Fraction proficiency and success in algebra

What does research say?

George Brown
Cold Springs Middle School, Reno, USA
Robert J. Quinn
University of Nevada, Reno, USA
<quinn@unr.edu>

Fractions and algebra are critically important components of the mathematics education of our youth. Unfortunately, however, students have typically struggled in these areas. For this reason, teachers and researchers have focused their attention on these topics for at least the past century. This article discusses what research shows regarding fractions and algebra, particularly, on issues related to when fractions should be taught, how fractions should be taught, and how competence with fractions affects the transition from arithmetic to algebra will be considered. Suggestions for teacher practice are included throughout the article.

The case for postponement

The first issue teachers and curriculum specialists must address is when fractions and rational numbers should be taught. Several researchers feel that the study of fractions and rational numbers often occurs before the student is ready. These researchers, who include Kieren and Freudenthal, suggest postponing the study of rational numbers until they can be taught within the context of algebraic ideas. Kieren (1976) feels that the experience base necessary for mature functioning with the complete rational number concept is best provided in a course of algebra and that, to sufficiently learn algebraic concepts that are intrinsic to rational number concepts, a student must experience and master the diverse interpretations of fractional numbers. Therefore, he recommends that an in-depth consideration of rational numbers be postponed until such time as the student studies algebra. Similarly, Freudenthal argues that teaching addition of rational numbers should be postponed until the concepts arise from algebraic ideas (as cited in Kieren, 1980).

Other researchers have studied how well students learn fractions when they are taught, as is usually the case, prior to the teaching of algebra. One such study tried to gather information about how much average students can learn about fractions under the best conditions. The results showed that in well-to-do junior high schools, instruction, even under optimal conditions, did not provide students with the necessary fractional skills. While students understood the fraction concept, they showed a poor understanding of the structure of the rational number system (Ginther, Ng & Begle, 1976). For example, only 30% of the students were able provide the correct number to make a true statement (Ginther et al., 1976, p. 4). Moreover, the results of the computation test were unsatisfactory and the students did poorly on simple word problems that involved fractions. This study suggests that without understanding structure, competent manipulation of fractions will not occur, since too much of the content depends on rote learning of algorithms, which make little sense to the learner and are too often misapplied. Is this because the concepts are being presented too early in a child's cognitive development? The study does not attempt to answer this question, but concludes, "Much of the work on fractions should be postponed to secondary school," (p. 9).

Approaches to teaching that allow students to construct their own knowledge can be powerful. The writings of Henry Margenau regarding the scientific method (1961) can shed light on what conditions are necessary in order for a constructivist approach to succeed and for students "to really know." He breaks knowledge into the elements of fact and construct and goes on to describe and redefine each element. Facts function as protocols, the "first draft of an experience later to become formalised knowledge" (p. 5). Constructs are the result of the processes of generalising and logical reasoning that lead to abstraction and ownership of complex concepts through a "long chain of activity" (p. 5). Protocols are collections of facts and related experiences that an individual brings to bear upon a problem. If the facts and related experiences can be connected effectively, then the individual is able to construct their own knowledge. If the facts are isolated and related experiences are not present, then one is unable to make the necessary connections to form a valid construct. These connections are like pathways that the learner logically negotiates to link relevant protocols and established constructs, which can be applied to a problem or to new learning (p. 16).

Margenau's thoughts can be applied directly to the rich and complex concept of rational numbers. Kieren (1980) asserts that the number of disjointed protocols a learner must control to form the rational number concept is extensive. Too often an algorithm has simply been taught, providing no connections for understanding, and leaving the student clinging to a prescribed step-by-step set of instructions. Algorithms that are taught when the concept is beyond the learner's cognitive development, force the learner to abandon their own thinking and resort to memorisation — doing without understanding.

If the algorithm is forgotten, the learner must retreat to familiar protocols (procedures), which can be applied in the given situation. For example, the individual may try to apply a natural number protocol for addition of fractions, adding both numerators and denominators, since addition of natural numbers arises from the natural activity of children (Kieran, 1980, p. 102).

Postponement and developmental readiness

Piaget's theory of cognitive development (Wadsworth, 1996) concludes that, in general, school-age children are either in the concrete operational stage (ages 7–11) of development or in the stage of formal operation (ages 11–16). The child in the concrete stage "must deal with each problem in isolation" (p. 112) and is unable to construct new knowledge from internal reflection alone. Formal thinkers are able to generalise and use internal reflection that "can result in new knowledge — new construction" (p. 118). In terms of Margenau (1961), this suggests that the individual in the stage of concrete operations does not progress very far into the constructional domain, yet is able to develop and connect simple protocols that are closely related to the individual's experience (p. 11).

The concrete operational child is capable of learning the basic part-whole relationship of rational numbers, but this is not enough for complete understanding of the rational number concept (Lamon, 1999). If instruction proceeds directly to computation procedures, then the child neither has the time nor the cognitive development to construct understanding.

Two important formal operational schemes are proportion and probability (Wadsworth, 1996). Both of these schemes are elemental to the rational number concept. Susan Lamon (1999) states "instruction needs to take an active role in facilitating thinking that will lead to proportional thinking" (p. 4). Mathematical topics that are related to proportions are fractions, decimals, ratios, percents, probability, similarity, linear functions, equivalence, measurement, and many others (p. 9). Consequently, a sizable gap exists in an individual's rational number concept, a gap that will become even more apparent as the individual begins to tackle a course in algebra.

McBride and Chiappetta (1978) investigated the relationship between proportional thinking and a student's ability to understand concepts related to simple machines and equivalent fractions. They reasoned that, in order to understand equivalent fractions, a student would need to think at Piaget's stage of formal operations. Consequently, students in the concrete operational stage could not be expected to demonstrate understanding in equivalent fractions after studying them in school. The reasoning ability of these students would limit their understanding of this concept. Since Piaget's stage of formal operational thinking begins around age 11 or 12, few students below this age level should be expected to display comprehension of the concept of equivalent fractions.

Several other studies provide support for this assertion. In one such study, only 7% of 9 to 12 year old students who had studied equivalent fractions were able to demonstrate understanding (Novillis, as cited in McBride & Chiappetta, 1978). In another study of 9 to 12 year olds, only 50 % were able to show comprehension of equivalent fractions, leading the authors to conclude that formal operational thinking was necessary for success with this topic (Steffe & Parr, as cited in McBride & Chiappetta, 1978). Similar findings by McBride and Chiappetta (p. 8) led them to conclude that proportional thinking is an underlying factor associated with achievement in equivalent fractions, supporting the hypothesis that postponement of teaching certain rational number concepts until secondary school is a viable alternative.

An extensive study of common fraction understanding and decimal fraction understanding was undertaken in upper socio-economic suburbs of Hobart, Tasmania. This study examined contexts related to diagrams, algorithms, and problem solving. The results were compiled for three levels of students, grades five and six, grades seven and eight, and grades nine and ten. With regard to diagrams, the results indicated that students at all three levels had a better concrete understanding of common fractions than of decimal fractions. In the problem-solving context, students at each level performed essentially the same (very poorly) for both common and decimal fractions (p. 10). The algorithmic results, however, indicated that students below grade nine were substantially better at applying decimal fraction algorithms than common fraction algorithms (Watson, Collis, and Campbell, 1995). Kieren (1976) supports these results, since operations on decimal fractions "form a natural extension to the whole numbers" (p. 102). However, by grades 9 and 10 the ability to manipulate common fractions was slightly better than the ability to manipulate decimal fractions (Watson, Collis & Campbell, 1995), lending credibility to the argument for postponement of teaching common fraction operations until secondary school

Some possible solutions

Currently, there is no indication that the mathematics curriculum will be modified to accommodate the previously discussed suggestions for postponement, despite an apparent lack of developmental readiness. Therefore, the logical alternative is that teachers need to provide better and more meaningful instruction of fraction concepts. Calls for improvement in this area are not new. More than twenty-five years ago, Ginther, Ng, and Begle (1976) suggested that research be done to discover whether or not better instruction would result in improved student learning of fractions. Research has been done, methods have been refined, but there has been little if any improvement as indicated by the data. If fewer than half of the adult population are able to reason proportionally (Lamon, 1999), then even the best instruction coupled with experience can be expected to have little effect on developing sophisticated mathematical reasoning (p. 5).

One positive finding with regard to the teaching of fractions suggests that providing increased time on this topic may be the answer. Studies have shown that if children are given the time to develop their own reasoning for at least three years without being taught standard algorithms for operations with fractions and ratios, then a dramatic increase in their reasoning abilities occurred; including their proportional thinking (Lamon, 1999). How fractions should be taught is inexorably linked to when the concepts are being presented and what impact the learned concepts will have on future mathematics courses such as elementary algebra.

Re-teaching the definition of fractions is one approach that can be effective when students experience problems with fractions. De Morgan (1910) suggests that a student having difficulties with fractions should return to the original definition and reason upon the suppositions, neglecting the rules until he or she can cognitively establish them by reflection upon familiar instances (p. 40). In this brief statement, De Morgan illuminates two of the major problems with the teaching of fractions. First, the concept of a fraction is never clearly defined (Wu, 2001); thus, returning to an original definition is impossible. Second, more time is needed to allow students to invent their own ways to operate on fractions rather than memorising a procedure (Huinker, 1998). An awareness of these shortcomings in the present approach to teaching fractions can be beneficial to teachers. If teachers make sure they provide a sound definition of a fraction and provide additional time for student exploration with fractions they may find that their students perform better.

Pedagogical reform

Based on the research already discussed, it seems clear that teachers need to reform the pedagogies by which they teach fractions. Additional support for change is provided by the results of *The National Assessment of Educational Progress* (NAEP, Mullis et al., 1990), a United States report.

This study indicated that only 46 % of twelfth grade students demonstrated success with decimals, percents, and fractions. Similarly, the 1999 NAEP reports that twelfth grade students responded correctly to test items related to the operations on fraction numbers only fifty per cent of the time (NCES, 1999). The remainder of this section will be devoted to discussing some potential pedagogical reforms that could serve to improve teaching methods in this area.

A change in emphasis from the development of algorithms to perform operations to the development of quantitative understanding based on students' experiences with physical models that emphasise meaning rather than procedure may be warranted (Bezuk & Cramer, 1989, p. 157). An added focus on problem solving is another potentially beneficial pedagogical technique. A problem-solving approach to teaching fractions was tested on fifth-graders in an urban school (low SES, Huinker, 1998). The students were not taught how to add, subtract, multiply, divide, or compare fractions, but instead, were left to develop meaning for fraction operations within the context of solving problems. The students in this four-week study "constructed intuitive quantitative understandings of fraction concepts and operations in the context of solving and posing realistic problems" (Huinker, 1998, p. 181). Carefully directed lessons can be designed to encourage students to form their own algorithms for adding and subtracting fractions. These student-invented algorithms are often very efficient and, with direction from the teacher, can be generalised to become powerful mathematical tools (Lappan & Bouck, 1998, p. 184).

Effective methods for the teaching of understanding of fractional numbers must be concerned with allowing students the time to construct their own understanding as teachers direct them toward accurate and meaningful student-invented algorithms. Bezuk and Cramer (1989) offer a few general recommendations, which are echoed in much of the literature concerned with the teaching of fraction concepts. These are:

1. the use of manipulatives is fundamental in developing students' understanding;
2. the majority of the time spent on fractions before grade 6 should be devoted to developing a conceptual base of fraction relationships;
3. operations on fractions should be delayed until students have a solid understanding of order and equivalence of fractions; and
4. the size of the denominator for computational exercises should be 12 or below (p. 158).

Teacher content knowledge and additional pedagogical considerations

A study conducted by Putt (1995) shows that a relationship exists between teachers' knowledge of mathematics and student learning. This study found that misconceptions about rational number concepts held by students were also evident among teachers (p. 11). The error patterns that are passed from teacher to student year after year create confusion and math anxiety, which too often begins right after introduction to fraction computation. Wu (2001) adds that teachers must have the necessary mathematical knowledge to be able to correctly guide their students through the subject and that textbooks must be written that treat fractions logically (p. 6).

Susan Lamon's book, *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*

(1999) provides a valuable resource that can help pre-service teachers acquire the requisite knowledge to teach fractions effectively. This book is designed to develop the rational number concept among this audience. Lamon underscores the rich "constructional domain" (Margenau, 1961, p. 9) for a simple fraction such as 3/4 supporting the position that fractional numbers might best be taught in the context of problems. There are many more interpretations of 3/4 than the simple, and single meaning as three parts of a four-part whole (p. 32).

Other researchers take a somewhat different position. Wu (2001), for example, feels that conceptual complexities are too often emphasised "at the expense of the underlying simplicity of the concept" (p. 2). When students are led through a multitude of interpretations, the simplicity is lost and the students are deprived of an essential component of doing mathematics: the ability to abstract. Wu's position is that prior to the fifth or sixth grade, children should become acquainted with fractions in an intuitive way through explorations, collecting data without concern for meaning; but then Wu (2001) goes on to state that, "when confronted with complications, [students] try to abstract in order to achieve understanding" (p. 5). He believes that the processes of abstraction should be introduced as soon as possible in the school mathematics curriculum, and that the teaching of fraction computation would be "as soon as possible," since at the age of eleven or twelve children are moving into formal operations and are capable of employing "reflective abstraction" (Wadsworth, 1996). "By giving abstraction its due in teaching fractions, we would be easing students' passage to algebra as well" (Wu, 2001, p. 6).

Sharp (1998), like Wu, believes that algebraic thinking can be developed as students are taught fractions. She suggests a method for teaching division of fractions that uses an algorithm that follows directly from whole number operations and fraction concepts. Since much of algebra is generalised arithmetic, prior practice in generalising previously developed algorithms can begin to build the type of thinking that is necessary for the transition from arithmetic to algebra (p. 203). If the logical development of algorithms for rational number operations, supported by fraction concepts, promotes algebraic thinking, then it would follow that students who have constructed a viable rational number concept would be successful in algebra.

The relationship between fractions and algebra

There are at least three critical achievements in the mathematical life of a student: mastering the idea of ten as a unit, understanding fractions, and grasping the concept of the unknown. Consequently, when attempting to learn algebra without the aid of understanding fractions, "it is no wonder that many students' seeming mastery of fractions begins to fall apart" (Driscoll, 1982, p. 107).

Rotman (1991) contends that although an arithmetic course need not be prerequisite for a first-year college algebra course, the understanding of "fraction concepts deserve[s] to be singled out, because algebra typically uses fractional notation to indicate a quotient" (p. 8). Similarly, Wu (2001) asserts that the study of fractions has the potential for being the best kind of pre-algebra and argues that unless the way in which the teaching of fractions and decimals is radically changed, then the failure rate in algebra will continue to be high (p. 10). He claims that adding fractions has become a conceptual preoccupation, but that understanding the concept is not sufficient; there is the need for fluency in computation. Wu insists that such fluency — the ability to efficiently manipulate fractions — is "vital to a dynamic understanding of algebra" (p. 17). Vague fraction concepts and misunderstood fraction algorithms will ultimately be generalised into vague algebraic concepts and procedures. The lack of precise definitions and reliance upon shortcuts that are thoughtlessly given to students are likely to hinder performance in algebra. Additional support for this position is provided by Laursen (1978) who found that many of the errors that students make in first-year algebra are due to an incomplete understanding of fraction operations and the subsequent misapplication of imprecise algorithms, which were previously taught as shortcuts.

Kieren (1980) suggests that there are algebraic aspects of operations on fractions, but that most school curriculum materials simply treat fractions as objects of computation. Rational numbers present the student with algebraic problems. The student must:

1. understand the notion of equivalence;
2. deal with an addition operation based in axiomatic reasoning rather than the natural extension of whole number addition;
3. work with a multiplication operation that is distinct from addition and is abstractly defined; and
4. cope with abstract properties and the concept of an inverse (p. 102).

If students have not had opportunities to learn how to abstract prior to an elementary algebra course, then they may opt for rote learning — the memorisation of algorithms without any conceptual basis — that allowed them to appear to be successful with fraction computation.

Generality and abstraction are characteristics of algebra that must eventually be expressed in symbolic notation (Wu, 2001). Fluent computation with numbers lies at the foundation of the ability to perform symbolic manipulations (p.13). When teaching addition of fractions, without the concept of the lowest common denominator, Wu suggests that the operation be clearly defined as . This formula can be first used for cases where a, b, c, and d are small numbers and then slowly built up from these specific cases to the general case. Wu insists that without such a foundation in fractions, students will be severely hindered when they come to study rational expressions in algebra (p. 14).

Discussion

There is no escaping fractions in algebra. From linear equations to completing the square, from solving systems of linear equations to solving rational equations, and from simple probabilities to the binomial theorem, algebra is replete with examples that are directly and indirectly related to fractions. Much of the basis for algebraic thought rests on a clear understanding of rational number concepts (Kieren, 1980; Driscoll, 1982; Lamon, 1999; Wu, 2001) and the ability to manipulate common fractions. "With proper infusion of precise definitions, clear explanations, and symbolic computations, the teaching of fractions can eventually hope to contribute to mathematics learning in general and the learning of algebra in particular" (Wu, p. 17).

As this article suggests, extensive research has been done in the areas of fractions and algebra, much of which considers the relationships between these two difficult, but important, topics. Since no definitive conclusions can yet be drawn, it is incumbent upon teachers and researchers alike to implement innovative strategies and to study the efficacy of these strategies with the ultimate goal of improving instruction in these critical areas. We continue this argument in a subsequent article that will discuss data from our research, and consequent implications for classroom teaching.

References

Bezuk, N. S. & Cramer, K. (1989). Teaching about fractions: What, when and how? In P. R. Trafton & A. P. Schulte (Eds), *New Directions for Elementary School Mathematics* (pp. 156–167). Reston, VA: National Council of Teachers of Mathematics.

De Morgan, A. (1910). *On the Study and Difficulties of Mathematics (3rd ed.)*. Chicago, IL: The Open Court Publishing Company.

Driscoll, M. (1982). *Research within Reach: Secondary School Mathematics. A Research Guided Response to the Concerns of Educators*. Missouri: (ERIC Document Reproduction Service No. ED 225 842).

Ginther, J., Ng K. & Begle, E. G. (1976). *A Survey of Student Achievement with Fractions. SMESG Working Paper No. 20*. Stanford, CA: School Mathematics Education Study Group. (ERIC Document Reproduction Service No. ED 142 409)

Huinker, D. (1998). Letting fraction algorithms emerge through problems solving. In L. J. Morrow & M. J. Kenny (Eds), *The Teaching and Learning of Algorithms in School Mathematics* (pp.170–182). Reston, VA: National Council of Teachers of Mathematics.

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Number and Measurement* (pp. 101–144). Columbus: Ohio State University. (ERIC Document Reproduction Service No. ED 120 027).

Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent Research on Number Learning* (pp. 125–149). Columbus: Ohio State University. (ERIC Document Reproduction Service No. ED 212 463).

Lamon, S. J. (1999). *Teaching Fractions and Ratios for Understanding: Essential Knowledge and Instructional Strategies for Teachers*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lappan, G. & Bouck, M. K. (1998). Developing algorithms for adding and subtracting fractions. In L. J. Morrow & M. J. Kenny (Eds), *The Teaching and Learning of Algorithms in School Mathematics* (pp. 183–197). Reston, VA: National Council of Teachers of Mathematics.

Laursen, K. W. (1978). Errors in first-year algebra. *Mathematics Teacher, 71*(3), 194–195.

Margenau, H. (1961, reprint 1983). *Open Vistas*. Woodbridge, CT: Ox Bow Press.

McBride, J. W. & Chiappetta, E. L. (1978). *The Relationship Between the Proportional Reasoning Ability of Ninth Graders and their Achievement of Selected Math and Science Concepts*. Toronto, Canada: National Association for Research in Science Teaching. (ERIC Document Reproduction Service No. ED 167 351).

Mullis, Ina V. S., Dossey, J. A., Owen, E. H. & Phillips, G. W. (1991). *The State of Mathematics Achievement: NAEP's 1990 Assessment of the Nation and the Trial Assessment of the States*. Princeton, N.J.: Educational Testing Service. (ERIC Document Reproduction Service No. ED 330 545).

National Center for Educational Statistics (2000, August). *The NAEP 1999 Long Term Trend Mathematics Summary Data Tables for Age 17 Student*. Accessed http://nces.ed.gov/naep3/tables/Ltt1999/NTM31031.PDF

Putt, J. I. (1995). Preservice teachers ordering of decimal numbers: When more is smaller and less is larger! *Focus on Learning Problems in Mathematics, 17*(3), 1–15.

Rotman, J. W., (1991). *Arithmetic: Prerequisite to Algebra?* Lansing, MI: Annual Convention of the American Mathematical Association of Two-Year Colleges. (ERIC Document Reproduction Service No. ED 338 279).

Sharp, J. (1998). A constructed algorithm for the division of fractions. In L. J. Morrow & M. J. Kenny (Eds), *The Teaching and Learning of Algorithms in School Mathematics* (pp. 198–203). Reston, VA: National Council of Teachers of Mathematics.

Wadsworth, B. J. (1996). *Piaget's Theory of Cognitive and Affective Development (5th ed.)*. White Plain, NY: Longman Publishers.

Watson, J. M., Collis, K. F. & Campbell, K. J. (1995). Developmental structure in the understanding of common and decimal fractions. *Focus on Learning Problems in Mathematics, 17*(1), 1–24.

Wu, H. (2001). How to prepare students for algebra. *American Educator, 25*(2), 10–17.