Investigating the relationship between fraction proficiency *and* success in algebra

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Editors' note: This article continues the discussion of fractions and algebra by the same authors, whose review of the relevant literature was published in Issue 3, 2007. The authors now describe the results of a study that investigated the relationship between algebra students' proficiency with fractions and their success in algebra.

Teachers all over the world are aware that students struggle with fractional concepts and with elementary algebra. Support for this assertion can be found in a variety of research reports. The National Assessment of Educational Progress (NAEP), a United States report, indicates that students have recurrently demonstrated a lack of proficiency in these areas over the past twenty years (NCES, 2000). An analysis of the 1990 NAEP in mathematics achievement found that only 46 percent of all high school seniors demonstrated success with a grasp of decimals, percentages, fractions, and simple algebra (Mullis, Dossey, Owen & Phillips, 1991). The inability to perform basic operations on common fractions has led to error patterns that emerge in learning algebra. Problems can arise when students attempt to apply misunderstood shortcuts, learned with fractions, to situations involving algebra (Laursen, 1978). Cross-multiplying when dividing fractions, for example, has the potential for creating future problems. These difficulties suggest that understanding the structure of arithmetic should be a prerequisite to understanding the structure of algebra.

Elementary algebra is built on a foundation of fundamental arithmetic concepts. Knowing and understanding these concepts is essential, since algebra is the generalisation of arithmetic and the first experience in symbolic representation of numbers (Wu, 2001). Students are asked to manipulate variable expressions as if they were numbers: add, subtract, multiply and divide. The task for the elementary algebra student is difficult; being asked to abstract arithmetic concepts for the first time is both confusing and discouraging. If algebra is for everyone, then a bridge must be built to span the gap between arithmetic and algebra. The building materials are conceptual understanding and the ability to perform arithmetic manipulation on whole numbers, decimal fractions, and common fractions.

Elementary algebra is the gate for higher mathematics and the ability to understand and manipulate common fractions might be vital to success in algebra. The elementary algebra curriculum is replete with new constructs that rely on fraction concepts. An introduction to the rational number system is taught very early in the elementary algebra curriculum, expanding the concept of the common fraction. In simple one-step equation solving the notion of the reciprocal is introduced. Combining like terms is a concept used while learning addition and subtraction of fractions. Multiplying an equation by a constant to clear denominators employs understanding of a basic fraction concept. Proportional equations use constructs that have their basis in equivalent fractions. For example, percent increase and decrease, direct variation and inverse variation, and basic probability rely on understanding equivalent fractions (McBride & Chiapetta, 1978). The entire study of linear equations is dependent on the slope of a line, a fraction representing the rate of change. Solving systems of linear equations is dependent on the ability to form equivalent equations and manipulate fractions, which often are part of the solution. To solve rational equations and simplify rational expressions it is necessary to apply generalised common fraction concepts. Solving quadratic equations by completing the square requires fluency with fraction manipulation; a skill required in the study of conic sections. The list of algebraic generalisations that rely on fractional constructs grows as students move to each subsequent level of mathematics.

Much of the content of elementary algebra and intermediate algebra depends on an understanding of fractional concepts and the ability to demonstrate that understanding when solving various algebraic equations. If students do not understand basic fraction concepts and lack computational fluency with fractions, then learning new algebraic concepts that assume fraction proficiency becomes even more difficult. For many students this is a constant source of frustration. Too often the frustration ends in failure or a poor understanding of the structure of algebra, as students employ memorised algorithms to cope with their anxiety. Unfortunately, the result is that algebra becomes a cumbersome collection of unrelated facts and algorithms, which students use in a haphazard approach to solve problems. If algebra is the gate to higher mathematics and if much of algebra is dependent upon the ability to understand and generalise fraction concepts and operations, it seems reasonable to assume that the ability to manipulate common fractions is essential for the typical student to be successful in elementary algebra and subsequent mathematics courses.

The present study

The purpose of this study was to determine if a relationship existed between proficiency with common fractions and success in algebra. Proficiency in this context indicates that not only is a student able to understand fraction concepts, but also that the student is able to manipulate fractions for accurate computation without the aid of a calculator. Wu (2001, p. 1) states that "there are at least two major bottlenecks in mathematics education of grades K–8: the teaching of fractions and the introduction of algebra." It is the intent of this study to determine whether or not the two "bottlenecks" are related by comparing student success in applying fraction concepts and performing fraction computations with student success in algebra.

In this study an *ex post facto* design was used to compare students' performance in elementary algebra and intermediate algebra with their ability to understand and manipulate fractions. The students' performance in elementary algebra was based on the score attained on the first semester final examination, while students' performance in intermediate algebra was based on the average of several tests given during the course. For competency with fractions, a 25-question test was prepared using questions from previous research (Ginther, Ng & Begle, 1976; Rotman, 1991), and questions devised by the researcher. The questions were designed to test concept knowledge and computational fluency and were divided into six categories:

1. algorithmic applications,
2. applications of basic fraction concepts and word problems,
3. elementary algebraic concepts,
4. specific arithmetic skills prerequisite to algebra,
5. comprehension of the structure of rational numbers, and
6. computational fluency.

All questions on the test are prerequisite for developing a complete rational number concept. Decimal fractions were not included on the fraction test since understanding the initial decimal fraction concept is more complex than that for common fractions (Watson, Collis & Campbell, 1995).

The participants in this study consisted of 191 students in seven classes taught by one of three teachers. Five elementary algebra classes (138 students) and two intermediate algebra classes (53 students) were included. All participants were enrolled in a four-year high school, located in the southwestern United States, that serves a predominately white (81%) upper middle class population. There is strong parental support and involvement as evidenced by an active booster club, that contributes to every aspect of the school. Approximately 50% of the 2001 graduating class took college entrance exams. The average score among this population exceeded the national average.

Elementary algebra students are mostly ninth graders with a few tenth graders. Typically these students are of average mathematical ability. Generally, a student in elementary algebra has taken one of the following routes:

1. passed elementary algebra in the eighth grade but opts to retake the course to bolster confidence,
2. passed an eighth grade regular mathematics course with a "C" or better, or
3. failed elementary algebra in either eighth or ninth grade.

All of these students would be required to take geometry and an intermediate algebra course prior to graduation.

Most of the intermediate algebra students are the typical elementary algebra student, two years later; all have passed both elementary algebra and a course in geometry. However, some of the students in intermediate algebra are not of average ability. Nearly one quarter of the students are in ninth or tenth grade. These students have been accelerated in mathematics, possessing above average mathematics ability and will have the opportunity to take calculus in high school. On the other end of the spectrum there are twelfth grade students who have struggled through elementary algebra and geometry, but need one more mathematics credit to graduate. Since no other option exists these students must take intermediate algebra.

Data analysis and results

The 25-question fractions test was administered to all participants. It was given to the elementary algebra classes eight weeks into the first semester and to the intermediate algebra classes six weeks into the second semester. Students were not permitted to use calculators on this paper and pencil test but were encouraged to show all of their work. The researcher scored all tests on a scale from 0 to 100.

The descriptive statistics for the elementary algebra sample are shown in Table 1. For this group, the standard deviation for both the fraction test and the first semester final exam was almost the same (SD ≈ 18). In contrast the difference between the means for these two tests was considerable. The mean for the fraction test (M = 51.91) was nearly 25 points lower than the mean for the final exam (M = 76.62). Since the fraction test was scored out of a possible 100 points, a mean of approximately 52 describes the "average" elementary algebra student, of the sample population, as neither proficient nor familiar with fraction operations and fraction concepts. The implication is that in their preparation for algebra the mastery of fraction concepts was overlooked. Anecdotal evidence suggests that many of these students use a calculator to perform all computations and may have forgotten the once memorised algorithms.

The descriptive statistics for the intermediate algebra students are shown in Table 2. These students have successfully completed both elementary algebra and geometry and are generally two years older than the elementary algebra students, with another two years of high school mathematics experience. The mean for the average of test scores (M = 74.09) is only slightly higher than the mean for the fraction test (M = 72.08). The standard deviation for the fraction test (SD = 17.47) was similar to the standard deviation for the average of the test scores (SD = 15.4).

Comparing the mean for the fraction test in Table 1 (M = 51.91) with the mean for the fraction test in Table 2 (M = 72.09) it is not surprising that the latter mean was twenty points higher. Two factors that might have contributed to the increase are class composition and increased experience with the rational number construct. First, intermediate algebra classes have a greater number of above average mathematics students then

Table 1. Descriptive statistics for elementary algebra.

VARIABLES	FINAL EXAM	FRACTION TEST
Valid Data (N)	138	138
Mean	76.62	51.91
Variance	330.78	332.43
Standard Deviation	18.18	18.23

Table 2. Descriptive statistics for intermediate algebra.

VARIABLES	TEST SCORE AVERAGE	FRACTION TEST
Valid Data (N)	53	53
Mean	74.09	72.08
Variance	237.28	305.07
Standard Deviation	15.4	17.47

elementary algebra classes do. Second, experience with fractional numbers is gained as students are confronted with not-so-nice fractional solutions, rational expressions and equations, rationalising the denominator of an irrational fraction, and numerous applications of proportional reasoning in the study of similar figures.

The Pearson Correlation Coefficient was calculated to determine if a statistically significant relationship exists between proficiency with fractions and success in algebra for each of these groups (see Table 3). For the 138 elementary algebra students, a Pearson Correlation Coefficient of $r = .58$ was calculated. This value was significant at the alpha = 0.05 level indicating that a statistically significant relationship exists between the score on the fraction test and the score on the first semester final examination. For the 53 intermediate algebra students, a Pearson Correlation Coefficient of $r = .35$ was calculated. This value was significant at the alpha = 0.05 level indicating that a statistically significant relationship exists between the score on the fraction test and the average of the scores achieved on tests covering polynomial functions, rational functions, radical functions, and conic sections. Thus, in both the elementary and intermediate algebra samples, proficiency with fractions and success in algebra are positively related.

The t-value used to test the significance of the correlation coefficient was obtained using the following formula (Jaccard & Becker, 1990, p. 333):

$$t = \frac{r - \rho}{s_r}; \rho = 0$$

The estimated standard error of r, with sample size N, was obtained using the following formula (p. 334):

$$s_r = \sqrt{\frac{1 - r^2}{N - 2}}$$

In addition to the t-test procedure, the significance of the correlation coefficient was confirmed using a procedure comparing the observed value of r with the critical values of r (p. 334).

This study reveals a significant relationship between an individual's ability to understand and perform fraction operations and his or her test scores in algebra. The fact that the structure of arithmetic is the basis for the structure of algebra leads one to expect such a correlation to exist. Since a relationship does exist, understanding the subject of fractions would appear to be prerequisite to the study of algebra (Rotman, 1991; Wu, 1999).

In attempting to explain the poor performance of elementary algebra students on the fraction test, one could argue that by ninth and tenth grade, students have simply forgotten what they previously learned regarding fractions, since the subject was formally taught in fourth, fifth, and sixth grade. There is veracity in this argument that points to a lack of

Table 3. Pearson correlation (r) for elementary and intermediate algebra.

	ELEMENTARY ALGEBRA	INTERMEDIATE ALGEBRA
degrees of freedom (N–2)	136	51
correlation coefficient (r)	0.58	0.35
t-value	8.27	2.67
alpha level	p < 0.0001	p = 0.0103

meaningful follow up in grades seven and eight, but it fails to address the problem. The NAEP 1999 Long-term trend mathematic summary data (NCES, 2000) indicates that age seventeen students have mastered whole number operations, which were first taught in first, second, and third grade. Students have learned whole number operations, yet it is apparent that they are unable to extend their learning to the subject of fractions. Mathematics educators must consider the evident conclusion; a pedagogical problem exists relating to the subject of fractions.

Students, who have not mastered fraction concepts and do not possess computational fluency regarding fraction operations are nevertheless, expected to master the subject of algebra. "Algebra for everyone" is an empty slogan unless; "fractions for everyone" pre-exists. It is the position of this study that when taught correctly, the subject of fractions is able to prepare students for the level of generalisation that is necessary for understanding algebraic concepts. Fractions should not be the bane of algebraic manipulations, but rather a familiar subject providing a foundation, which makes the understanding of algebra possible.

Implications

The results of this study raise many questions that require further consideration by teachers and researchers alike. Although the results were significant, it is not the intention of this study to generalise the findings to a broader population. Research examining existing conditions for other samples of elementary and intermediate algebra students may well support the position that a relationship exists between proficiency with fractions and success in algebra, but that is not the recommended course for further study. It is the intention of this study to shed light on the existence of a problem in the learning of mathematics that must be rectified. It is the position of this paper that further consideration must be given to finding interventions that serve to eliminate or mitigate the difficulties students have in algebra due to a lack of proficiency with fractions. In particular, teachers need to attempt new strategies that may unveil potential solutions to this problem and, in turn, researchers must study the efficacy of these potential solutions.

The recommendations below were prompted by the researchers previous review of the literature (Brown & Quinn, 2007) and the findings of this study. These recommendations provide ideas for teachers to consider as they decide how best to teach fraction concepts to their students. Additionally, they provide fodder for researchers to develop longitudinal studies that track the development of the fraction construct over time and determine the long-term effect of these methods on learning algebra.

1. Children in grades one through four should be allowed the time to develop whole number concepts and whole number operations informally with abundant concrete referents. Arabic symbols should be used for counting purposes only and always connected to concrete objects or pictorial representations. Informal practice with fraction concepts should be limited to experiences that arise naturally, like fair sharing or situations that involve money.
2. Fifth grade students should be given experiences that extend the whole number concept with an eye toward algebra, involving an informal treatment of the field properties. These students need to be provided with experience in partitioning as a method for solving verbal

problems involving fractions (Lamon, 1999; Huinker, 1998). The informal treatment of fractions should include manipulation of concrete objects and the use of pictorial representations, such as unit rectangles and number lines. Fraction notation must be developed, but formal fraction operations using teacher-taught algorithms should be postponed. Learning the subject of fractions will revolve around informal strategies for solving problems involving fractions. The objective at this level is to build a broad base of experience that will be the foundation for a progressively more formal approach to learning fractions.

3. In grades six and seven, the development of fraction operations as an extension of whole number operations must provide experiences that guide and encourage students to construct their own algorithms (Lappan & Bouck, 1998; Sharp, 1998). Progressively the development should lead to more formal definitions of fraction operations and algorithms that prepare students for the abstractions that arise in the study of algebra (Wu, 2001).

The potential effect that understanding the structure of arithmetic could have on learning the structure of algebra cannot be overstated. Consequently, research that examines variables that either promote or inhibit an accurate perception of arithmetic structure has the potential to affect achievement in algebra. One such variable that may have an effect on learning the structure of arithmetic as applied to fraction concepts is the scientific calculator. Does applying the correct button to add two fractions promote understanding of the fraction construct? Do calculators inhibit computational fluency and familiarity when used to calculate fraction operations? If students know that they can use a calculator for fraction operations, will this affect their motivation for becoming proficient with frac-

Answer each of the following questions to the best of your ability.
If you simply do not know how to answer a question, then leave the box blank.

1. Find the sum of $\frac{5}{12}$ and $\frac{3}{8}$
2. Subtract $\frac{3}{5}$ from 8
3. Find the product of $\frac{1}{2}$ and $\frac{1}{4}$
4. The quotient of $\frac{1}{2} \div \frac{1}{3}$ is greater than (>) or less than (<) $\frac{1}{2}$?
5. Find $\frac{6}{7} \times \frac{2}{3} \times \frac{7}{4} =$
6. What is $\frac{1}{2}$ of $\frac{2}{3}$?
7. Write $3\frac{5}{6}$ as an improper fraction.
8. $\frac{12}{13} - \frac{3}{7}$ is closest to:
   a) 1
   b) $\frac{1}{2}$
   c) 0
   d) I don't know
9. Solve $x + \frac{1}{3} = 7$.
10. Write the fractions $\frac{4}{7}, \frac{5}{9}, \frac{3}{5}$ in order from least to greatest.
11. Write $5\frac{2}{7}$ as a sum.
12. Reduce $\frac{24}{36}$ to lowest terms.
13. If $\frac{5}{8} = \frac{x}{24}$, then find x.
14. $\frac{1}{3} \times a =$
15. Find what $\dfrac{7}{\frac{3}{5}}$ is equal to.
16. How many twelfths does $2\frac{1}{4}$ equal?
   a) 28
   b) 27
   c) 25
   d) 16
   e) 12
17. One half the students of a school are going to a concert. These students will be taken on 5 buses. What fraction of the students of the school will ride each bus?
18. If you have a half ball of string and each kite needs an eighth of a ball of string, how many kites can you fly?
19. Find the sum $\frac{7+5}{3+5} + \dfrac{\frac{5}{6}}{\frac{5}{3}}$
20. Find $\frac{18}{0}$
21. Reduce $\frac{3+4}{2}$
22. If n gets very large, then $\frac{1}{n}$,
   a) gets very close to 1
   b) gets very close to 0
   c) gets very large too
23. Simplify $\dfrac{1}{\frac{1}{2} \times \frac{1}{3}}$
24. Adrian has conquered only 6 giants in his new video game, Giant Trouble, but this is only two-fifths of the giants that he must conquer. How many giants are there in the new video game?
25. In a school election,
   candidate A got $\frac{1}{3}$ of the votes,
   candidate B got $\frac{9}{20}$ of the votes, and
   candidate C got $\frac{2}{15}$ of the votes.
   What fraction of the votes did candidate D get?

tions? What are appropriate applications of calculators with regard to the development of fraction concepts? These and other questions should be considered anecdotally by teachers and formally by researchers.

Conclusion

The findings of this study stand as a reminder that if students are expected to perform better in algebra and subsequent mathematics courses, they must be better prepared. This preparation should enable students to be proficient with all facets of the fraction construct. If algebra is for everyone, then all students must first become familiar and fluent with fractions.

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