Year 4 students could represent fractions, 71.4% gave incorrect explanations. Example explanations (Table 4) and trends (Table 3) reflect students' levels of understanding, along a developmental continuum of learning fractions, across the primary–early-secondary years. Pedagogically, teachers develop and consolidate student understanding of equivalence and extend fractions to include halves, quarters and eighths in Stage 2 (Years 3 and 4) through modelling and pictorial representations, with expansion to thirds, sixths and twelfths in Stage 3 (Years 5 and 6), whilst increasingly more sophisticated justifications are expected from Stage 4 (Years 7 and 8) students (NSWBOS, 2002).

Table 3. Question 5: Student Responses.

Criteria	Year 8	Year 6	Year 4
Represent and correctly explain	95.5%	83.3%	28.6%
Represent and incorrectly explain	4.5%	16.7%	71.4%

Table 4. Question 5: Correct and Incorrect Explanations.

Students	Correct explanations	Incorrect explanations
Year 8	"They are equal. This is because the amount shaded in the same, it's just broken up into different fractions of each circle. Also, $\frac{4}{6}$, when simplified, is $\frac{2}{3}$, the same as the first circle." "They are equal because the area shaded is the same."	"No they are not equal they are added (doubled) on."
Year 6	"They occupy the same area. Also you can see two sixths equal one third." "Yes, because $\frac{4}{6}$ can be $\frac{2}{3}$ by going $4 \div 2 = 2$ and $6 \div 2 = 3$."	"No one of them has 2 shaded and the other has four shaded."
Year 4	"It is equal because it has same amount but it has been cut it into different pieces. And another way of knowing to compare these circles is that 1 piece of the $\frac{2}{3}$ is 2 pieces of the $\frac{4}{6}$." "In each circle the shaded part is the same area." "The shaded parts are equal but one has smaller parts."	"They aren't equal because A has $\frac{2}{3}$ shaded and B has $\frac{4}{6}$ shaded and they're different fractions." "No, because they have different numbers of pieces shaded and they have different pieces in total." "They are not equal because A has two out of three pieces shaded and B has four out of six pieces shaded."

Overall, some students have correct visual spatial representations of equal areas, and therefore of equivalent fractions (column 2, Table 4) In contrast, incorrect explanations indicate that some students view fractions additively as two unrelated whole numbers, where more pieces means the two areas, and therefore fractions are different (column 3, Table 4). The correct explanations (column 2, Table 4), in contrast, indicate some (Years 6 and 8) students could justify equivalence, not only geometrically, but also procedurally through simplification and division. The number and sophistication of interpretations progressively increase from Years 4 to 8.

Interview results

Question 1. How would you explain to your friends what a fraction is?

Responses varied with most referring to "parts of a whole," while one mentioned "part of a number" and another "one number over another" (Table 5, each letter (A, D, E, F, G, H and J) represents a student).

Year 4 students related "whole" to examples such as pizza, circles and orange as evident below:

"OK, well, a fraction is basically two thirds or say you had a piece of like a circle things and well we have a whole and we can cut it into half to make it two and that would be two, uh... You cut it into four that would be four, uh..." (E)

"Uh... a fraction is somewhat like you have an orange: you cut them into half and you could cut into certain amount of pieces and however many pieces out of one that would be say you cut them into six pieces, that would be um..." (F)

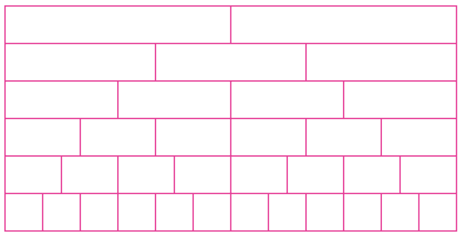
Table 5. Interview Question 1 Responses.

Description	Year 8	Year 6	Year 4
Part of a whole	J	A	D, E & F
Part of a number	H	–	–
One number over another	J	–	–
Does not know or could not say	G	–	–

For Year 8, J described a fraction as “one number on another… a part of whole,” while H said, “a part of a number, a part of a… like a section part.”

Students (J, A, D, E and F) predominantly described fractions as “part of a whole,” experientially by Year 4 (E and F) while Year 8 students used other descriptors such as “part of a number” (H), suggesting a connection between fractions and numbers, and “one number over another” (J), which acknowledges the numerical notation $\frac{a}{b}$. One Year 8 student could not define fractions.

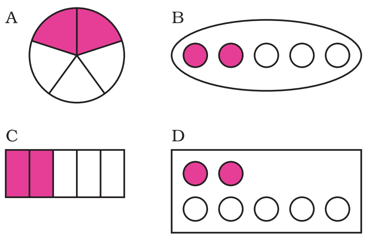
Question 2. Have you come across this picture somewhere? What does it tell you?



Years 4 and 6 student remembered seeing the picture either in textbooks or on classroom walls. Most students did not discuss equivalent fractions unless prompted. Instead, their observations were limited to stating halves, thirds and sixths. One Year 6 student (A) explained equivalent fractions when looking at the chart: “Oh it shows equivalent fractions, so like one half, like one out of two is equal to two out of four and eight out of… Oh, four out of eight and six out twelve. So that is exactly the same.”

No Year 8 student remembered seeing the chart, suggesting they had not used it recently. For example, “Um… the numbers… they are um… I don’t know really…” (H). The wall chart shows linear models of fractions as lengths (i.e., models are line segments), or arguably, a stack of rectangle area models (models are rectangles).

Test Question 2 revisited: Which of these pictures would help you know what the fraction $\frac{2}{5}$ is?



Responses indicated misconceptions particularly with D. They all stated it was two out of five because of the two shaded circles on top of the five unshaded circles. For example:

“Um… I found it harder to use because two shaded ones are at the top not in like five…” (J, Year 8)

“That one’s more than five because it’s two on top of five and that makes more sense.” (F, Year 4)

"Because, uh… on this one it is five there and on this one it's got two and then five down the bottom. It's easier for me to say that there is five there with this two shaded, so it is two shaded out of five here." (E, Year 4)

E's explanation, although a misconception, suggests another viable description for $\frac{2}{5}$, namely, "two out of five"; one that descriptively emphasises the part–whole relationship. Some students may have been distracted by the oval and box around the discrete models (B and D). However, this appeared unlikely in this survey as no student raised it as a concern, either through the test or in the interviews.

Discussion

Partial results from this study showed geometric area models, representing "part of a whole" were the most familiar to the participating students, particularly circles and, to a lesser extent, rectangles. The discrete model was less common while most (of those interviewed) had not recently used or seen a fraction wall chart. Kerslake (1986) and Cramer and Henry (2002) also found children predominantly selected circle models over discrete and linear models in representing fractions.

Responses showed some students perceive the numerator and denominator as two separate, unrelated whole numbers, which subsequently led to misconceptions when comparing area and numerical representations of equivalent fractions.

Results reported here showed students found it easy to accept equivalent fractions (halves, thirds and sixths) when presented geometrically (test questions 3 and 5), but misconceptions emerged when comparing thirds and sixths numerically ($\frac{2}{3}$ and $\frac{4}{6}$, test question 4a) and explaining equivalence (test questions 4b and 5). For example, students reasoned circles divided into thirds and sixths are the "same" but some felt that, numerically, "$\frac{4}{6}$ is double $\frac{2}{3}$", suggesting the student probably meant "double" the number of parts but leaves unstated the size of the part (i.e., does the student mean: $\frac{4}{6} = 2(\frac{2}{3})$, $\frac{4}{6} = \frac{2}{2}(\frac{2}{3})$, or $2(\frac{1}{6}) = \frac{1}{3}$?) and, "they are not equal they are added (doubled) on," indicating misconceptions of fractions as representing double counts of unrelated numbers and also their transitional understanding between developmental stages of the continuum of learning fractions. These misconceptions suggest that, pedagogically and conceptually, there is a need to develop and consolidate students' understanding of (i) fractions, using multiple models, as representing a relationship between the numerator and the denominator and in addition, when comparing fractions, and (ii) the relationship between number of parts, and size of parts.

These findings support those by Kerslake (1986), namely, students recognised instances of equivalent fractions when presented in geometric form, however, there was some conflict between the awareness that, for example, $\frac{2}{3}$ and $\frac{10}{15}$ were the "same," and the feeling that $\frac{10}{15}$ was bigger than $\frac{2}{3}$ because $\frac{10}{15}$ was 5 times bigger than $\frac{2}{3}$. Larson (1980) also indicated that, for seventh graders $\frac{2}{6}$ was not seen as having the same meaning as $\frac{1}{3}$.

The results also reflected the timing of the implemented curriculum on fractions (NSWBOS, 2002). For example, an increasing proportion of students from Years 4 to 8 correctly explained equivalence between circle-models of $\frac{2}{3}$ and $\frac{4}{6}$ with students showing the least percentage in Year 4, the level where the equivalent fraction concept is formally introduced, and the highest at Year 8. Also the lesser (B) and least common (fraction wall) models indicate classroom pedagogy across the levels. It is recognised that students' ability and capacity to define fractions and explain equivalence are conceptually different depending on where they are along the continuum of learning fractions practised in classrooms and promoted by the syllabus.

Implications

Results revealed there is a tendency by some students to perceive the double count of shaded parts and total parts as two unrelated quantities. Attention needs to be paid to developing and consolidating the notion of a fraction as representing a relationship between the two counts.

Results also revealed students limited the idea of fractions to the "part–whole" model. They linked fractions to pictures of shaded parts of a model such as circles or rectangles and less frequently to part of a group. To challenge and extend student understanding, multiple contexts and representations should be used to develop flexible interpretations and consolidate understanding of fractions.

The results highlighted the pedagogical importance of developing students' conceptual understanding of the basic ideas of fractions and equivalent fractions, namely, the part–whole relationship, number of parts (partitions), and size of parts (units) using multiple representations. Having students talk and write about how they create or recognise equivalent fractions can strengthen their understanding and provide valuable information to teachers (Niemi, 1996).

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From Helen Prochazka's

Scrapbook

Constant

Throughout my life, my brother had been the one person I could rely on. Even when it seemed we had absolutely nothing in common, I knew he was as reliable as a mathematical formula.

Augusten Burroughs in his best selling memoir, Running With Scissors, (Picador, 2002).