Algebra students' difficulty with Fractions

An error analysis

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The National Assessment of Educational Progress (NAEP), a United States report, raises concerns regarding trends in student achievement over the past twenty years (NCES, 2000). The results indicate that students of age seventeen recurrently demonstrated a lack of proficiency with fraction concepts. An analysis of the 1990 NAEP mathematics achievement by Mullis, Dossey, Owen, and Phillips (1991) found that only 46 percent of all high school seniors demonstrated success with a grasp of decimals, percentages, fractions, and simple algebra. If algebra is for everyone, then a bridge must be built to span the gap between arithmetic and algebra. The building materials are conceptual understanding and the ability to perform arithmetic manipulation on whole numbers, decimal fractions, and common fractions.

Augustus De Morgan, writing in *Study and Difficulty of Mathematics* (1910) acknowledges that the learning of fractions is expected to "present extraordinary difficulties." This was true in the nineteenth century and it is still true today. Consider the following (p. 41):

> What is $\frac{1}{4}$ of $\frac{2}{7}$ of a foot? What is $\frac{2}{5}$ of $\frac{1}{3}$ of $\frac{3}{4}$ of a foot? Into how many parts must $\frac{3}{7}$ of a foot be divided, and how many of them must be taken to produce $\frac{14}{15}$ of a foot? What is $\frac{1}{3}+\frac{1}{7}$ of a foot? and so on.

Is the above a logical and natural progression from operations on whole numbers? Examine the difficulty in finding the product of two common fractions. Multiplication is precisely defined as repeated addition, multiply 5 and 8 together and the product is either eight fives or five eights. Having become proficient at whole number multiplication with a solid understanding of the concept, consider the product of $\frac{2}{3}$ and $\frac{3}{5}$. The result is absurd (p. 34). For the sake of the learner the absurdity must be alleviated, the mystery of $\frac{1}{3}\times\frac{1}{7}=\frac{1}{21}$ must be resolved. When fractions should be taught, how fractions should be taught, and how competence with fractions affects the transition from arithmetic to algebra, are questions that mathematics educators and researchers have examined for the past century.

Kieren (1980) suggests that the instruction of rational numbers be postponed until the student has reached the stage of formal operations. He reasons that five concepts of fractional numbers must be both differentiated and connected to form a cogent rational number construct. The five ideas, (1) part–whole relationships, (2) ratios, (3) quotients, (4) measures, and (5) operators, represent "five separate fractional or rational number thinking patterns" (p. 134).

The rational number concept is rich and complex. Kieren (1980) asserts that the number of disjointed protocols a learner must control to form the rational number concept is extensive. Too often simply an algorithm has been taught, abandoning the student deep in the rational number construct. This provides no connection for understanding, and leaves the student clinging to a prescribed step-by-step set of instructions. If the algorithm is forgotten, then the learner must retreat to familiar protocols, which can be applied in the given situation. For example, the individual may try to apply a natural number protocol for fraction addition, adding both numerators and denominators, since addition of natural numbers arises from the natural activity of children (p. 102). Algorithms that are taught when the concept is beyond the learner's cognitive development force the learner to abandon his or her own thinking and resort to memorisation — doing without understanding. Lamon (1999) insists that the consequences of doing rather than understanding affect both a student's enjoyment of and motivation for learning mathematics (p. xi).

This article will investigate error patterns that emerge as students attempt to answer questions involving the ability to apply fraction concepts and perform operations on fractions. This analysis will provide a source that can assist teachers in detecting and correcting common mistakes students make when manipulating fractional numbers.

Methodology

A twenty-five-question test was prepared to analyse competency with fractions. This test was developed using questions from previous research (Ginther, Ng & Begle, 1976; Rotman, 1991), and questions devised by the researcher. The test was a pencil and paper instrument in which calculators were not allowed. Students were encouraged to show all of their work. The questions were designed to test concept knowledge and computational fluency and were divided into six categories. All questions on the test are prerequisite for developing a complete rational number concept.

The test was administered to five elementary algebra classes (N = 143). The students in all five groups finished the fraction test within a thirty-minute time period. The researcher scored all tests and analysed each test item. The test items were categorised and errors were analysed by type and frequency.

All participants were from the same four-year high school that serves a predominately white (81%) upper middle class population where there is strong parental support and involvement. Approximately 50% of the 2001 graduating class took college entrance exams. The average score among this population exceeded the national average.

The elementary algebra students are mostly ninth graders with a few tenth graders; typically these students are of average mathematics ability. Generally, a student in elementary algebra has taken one of the following routes:

1. passed elementary algebra in the eighth grade but opts to retake the course to bolster confidence;
2. passed an eighth grade regular mathematics course with a "C" or better; or
3. failed elementary algebra in either eighth or ninth grade.

All of these students will be required to take geometry and an intermediate algebra course prior to graduation.

Error analysis

A close examination of the specific errors made on the fraction test demonstrates the degree of familiarity with fraction concepts and operations that exists among the students who participated in the study. The following discussion consists of several dynamics. It is essential to note that each of the twenty-five problems on the fraction assessment was selected or developed for specific reasons. The researcher was looking for understanding on many different levels. Consequently, the questions have been broken into six general categories. Some of the problems could fit into more than one category, but for the sake of succinct analyses each problem was assigned to a single category. Each of the categories has been explained in terms of the researcher's intent. Several of the questions have been used in related studies (Benander & Clement, 1985; Ginther, Ng & Begle 1976; Rotman, 1991).

Each of the twenty-five problems is briefly analysed and examples of common errors and unique errors are discussed. The purpose of the discussion is to provide quick detection of specific error types, which can serve as a guideline for remediation and re-teaching. Since the students were encouraged to show all of their work, the researcher was able to discover several misconceptions relating to the rational number construct and several misapplications of algorithms used for fraction computation.

The following is a question-by-question analysis from the fraction test.

Category I: Algorithmic applications

The following six examples were selected to check the algorithms that students use for finding sums, products, and differences, and for reducing fractions to lowest terms.

A. Find the sum of $\frac{5}{12}$ and $\frac{3}{8}$

Nearly 48% of the students were unable to find the correct sum.

1. Twenty-seven students were unable to find a common denominator.
2. Nineteen of the 27 added the numerators and added the denominators.
3. Six students demonstrated misconceptions related to equivalent fractions, e.g., $\frac{3}{8}$ became $\frac{7}{12}$ by adding 4 to numerator and denominator.

Driscoll (1982) noted the second error as one of the two most common errors in his study (p. 110). This error is easy to detect, but is difficult to remedy. This particular mistake is a logical extension of natural number addition. Wu (2001, p. 13) suggests avoiding the concept of the lowest common denominator and assisting the student, over time, in developing the formula

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

B. Subtract $\frac{3}{5}$ from 8

Sixty-seven percent of the students gave an incorrect response.

1. Twenty-one students did not find the lowest common denominator.
2. Sixteen students rewrote the example as $\frac{8}{5}-\frac{3}{5}$ or $\frac{3}{5}-\frac{8}{5}$, getting answers of $\pm\frac{5}{4}$ or 1.
3. Sixteen students subtracted the numerators and subtracted the denominators.

For the first error, Wu (2001) would argue that there is no need to find a lowest common denominator. The second error indicates a lack of clear understanding about the relationship between a natural number and a fraction. This error is a student-developed modification of the poorly understood short-cut algorithm for renaming a mixed number as an improper fraction (see Category I example E). These students will need re-teaching, that should include visual and verbal reasoning activities that build a conceptual understanding using the part–whole relationship of rational numbers (Lamon, 1999). The third error is an attempt to extend the subtraction algorithm for natural numbers and apply it directly to fractions.

Of the 47 students that answered correctly, only 13 students did not resort to an algorithm; instead, they simply recognised that $8-\frac{3}{5}=7\frac{2}{5}$.

C. Find the product of $\frac{1}{2}$ and $\frac{1}{4}$

Fifty-eight percent of the students did not find the correct product.

1. Thirty-seven students misapplied the standard multiplication algorithm.
2. Eight of the 37 students added the denominators and multiplied the numerators.
3. Another eight of these students found the least common denominator before multiplying.

Each of the three errors indicated a strict algorithmic approach to operations on fractions. The most frequent error was either the overgeneralisation of a cross-multiplication algorithm or the overgeneralisation of the division algorithm (Benander & Clement, 1985). The former error yields a product of $\frac{1}{2}$ and the latter should yield an answer of 2. The first overgeneralisation could be eliminated if students were given the time to develop the formula

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

Once the product is written in the form

$$\frac{a\cdot c}{b\cdot d}$$

it can be reduced using the commutative property and then multiplied. The overgeneralisation of the division algorithm stems from the conflict between the visual and the algorithmic approach to fraction operations (Driscoll, 1982) — students are literally unable to see how the algorithm works. To eliminate the conflict, Sharpe (1998) and Lamon (1999) offer methodologies that use a visual approach to allow students to develop a division algorithm for fractions; the algorithm extends logically from the division of natural numbers. Sharpe uses pattern blocks and fraction circles and the concept of repeated subtraction to facilitate a student-invented division algorithm (pp. 198–203).

The third error is symptomatic of students applying teacher-taught algorithms without a proper conceptual base.

D. Find what $\frac{6}{7}\times\frac{2}{3}\times\frac{7}{4}=?$

Forty-eight percent of the students answered this incorrectly.

1. Thirty-five of the students demonstrated no understanding of finding the product of more than two fractions.
2. Sixty-one students failed to reduce before multiplying.

More students were able to find the correct product on this example than on the previous problem. This should be expected because there is no division algorithm for more than two fractions; therefore, overgeneralisation did not occur in this example. Most of the students multiplied the numerators together, multiplied the denominators together, and then reduced the fraction to lowest terms. This works, but is not very efficient. Twelve of the students, having an informal concept of the commutative property, recognised common factors and reduced before multiplying. All students need to be able to visualise or rewrite the given product,

$$\frac{6}{7}\times\frac{2}{3}\times\frac{7}{4}$$

as

$$\frac{6\cdot 2\cdot 7}{7\cdot 3\cdot 4}$$

then reduce and find the product.

E. Write $3\frac{5}{6}$ as an improper fraction.

Twenty-five percent of the students did not rename the mixed number correctly.

1. Eighteen students indicated that they did not know or their process demonstrated a complete lack of understanding.
2. The other 125 students applied the shortcut algorithm for renaming a mixed number as an improper fraction.
3. Thirteen of these 125 applied seven different incorrect forms of the shortcut algorithm.

Out of the 143 students who attempted to rename the mixed number, none employed partitioning strategies. Partitioning can be defined as a general strategy for dividing a given quantity into a given number of equal parts (Kieren, 1980). Although 75% of the students were able to rename the mixed number correctly, all should be taught to use partitioning, since the shortcut method, “6 × 3 + 5 over 6” is not easily generalised in algebra. Students should learn to partition the three into eighteen-sixths and then add this to five-sixths, e.g.,

$$3\frac{5}{6}=3+\frac{5}{6}=\frac{18}{6}+\frac{5}{6}$$

Learning to partition whole numbers could eliminate mistakes similar to those made in example B of Category I.

F. Reduce $\frac{24}{36}$ to lowest terms

Twenty-seven percent of students did not correctly reduce the fraction.

1. Thirteen students did not demonstrate a method for reducing fractions.
2. Twelve students committed basic division errors.
3. Fourteen students did not completely reduce the fraction.

The algorithm employed by the students is dependent upon their ability to exhaust all of the factors that are common to both numerator and denominator. Most of the students were able to complete the task without any errors. However, reducing fractions into lowest terms is only one of the components of a complete concept of equivalence. Students may be proficient in reducing fractions, but they need to be provided with problem solving experiences in which they must decide which equivalent form of the fraction is best suited for the given situation.

Category II: Applications of basic fraction concepts in word problems

The next three problems are word problems involving basic fraction concepts and simple fraction computations. The objective was to determine if students could recognise how a fraction should be used as an operator in each of the three contexts and then correctly apply the operator.

A. One half the students of a school are going to a concert. These students will be taken on 5 buses. What fraction of the students of the school will ride each bus?

Sixty-six percent of the students did not solve the problem correctly.

1. Twenty students made errors by choosing the wrong operator. For example some students found the product of $\frac{1}{2}$ and 5.
2. Fourteen students applied the correct concept, but left their answer as $\frac{1}{5}$ of half the school or $\frac{1}{2} \div 5$.
3. Four students simply gave an answer of $5\frac{1}{2}$.

All three errors can be addressed by encouraging students to use pictorial representations and by providing a variety of partitioning experiences, especially when the unit to be partitioned is a fraction (see Lamon, 1999, pp. 75–109).

B. If you have a half ball of string and each kite needs an eighth of a ball of string, how many kites can you fly?

Thirty-nine percent of the students gave the correct response to the question.

1. Thirteen of the students made errors resulting in answers that did not make sense. For example one student incorrectly applied the division algorithm, getting
$$\frac{1}{2} \div \frac{1}{8} = \frac{0}{}$$
but did not correct the obvious error.
2. Forty-two students did not attempt the problem.

The errors that were made and the failure to attempt the problem by so many students indicate a lack of experience with word problems. Students need to be able to employ drawings so that they can construct a relationship between the visual and the verbal. Once again, an abundance of partitioning activities in the context of verbal problems can provide every student with a strategy to be able to at least attempt to find a solution to the problem.

C. Adrian has conquered only 6 giants in his new video game, Giant Trouble, but this is only two-fifths of the giants that he must conquer. How many giants are there in the new video game?

Forty-nine percent of the students were unable solve this problem.

1) Ten students did not use the correct fraction operator.
2) Fifteen students applied disjointed algorithms using different combinations of 2, 5, and 6; e.g., five students first wrote $6\frac{2}{5}$ and then used the shortcut algorithm to get $\frac{32}{5}$, concluding that there are 32 giants.
3) Twelve students divided the six giants into fifths; the result was 30 giants.

The first two error types are indicative of students who have collected a series of fragmented algorithms, but do not understand fraction operations. The usual cues for deciding which algorithm to use are not obvious in a verbal problem; consequently, these students arrange the numbers in a familiar form and then apply an algorithm that they believe fits the form. This allows them to cope with the confusion of fractions and to at least get an answer. Since these students do not have a basic understanding of fraction concepts, they have no mechanism for checking the reasonableness of their answers. Offering a variety of experiences to allow students to develop partitioning skills and a de-emphasis on teacher-taught algorithms will provide a few more avenues for the student to take when solving problems (see Category II, examples A & B).

The third error type stems from viewing fraction relationships as strictly part–whole comparisons in which one portion is a fixed size: there are six giants and since the number of giants is being divided into fifths, then there must be thirty giants. Lamon (1999) suggests that students need a variety of experiences in order to develop the cognitive process of unitising: a mental skill that allows the individual to decide upon the most useful operating unit to apply to a given set of conditions (p. 42). In this example unitising would allow the student to think of the operating unit as either one fifth or as three giants, both approaches illustrate that for every fifth there are three giants.

Category III: Elementary algebraic concepts

The next three examples include algebraic concepts that are covered in first semester elementary algebra. The two equations could have been solved intuitively or by using an equation-solving algorithm. In the first example students could have recognised that $6\frac{2}{3} + \frac{1}{3} = 7$ and simply written down the answer, $6\frac{2}{3}$. In the second example students could have used equivalent fractions, recognising that $\frac{5}{8} = \frac{15}{24}$.

The rationale for the third example was to determine if students could generalise a product like $\frac{1}{3} \times 5 = \frac{5}{3}$ and apply the results to $\frac{1}{3} \times a = \frac{a}{3}$.

A. Solve $x + \dfrac{1}{3} = 7$

Fifty percent of the students did not find the correct value for x.

1. Seventy-seven students used algebra to solve the equation, i.e., they subtracted one-third from both sides of the equation. Thirty-eight of these students were unable to correctly subtract one-third from seven (see discussion in Category I, example B).
2. Thirty-two students were able to recognise that $6\frac{2}{3} + \frac{1}{3} = 7$.

Of the 39 students who used algebra and found the correct value for x, 27 of these students did not immediately recognise that $7 - \frac{1}{3} = 6\frac{2}{3}$, and used $\frac{21}{3} - \frac{1}{3}$ to get $\frac{20}{3}$. This observation indicates that students may perceive fraction operations as a series of isolated algorithms rather, than as a set of connected ideas that form one complete concept. Partitioning a unit rectangle into the given fractional unit, as a method for subtracting common fractions from whole numbers, offers visual clues that a pure algorithmic approach is unable to provide.

B. If $\dfrac{5}{8} = \dfrac{x}{24}$, then find x

Eighteen percent of the students did not find the correct value for x.

1. Twenty-seven students either made arithmetic errors or did not attempt to solve the equation.

Most of the students used the cross-product algorithm to solve the proportion. Forty-six students recognised equivalent fractions and concluded that x was equal to 15, without employing the cross-product algorithm. It should be noted that finding the correct answer in this example does not indicate competence in proportional reasoning; there is more to proportional reasoning than recognising the right scenario in which to apply a familiar algorithm.

C. $\dfrac{1}{3} \times a = ?$

Sixty-two percent did not answer correctly with either $\frac{1}{3}a$, $\frac{a}{3}$ or $a\frac{1}{3}$.

1. Forty-one students wrote the answer incorrectly as $\frac{1}{3a}$.
2. Nine tried to solve a non-existent equation for a.

Benander & Clement (1985) in their catalogue of error patterns concluded that the first error is because students tend to view multiplication of fractions as non-commutative. The second is a common error in elementary algebra. Students are bothered by variables and go to great lengths to assign them a value; as a result, many will attempt to solve algebraic expressions as if they were equations. Offering students experiences in generalising fraction operations could help to eliminate such errors. For example, allow students to induce the formula

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

by observing several examples in which different positive integers are substituted for a, b, c, and d.

Category IV: Specific arithmetic skills that are prerequisite to algebra

Rotman (1991) believes that the foundation for understanding algebra is laid in the understanding of arithmetic that students encounter before they reach an algebra course. It is the understanding that is prerequisite to algebra, not the set of computational skills (p. 8). The next three exercises are samples involving rational number concepts that Rotman believes are prerequisite to algebra.

A. Reduce $\frac{3+4}{2}$

Eleven percent of the students failed to reduce the fraction correctly.

1. Ten chose to use an algorithm to rename $\frac{7}{2}$ as a mixed number.
2. Two divided four by two and added the quotient to three.

This example emphasises fraction concepts and de-emphasises using rote algorithms (Rotman, 1991). The first error type occurred when students applied a rote algorithm to convert the improper fraction to a mixed number. These students correctly found the fraction to be $\frac{7}{2}$ and then renamed the improper fraction incorrectly. Leaving the answer as $\frac{7}{2}$ is often the preferred form in algebra, but many students think that reducing an improper fraction means renaming it as a mixed number.

The second error indicates a basic misunderstanding with regard to fraction addition. Since students were not expected to add two fractions, the impulse to find common denominators was not triggered. Without having to consider common denominators students may have viewed

$$\frac{3+4}{2}$$

and

$$\frac{3}{2}+\frac{4}{2}$$

as two distinct ideas, rather than as two forms of the same idea. Consequently, $4 \div 2 + 3 = 5$ would seem to be a logical solution.

Anecdotal evidence indicates that this is a frequent error made by students in elementary algebra.

B. Write $5\frac{2}{7}$ as a sum

Fifty-eight percent of the students were unable to give a correct response.

1. Eighty-three students either stated that they did not know or demonstrated that they did not know.
2. Eleven of the 83 gave an answer of 37, $(7 \times 5 + 2 = 37)$.

This example focusses on the meaning of symbols (Rotman, 1991). Accordingly, students who were unable to write the mixed number as a sum do not understand the meaning of mixed number notation. Students, who have only learned the shortcut algorithm for renaming mixed numbers as improper fractions, may not know that a mixed number is a sum. This possibility is indicated by the results of this example (see discussion on Category I, example E).

C. Find $\frac{18}{0}$

Eighty percent of the students failed to indicate this operation as undefined.

1. Eighty-six students responded with an answer of 0.
2. Twenty-one students gave an answer of 18.

Rotman's (1991) rationale for this example is that (1) fractions show division, (2) division by zero is not defined, and (3) it avoids employing a mechanical process (p. 12). The idea that division by zero is undefined is too often taught as an isolated fact, that students are asked to simply believe. The results indicate that they really do not believe it.

For students, a quotient of 0 is a logical choice. They do not see why you cannot have a 0 in the denominator if a 0 in the numerator is acceptable (Benander & Clement, 1985).

The second result, 18, is also logical. If you have a number and you "don't divide it by anything" then its value should not change. Besides, division is repeated subtraction; therefore, the reasoning is understandable, i.e., subtract 0 from 18 and it remains 18, subtract 0 from 18 repeatedly and it still remains 18.

Division by 0 is an abstract notion that needs to be connected to the complete rational number concept. Since the solution is undefined, there is no process for determining the outcome. However, the results can be logically developed using the following pattern.

a) $\frac{18}{9}=2$ therefore $2\times9=18$

b) $\frac{18}{2}=9$ therefore $9\times2=18$

c) $\frac{18}{6}=3$ therefore $3\times6=18$

d) $\frac{18}{3}=6$ therefore $6\times3=18$

e) $\frac{18}{1}=18$ therefore $18\times1=18$

f) $\frac{18}{18}=1$ therefore $1\times18=18$

g) $\frac{18}{0}=0$ but $0\times0\neq18$

h) $\frac{18}{0}=18$ but $18\times0\neq18$

The process that was employed above is the same process that students used to check their quotients when they first encountered whole number division.

Category V: Comprehension of the structure of rational numbers

The five problems in this category were intended to ascertain a student's conceptual understanding of the basic structure of rational numbers. One aspect that was assessed was the ability to use the relationship between the numerator and denominator in determining the relative size of two or more fractions. Another facet that was checked was the capability to use fractions as operators. By simply applying the appropriate concept, each of these questions could have been solved. None of these problems required computation or application of an algorithm.

A. The quotient of $\frac{1}{2}\div\frac{1}{3}$ is greater than (>) or less than (<) $\frac{1}{2}$?

Forty-six percent of the students gave the incorrect response.

1. Sixty-seven students responded incorrectly.

The error indicates that students are probably attempting to extend the concept of natural number division, in which the quotient is always smaller than the dividend. These students need to be provided with experiences that can be represented pictorially, allowing them to see and count the number of times a given fraction can be divided into a given amount. Initially the student should start working with dividing whole numbers by proper fractions to develop the notion that division does not always yield a quotient that is smaller than the dividend. This concept should be developed with and connected to division by one, division by zero, and division by numbers between zero and one.

B. What is $\frac{1}{2}$ of $\frac{2}{3}$?

Sixty-four percent of the students did not answer this question correctly.

1. Twenty students did not recognise that the solution is a product.
2. Thirty-seven students used the cross-multiplication algorithm (see the discussion in Category I, example C).

The first error is another example of improper extension of natural number operations, where "of" implies a division process. The results of this example were unexpected. None of the students who attempted this problem demonstrated a basic knowledge of the concept that one-half of two-thirds is simply one-third. Those who attempted the problem relied solely on the multiplication algorithm.

C. $\frac{12}{13}-\frac{3}{7}$ is closest to...?

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) I don't know

Forty-eight percent of the students did not choose the correct response.

1. Twenty-two students selected (a) or (c).
2. Forty-six students indicated that they did not know.

The students were being asked to estimate the answer; therefore, many did not attempt to apply the subtraction algorithm. Roughly half of the class did not demonstrate a scheme for estimating fractions. One element of an informal treatment of fraction operations involves activities in which students are given fractions and asked to indicate whether the given fraction is closer to 0, $\frac{1}{2}$, or 1. Estimating activities that allow the student to determine the relative size of fractions should later be extended to estimating sums, differences, products, and quotients of common fractions. These informal estimating activities are valuable precursors to a more formal treatment of fraction operations. Essential to the development of computational algorithms is the ability to estimate the outcome of basic operations on fractions; students need to be able to predict a sum, difference, product, or quotient with a relative degree of accuracy.

D. Write the fractions $\frac{4}{7}, \frac{5}{9}, \frac{3}{5}$ in order from least to greatest.

Forty-three percent of the students did not order the fractions correctly.

1. Of the 62 students who ordered the fractions incorrectly, 39 reversed the order.

Seven students of the 81 who ordered the fractions correctly used a least common denominator, while the rest of the students relied upon their intuition. It is interesting to note that most students saw this example as an exercise to estimate the relative size of fractions, but failed to see the same connection in the previous example.

E. If n gets very large, then $\frac{1}{n}$

a) gets very close to 1

b) gets very close to 0

c) gets very large too

Fifty percent of the students did not answer correctly.

1. Thirty of the students claimed that $\frac{1}{n}$ gets very close to one.
2. Thirty of the students claimed that $\frac{1}{n}$ gets very large.
3. Twelve did not even offer a guess.

The results indicate that half of the students either guessed correctly or were able to generalise the effect of a continually increasing denominator. All students need to be provided with experiences that are aimed at developing inductive reasoning skills. In the present example, students could experiment with several cases using ever-increasing whole number denominators while observing the decreasing nature of the quotients. Given this experience students would be ready to generalise the nature of their observations and make a logical conclusion, thus developing a skill that is at the heart of algebra (Wu, 2001, p. 14).

Category VI: Computational fluency

The following questions were specifically designed to determine if students have control over fraction concepts and algorithms that would allow them to demonstrate fluent computation in unfamiliar contexts. These five questions should be the strongest predictor for success in both elementary and intermediate algebra. Wu (2001) believes that fluent computation with numbers lies at the foundation of symbolic manipulation, which is an integral part of proficiency in algebra (pp. 10, 13). Students cannot rely upon memorised algorithms to solve these problems.

A. Find what $\dfrac{7}{\frac{3}{5}}$ is equal to.

Eighty-five percent of the students did not correctly interpret the complex fraction and perform the correct operations.

1. Four students wrote the equivalent fraction
$$\dfrac{14}{\frac{6}{10}}$$
2. Five students gave a response of $7\frac{3}{5}$.
3. Six students gave a response of $\frac{7}{15}$.

The first response is true but does not recognise the fraction as a division operator. The second response is indicative of the 109 incorrect responses, which demonstrated a complete lack of understanding. The third error type supposes a commutative property for division.

The results may be a symptom of inexperience. If students do not master basic operations with common fractions, then teachers are less likely to provide experiences that extend computation to operations on complex fractions. When an abundance of experience with complex fractions is neglected, students may be unable to acquire the familiarity and fluency that will be needed to manipulate algebraic fractions.

B. How many twelfths does $2\frac{1}{4}$ equal?

a) 28 b) 27 c) 25 d) 16 e) 12

Thirty percent of the students selected the correct response.

1. Forty-one students selected (e) as the correct answer, 24 students selected (d), nine selected (a), nine selected (c), and 17 did not select an answer.

In most cases, the students did not show their work, making it impossible to ascertain the type of error that was made, but the results indicate an overall lack of understanding.

This problem could be solved by simply applying basic partitioning strategies. The students could draw a number line using twelve intervals between each whole number, and then count the intervals from 0 to $2\frac{1}{4}$; no computation is necessary.

Looking ahead to learning algebra, students need to firmly understand two basic concepts involved in this question. First, they need to understand that $2\frac{1}{4}$ means $2 + \frac{1}{4}$ and second, if the fractional unit is given as 12, then the sum can be written as $\frac{24}{12} + \frac{3}{12}$; therefore, they can conclude that there are 27 twelfths. Eventually this process can be generalised and applied to algebraic fractions.

C. Find the sum $\dfrac{7+5}{3+5}+\dfrac{\frac{5}{6}}{\frac{5}{3}}$

Eighty-nine percent of the students could not find the correct sum.

1. Seventy students did not attempt to find the sum.
2. Fifty-seven made manipulation errors.

Of the 57 students who calculated an incorrect sum, there were forty different algorithmic variations that yielded over forty different sums. Most of the answers were unreasonable. It appears that these students do not possess the conceptual understanding necessary to make a reasonable estimate of the sum (see Category V, question C). Most of these students found that

$$\frac{7+5}{3+5}=\frac{12}{8}$$

but were unable to manipulate the quotient of fractions. The quotient

$$\frac{\frac{5}{6}}{\frac{5}{3}}$$

is written in a form that is not usually associated with the "invert and multiply" algorithm; consequently, students were confused, applying a variety of incorrect algorithms. The form $\frac{5}{6} \div \frac{5}{3}$ would be more likely to elicit the correct algorithmic response. Once again the emphasis should be on providing students with more opportunities in manipulating complex fractions (see discussion on Category VI, question A).

D. Simplify $\dfrac{1}{\frac{1}{2}\times\frac{1}{3}}$

Seventy-nine percent of the students could not simplify the complex fraction.

1. Twenty-three students committed algorithmic errors in finding the product of the denominator.
2. Fifty-three errors were made in trying to find the quotient of

$$\frac{1}{\frac{1}{6}}$$

The errors made in finding the product of the denominator involve the same error patterns that were evident in previous problems pertaining to the product of fractions (see the discussion in Category I, question C).

The error made in finding the quotient may be a failure to recognise the numerical definition of the reciprocal. The failure could stem from confusion relating to the familiar shortcut, "flip the fraction to find the reciprocal," which may serve the student temporarily, but ultimately provides no understanding of the essential concept. The reciprocal of x is defined as $\frac{1}{x}$. This definition can be developed over time, first with the natural numbers and then extended to rational numbers. This approach is preferred to the treatment of the reciprocal concept as an isolated step in the "invert and multiply" method for dividing fractions; a teacher-taught algorithm that students are required to memorise. Anecdotal evidence indicates that division of fractions makes little sense to both teacher and student alike.

Students with experience in partitioning and unitising activities, however, do not have to search their memories to find the correct algorithm. Their experience would direct them to find the number of $\frac{1}{6}$ units that are in 1 (Lamon, 1999, pp. 39–58).

E. In a school election,
candidate A got $\dfrac{1}{3}$ of the votes,
candidate B got $\dfrac{9}{20}$ of the votes,
and candidate C got $\dfrac{2}{15}$ of the votes.
What fraction of the votes did candidate D get?

Ninety-two percent of the students did not find the correct fraction of votes.

1. Forty-one students failed to find the correct number of votes after finding the least common denominator.
2. Ten of the previous 41 students got the correct fraction sum, but did not subtract the sum from one.
3. Only half of the students who answered correctly wrote the fraction in lowest terms.

It should be noted that only 36% of the students recognised that addition was one of the operations needed to solve the problem. These students were able to find a common denominator, but many could not apply the rest of the addition algorithm correctly. There were only 21 students who were able to apply the algorithm accurately and get the correct sum, but 10 of these students did not subtract the sum from one; giving the unreasonable answer of $\frac{11}{12}$, which is the fraction of votes for the other three candidates.

The fact that only 11 out of 143 students were able to solve this problem emphasises the need to provide numerous experiences that will allow students to develop strategies for solving word problems that involve fractions. Students at this stage in elementary algebra have been introduced to solving problems by defining a variable and then setting up an equation, yet none of the students in this sample opted to use algebra.

It should be noted that this problem could be solved visually, without applying the addition algorithm and without algebra. If students were able to partition a number line from 0 to 1 using 60 intervals, the solution can be reduced to the simple process of counting.

Discussion of error analysis

The previous error analysis indicated that most of the students in the sample had a fragmented understanding of fraction concepts and operations. In test items that could be solved by directly applying a concept (Category V), most of the students opted to use an algorithm. The results show that many who chose to use an algorithm were not really sure of the correct process, demonstrating several of the error patterns catalogued by Benander and Clement (1985). Many of the errors produced unreasonable answers, indicating that doing an operation on fractions is not connected to understanding the operation. For example, if a student insists that $\frac{1}{2}$ of $\frac{2}{3}$ is equal to $\frac{3}{4}$ by misapplying a shortcut division algorithm, then it is evident that he or she does not understand the relative size of fractions or the function of $\frac{1}{2}$ as a multiplicative operator.

Capable high school students often complain that they cannot do fractions; fluency with rational numbers means doing with understanding. Regrettably, many students are taught algorithms before they have had the time to develop the fundamental concept. Their only alternative when confronted with fraction operations is to match what is being presented with one of the disconnected, previously-memorised algorithms from earlier mathematics experiences. If the situation being presented is novel or is not in a recognisable form, then a student's best effort is no more than a good guess. The errors that were made repeatedly demonstrate that a good guess is not sufficient.

The results of the error analysis reveal an overall lack of experience with basic fraction concepts — experience that should have been gained through an informal treatment of fractions providing an abundance of concrete referents. Only a few students used pictorial representations to help them answer some of the questions. A few more students were able to apply the concept directly and provide the correct answer without resorting to an algorithm. For most of the students, however, the strategy of choice was to select an algorithm and then use it. This approach yielded a host of illogical answers, which more often than not went undetected by the student. Such inconsistent results demonstrate a lack of fluency with fraction computation, the fluency that becomes a necessity when students begin to work with algebraic fractions.

Implications for practice

The results of this analysis magnify the existence of a problem in the learning of mathematics that must be rectified. The error analysis revealed a large number of misconceptions that students have related to the subject of fractions. These misconceptions must be addressed. Research designed to examine the effect of treatment on a given population over an extended period of time needs to be organised. Below are recommendations which were prompted by the review of the literature and the findings of this study. For example, Bezuk and Cramer (1989) offer a few general recommendations, which are echoed in much of the literature concerned with the teaching of fraction concepts: (1) the use of manipulatives is fundamental in developing students' understanding, (2) the majority of the time spent on fractions before grade 6 should be devoted to developing a conceptual base of fraction relationships, (3) operations on fractions should be delayed until students have a solid understanding of order and equivalence of fractions, and (4) the size of the denominator for computational exercises should be 12 or below (p. 158).

The following recommendations are intended for implementation in longitudinal studies, which track the development of the fraction construct over time.

1. Children in the early primary grades should be allowed the time to develop whole number concepts and whole number operations informally with abundant concrete referents. Arabic symbols should be used for counting purposes only and always connected to concrete objects or pictorial representations. Informal practice with fraction concepts should be limited to experiences that arise naturally, like fair sharing or situations that involve money. Lamon (1999) claims that studies have shown that if children are given the time to develop their own reasoning for at least three years without being taught standard algorithms for operations with fractions and ratios, then a dramatic increase in their reasoning abilities occurred, including their proportional thinking (p. 5).
2. Upper primary students should be given experiences that extend the whole number concept with an eye toward algebra involving an informal treatment of the field properties. These students need to be provided with experience in partitioning as a method for solving verbal problems involving fractions (Lamon, 1999; Huinker, 1998). The informal treatment of fractions should include manipulation of concrete objects and the use of pictorial representations, such as unit rectangles and number lines. Fraction notation must be developed, but formal fraction operations using teacher-taught algorithms should be postponed. Learning the subject of fractions will revolve around informal strategies for solving problems involving fractions. The objective at this level is to build a broad base of experience that will be the foundation for a progressively more formal approach to learning fractions.
3. In middle school, the development of fraction operations as an extension of whole number operations should provide experiences that guide and encourage students to construct their own algorithms (Lappan & Bouck, 1998; Sharp, 1998). More time is needed to allow students to invent their own ways to operate on fractions rather than memorising a procedure (Huinker, 1998). Progressively this development should lead to more formal definitions of fraction operations and algorithms that prepare students for the abstractions that arise later in the study of algebra (Wu, 2001). How fractions should be

taught is inexorably linked to when the concepts are being presented and what impact the learned concepts will have on future mathematics courses.

Conclusion

The lack of experience with both fraction concepts and fraction computation is inexcusable. Prior to entering the ninth grade, students should have had at least two years for an informal treatment of fraction concepts and three years for the development of formal concepts and computational fluency. The rational number concept is pivotal to extending whole number concepts while building fraction concepts, which can then be extended to form algebraic concepts. The process is one of generalisation. Wu (1999) prescribes numerous experiences with generalising number concepts, but an individual cannot generalise a concept if the concept is not understood. This is a genuine problem in mathematics education. Scores from the NAEP (Mullis et al., 1991; NCES, 2000) indicate that the problem is perpetual. The "Nation's Report Card" persistently demonstrates extremely low achievement in both fraction and algebraic concepts for the age seventeen student.

It is the view of this study that the lack of experience is a problem that will not be resolved unless the philosophy of American mathematics education undergoes a dramatic reformation. A philosophy that actively promotes breadth of learning over depth of learning exacerbates a problem, which even the best pedagogy cannot overcome. A philosophy that seemingly ignores established guidelines regarding a child's cognitive development (Wadsworth, 1996) and forces children into the belief that learning mathematics is memorising facts and algorithms is worse than a problem. It causes children to lose control over numbers and to perceive doing mathematics as a drudgery.

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