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Capitalizing on Advances in Mathematics and K-12 Mathematics Education in Undergraduate Mathematics: An Inquiry-Oriented Approach to Differential Equations

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This paper provides an overview of the Inquiry-Oriented Differential Equations (IO-DE) project and reports on the main results of a study that compared students' beliefs, skills, and understandings in IO-DE classes to more conventional approaches. The IO-DE project capitalizes on advances within mathematics and mathematics education, including the instructional design theory of Realistic Mathematics Education and the social negotiation of meaning. The main results of the comparison study found no significant difference between project students and comparison students on an assessment of routine skills and a significant difference in favor of project students on an assessment of conceptual understanding. Given these encouraging results, the theoretical underpinnings of the innovative approach may be useful more broadly for undergraduate mathematics education reform.

Key Words: Undergraduate, Differential equations, Realistic Mathematics Education

The recent report by the International Commission on Mathematical Instruction on the teaching and learning of university level mathematics argues that universities are now facing many new challenges to which the community needs to respond in innovative and theoretically grounded ways (Holton, 2001). One major challenge comes from the fact that universities are now accepting a much larger and more diverse group of students. Consequently, the educational issues facing universities have changed, introducing new pedagogical challenges. One response to these challenges is to develop new curricular and instructional approaches based on contemporary theories of learning and instructional design. One such innovative approach, referred to as the Inquiry-Oriented Differential Equations (IO-DE) project, capitalizes on advances within the discipline of mathematics and on advances within the discipline of mathematics education, both at the K-12 and tertiary levels.

This paper provides an overview of the IO-DE project and reports on the main results of a study that compared students' beliefs, skills, and understandings in the IO-DE project to more traditional approaches to differential equations (TRAD-DE). Because universities have begun to adopt a role more like that of the school system and less like the elite

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institutions of the past (Holton, 2001), we argue that approaches such as the IO-DE project that seek to find commonality between undergraduate and K-12 mathematics education will contribute more, in the long run, to sustainable improvements in students' mathematical learning. Thus, an analysis of whether or not an innovation such as the IO-DE project is successful in terms of what students learn is significant because it may serve as a useful model for other undergraduate course reforms.

We next describe the theoretical underpinning for the IO-DE project and point the reader to a number of previously published illustrative examples of the instructional materials. This is followed by a description of the research methods and the main results of the comparison study.

Background Theory

From the Discipline of Mathematics

Taking a historical point of view reveals the centrality of finding closed form symbolic expressions for solutions to differential equations. This traditional approach, however, is restrictive in the types of differential equations that can be studied. In particular, most nonlinear differential equations (which are the most interesting for science and engineering applications) cannot be solved with the cadre of available analytic techniques. From a dynamical systems perspective, however, all differential equations, not just specially defined classes of equations amenable to analytic techniques, are candidates for analysis. Drawing on a dynamical systems point of view (Artigue & Gautheron, 1983; Blanchard, Devaney, & Hall, 2002; Kallaher, 1999; West, 1994), the IO-DE project treats differential equations as mechanisms that describe how functions evolve and change over time. Interpreting and characterizing the behavior and structure of these solution functions are important goals, with central ideas including the long-term behavior of solutions, the number and nature of equilibrium solutions, and the effect of varying parameters on the solution space.

For example, in one IO-DE instructional sequence students actually (re)invent a bifurcation diagram as a way to illustrate the impact of varying the parameter α in the differential equation $\frac{dP}{dt}=0.2P(1-\frac{P}{25})-\alpha$ CC that models a fish harvesting scenario. The bifurcation diagram reinvention culminates an instructional sequence in which students create and use graphs of autonomous differential equations (e.g, dP/dt vs. P) and phase lines to analyze the behavior of solutions (See Stephan & Rasmussen (2002), for an analysis that includes student reinvention of a bifurcation diagram). Also consistent with a dynamical systems point of view, the IO-DE approach spends a considerable amount of time on systems of differential equations, which are the appropriate method to study higher order differential equations

Addressing these central ideas draws on a variety of techniques made viable with the use of technology (Kallaher, 1999). These techniques utilize different graphical representations, such as slope fields for first order differential equations and vector fields for systems of differential equations, as well as numerical algorithms such as Euler's method for producing approximate solutions. The approach we take in the IO-DE project is that graphical and numerical approaches should not be taught as ends in and of themselves, but rather should emerge as tools for students as they solve challenging problems.

From the Discipline of Mathematics Education.

At the outset of the IO-DE project we conjectured that theoretical advances originating from K-12 classroom based research would be useful for informing and guiding the learning and teaching of undergraduate mathematics. There are two complementary lines of K-12 research from which we draw: the instructional design theory of Realistic Mathematics Education (RME) (Freudenthal, 1991; Gravemeijer, 1999) and the social production of meaning (Cobb & Bauersfeld, 1995).

Central to RME is the design of instructional sequences that challenge learners to organize key subject matter at one level to produce new understanding at a higher level. In this process, referred to as mathematizing, graphs, algorithms, and definitions become useful tools when students build them from the bottom up through a process of suitably guided reinvention (for illustrative examples and further theoretical development, see Kwon 2003; Rasmussen & Keynes, 2003; Rasmussen & King, 2000; Rasmussen & Marrongelle, in press; Rasmussen, Zandieh, King, & Teppo, 2005).

The mathematization process is embodied in the core heuristics of guided reinvention and emergent models. Guided reinvention speaks to the need to locate instructional starting points that are experientially real to students and that take into account students' current mathematical ways of knowing. One aspect of the reinvention principle entails examination of students' informal solution strategies and interpretations that might suggest pathways by which more formal mathematical practices might be developed. Another aspect of the reinvention principle involves looking at the history of mathematics for insights into how particular mathematical concepts and practices evolved over time as well as potential barriers and breakthroughs. The IO-DE project further expanded the reinvention principle to include possible starting points due to technology that historical examinations would not reveal (Rasmussen & King, 2000).

The learning environments of the past can be significantly different from those of today (Kaput, 1997) and therefore being open to the possibilities afforded by technology alerts one to prospects that otherwise might not be evident. We designed pedagogically oriented Java applets to support the development of a dynamical systems point of view, including the concepts of exact and approximate solutions and eigenvectors and eigenvalues. For example, Rasmussen and Keynes (2003) illustrate how students use a Java applet to reinvent the notion of eigenvalues and eigenvectors. One of the interesting findings reported by Rasmussen and Keynes is that students' activity with eigenvectors conceptually and symbolically precede eigenvalues. This is a reversal of the traditional and historical approach where eigenvalues precede eigenvectors.

Finally, the heuristic of emergent models highlights the need for instructional sequences to be a connected, long-term series of problems in which students create and elaborate symbolic models of their informal mathematical activity (Gravemeijer, 1999). The use of the term model is an overarching idea that encompasses students' evolving activity with a chain of symbols, such as number tables, algorithms, graphs, and analytic expressions. From the perspective of RME, there is not just one model, but a series of models where students first develop *models of* their mathematical activity in an experientially real task setting, which later becomes *models for* reasoning about mathematical relationships.

As researchers working within an RME approach emphasize, a coherent sequence of learning tasks does not guarantee that students will learn mathematics with understanding (Graveimeijer, Cobb, Bowers, & Whitenack, 2000; Treffers, 1987). In addition to theoretically informed design with extensive classroom testing, the IO-DE project works from the premise that the way in which instructional tasks are constituted is as important as the material itself, and it is toward this aspect that we now turn.

An explicit intention of IO-DE project classrooms is to create a learning environment where students routinely offer explanations of and justifications for their reasoning. Following Richards (1991) and Cobb, Wood, Yackel, and McNeal (1992), we define such learning environments as "inquiry-oriented." Because of the strong emphasis on argumentation in inquiry-oriented classrooms, we conjectured that the theoretical constructs arising from research in inquiry-oriented elementary school classrooms would be useful for learning advanced mathematics, such as differential equations. After all, mathematicians engage in similar forms of argumentation when creating new mathematics (Richards, 1991).

Specifically, we found the constructs of social norms and sociomathematical norms (Yackel & Cobb, 1996) useful because they offer a way of thinking about the multiple and complementary roles of argumentation as a means to conceptualize processes by which teaching mathematics for understanding can occur (see also Rasmussen, Yackel, & King, 2003; Yackel & Rasmussen, 2002). Social norms pertaining to explanation and justification are discursive regularities that are constituted and reconstituted through ongoing interaction. Being intentional about fostering such norms is a pedagogical goal of IO-DE project classroom teachers. Examples of social norms include students' routinely explaining their thinking, listening to and questioning others' thinking, and responding to others' questions and challenges. Sociomathematical norms refer to criteria for that what counts, for example, as an acceptable explanation, a different explanation, and as an elegant justification. Like social norms, sociomathematical norms are not rules set out in advance but rather emerge as joint accomplishments.

Research focusing on student conceptions in differential equations also pointed to a number of relevant issues for instructional design and teacher planning. For example, in one case study of a differential equations class that treated contemporary topics in dynamical systems, Rasmussen (2001) found that rather than building relational understandings (Skemp, 1987), students were learning analytical, graphical, and numerical methods in a compartmentalized manner. An important lesson from this research is that working with multiple modalities does not guarantee that students will build a coherent network of ideas. It is also important to have a long-term, coherent sequence of tasks, and our adaptation of RME was useful for this purpose.

Other informative research on student cognition in differential equations highlights students' concept images of Euler's method and students' informal or intuitive notions underlying equilibrium solutions, asymptotical behavior, and stability (Artigue, 1992; Rasmussen, 2001; Zandieh & McDonald, 1999). Knowledge of such informal or intuitive images was useful for the IO-DE project because it suggested task situations and instructional interventions that could engage and help reorganize students' informal and intuitive conceptions.

Method

We collected data on the IO-DE project and TRAD-DE students' beliefs, skills, and understandings in differential equations at two different US sites, one located in the Midwest and one located in the Northwest, and at one site in South Korea. Students at all sites were primarily engineering or mathematics majors. At the Northwest and Korean sites TRAD-DE and IO-DE students attended the same university. At the Midwest site IO-DE and TRAD-DE students came from nearby universities because no one university had both IO-DE and TRAD-DE classes at the time we conducted the study.

The instruments

The beliefs instrument used was adapted from the Views About Mathematics Survey (VAMS) developed by Carlson (1997; 1999). VAMS items are grouped in two broad dimensions: philosophical and pedagogical. The philosophical dimensions pertain to (a) the structure of mathematical knowledge, (b) the methods of mathematics, and (c) the validity of mathematical knowledge. The pedagogical dimensions pertain to (a) the learnability of mathematics, (b) the role of reflective thinking, and (c) personal relevance of mathematics. Each item consists of a statement followed by two contrasting alternatives which respondents are asked to balance on a five-point scale. A more detailed description of the validity and reliability of VAMS items are reported elsewhere (Halloun, Carlson, & Hestenes, 1996).

The survey was administered at the beginning and end of the semester to all IO-DE and TRAD-DE students. We also developed two instruments to inform us about student learning of important skills and concepts. The first instrument, the routine assessment, consisted of eight items that covered some analytic methods of solving differential equations and other routine topics. The second instrument, the conceptual assessment, also consisted of eight items designed to reflect relational understandings (Skemp, 1987). Readers interested in the items are referred to Kwon, Rasmussen, and Allen (2005) or they may contact the first or second author.

To the best of our ability we developed the routine and conceptual assessments so that they would be fair for all students. In particular, we did not include any items on the assessment instruments that reflected topics and ideas unique to a dynamical systems point of view. In keeping with this spirit, we asked several mathematicians with expertise in differential equations to review the two assessments. These reviewers informed us that the items we developed represented an important collection of skills and understandings for students, regardless of the nature of the curriculum used in class.

We administered the routine assessment items as part of students' regular final examination[1]. Our decision to incorporate the routine assessment items into students' final exam meant that students came prepared to take the exam and do their best. Another consequence of this choice is that since we were imposing on the goodwill of the teachers to put these items on their exams, we asked them to select as many of the eight items on the routine assessment that they felt were fair for their students. As a result, not every site used all eight items from the routine assessment, but we can be confident that those items that were used were relevant to students. The main disadvantage of this choice is that we had less control over the administration of the final exam. However, there was a great deal of similarity in test conditions across sites, minimizing the impact of any differences.

The conceptual assessment was administered to volunteers after the final exam and lasted 60 minutes. At the Korean site, three items were removed because the concepts assessed in these three items were not covered in the TRAD-DE class. At the US sites, all eight items on the conceptual assessment were used since the concepts and methods of analysis were, according to the teachers of the TRAD-DE and IO-DE classes, relevant to what students had learned.

After students completed the assessments we coded each paper and removed student names so that scoring of the papers was blind. We developed rubrics for scoring the routine and conceptual assessments and two project team members scored each problem. A third project team member resolved differences that could not be resolved by the two scorers.

Site Descriptions

IO-DE project classes at all sites typically followed an inquiry-oriented format in which students routinely offered explanations and justifications for their ideas as they cycled between small group work and whole class discussion. These types of social norms reflect the IO-DE project's explicit attention to argumentation, as previously described in the background theory section. Course materials for the IO-DE project grew out of several semester long classroom teaching experiments and represent an innovation adaptation of RME to the undergraduate level. The textbooks used at the TRAD-DE sites varied, but all TRAD-DE sites shared in common a traditional, lecture-style format. Thus, since our interest was in comparing student learning in the IO-DE approach versus traditional approaches, rather than one specific traditional textbook, we sought to find comparable groups of students at the same or similar institutions rather than to match the traditional textbook across sites. This methodological choice is consistent with other comparison studies (e.g., Huntley et al, 2000).

At the Midwest site, students in the IO-DE project attended a mid-sized public institution with an open admission policy. There were two IO-DE project teachers at the Midwest site. One of the teachers had regularly taught differential equations for more than 10 years, but this was his first time teaching with the IO-DE project materials. Prior to this course all of his teaching had been conducted using a traditional lecture-style format. The other IO-DE project teacher was a recent Ph.D in mathematics with post-doctoral experience in mathematics education and this was his third time teaching differential equations with IO-DE project materials. The two teachers at the Midwest TRAD-DE site also routinely taught differential equations. Unlike the project class that was a 3 credit hour course, the TRAD-DE class was a 4 credit hour course that included treatment of linear algebra. The textbook used at the Midwest TRAD-DE site was *Differential Equations and Linear Algebra* (2nd ed.) by Farlow, Hall, McDill, and West (2002). Approximately two-thirds of the course was devoted to differential equations, which is roughly equivalent to the amount of time the IO-DE project students spent studying differential equations. Students in both Midwest TRAD-DE classes were allowed to use calculators for all work outside of class and they often used a computer algebra system in class.

At the Northwest site, students in both the TRAD-DE class and the IODE project class attended a large state university. The IO-DE project teacher was a recent Ph.D. in mathematics education and this was her second time teaching differential equations with the IO-DE materials. A teacher with more than 10 years of experience teaching differential equations taught the TRAD-DE class and the course used the text, *Differential Equations* (2nd ed.), by Blanchard, Devaney, and Hall (2002), which treats the subject of differential equations from a dynamical systems point of view, but does not integrate the other innovative aspects of the IO-DE project. Students in the TRAD-DE class were assigned three lab projects during the term in which they used computer software specific to differential equations. This software was not used on their regular exams.

At the South Korean site, students in both the TRAD-DE and the IO-DE classes attended a prestigious woman's university in South Korea. The textbook used in the TRAD-DE class was *Fundamentals of Differential Equations and Boundary Problems* (3rd ed.), by Nagel and Saff (1993). The TRAD-DE teacher had 24 years of teaching experience on advanced calculus, real analysis, and differential equations. No technology of any type was used. The IO-DE project teacher had regularly taught differential equations for 10 years, and this was her third time with the IO-DE project materials.

Main Results

To ensure comparability in mathematical expertise, we collected baseline data on students' previous mathematical understanding. We were not able to collect baseline data on all students who took the assessments and therefore we have excluded these students from the analyses. At the Korean site, we used the students' mathematics score on their College Scholastic Ability Test (CSAT); at the Midwest site, we used

Table 1. *Mean Scores of IODE and TRAD-DE Groups' baseline data*

	Korean Site		Midwest Site		Northwest Site	
N	IO-DE	TRAD-DE	IO-DE	TRAD-DE	IO-DE	TRAD-DE
	15	20	12	30	38	25
M	91.69	88.97	580.83	612.33	2.98	2.97
(*SD*)	(5.29)	(7.68)	(104.83)	(80.42)	(.50)	(.69)
t	1.18		-1.05		.10	
Sig.	.25		.30		.92	

the students' mathematics score on their SAT; at the Northwest site, students were not required to take the SAT for college entrance and therefore we used their averaged differential and integral calculus grade on a four point scale. We conducted t-tests on the comparability of IO-DE and TRAD-DE students at each site and found there was no significant difference in students' mathematical background knowledge.

The *M* Row of Table 1 shows the mean percentile mathematics score on the CSAT for the Korean site, the mean score on the SAT for the Midwest site, and the averaged grade on calculus courses for the Northwest site. There were no significant differences between the two groups for each site respectively, providing evidence that the two groups had comparable mathematics background before the treatment.

Student Views About Mathematics

An analysis of variance on the pre-instruction VAMS data indicated that the two groups started their differential equations course with similar mathematical views about knowing and learning mathematics. As shown in Table 2, the mean of each group was 3.55, where a score of 5 indicates a high level of consistency with the views of professional mathematicians (experts). An analysis of covariance was conducted on the posttest scores using the pretest scores as a covariate, testing for the effects of group (IO-DE and TRAD-DE). Table 2 summarizes mean posttest scores, adjusted on the basis of pretest scores, for the two groups.

The adjusted mean scores in the IO-DE group are higher than in the TRAD-DE group, but this did not reach a level of any significance. However, in one of the IO-DE classes from this study, Ju and Kwon's (2003) classroom discourse analysis found that students' ways of talking about mathematics, especially in terms of how to do mathematics, changed from the view of mathematics as independent of, external to, and superior to the students' minds to that of mathematics as a product of their own engagements. Specifically, these researchers found a change in students' discourse patterns from a third person perspective to a first person perspective, indicating evolving ownership, interest, and relevancy of mathematics to the students. In other research in an IO-DE classroom not part of this study, Yackel and Rasmussen (2002) document how students' evolving beliefs about their ability to create mathematics and the role of explanation and justification in learning mathematics are reflexively related to the social and sociomathematical norms of their classroom communities. Thus, the beliefs instrument we used may not have been sensitive enough to detect such changes.

Routine Assessment

The routine assessment consisted of five problems of an analytic nature, one numerical problem used to compute two steps of Euler's method, one phase plane graph sketching problem, and one modeling problem. The maximum score on each problem was 5. Because of the emphasis on analytic techniques, the staple of conventional approaches to differential equations, we predicted that the TRAD-DE group would perform better than the IO-DE group. As shown in Table 3, however, there was no significant difference between the two groups when all eight problems were combined. IO-DE project students were equally proficient at routine problems, even though this was not a primary instructional focus.

An item-by-item analysis found that there was no significant difference (at the 0.05 level) on seven of the eight problems. On the one problem for which there was a

Table 2. *Pre- and Post-Scores of IODE and Comparison Groups on VAMS (All Sites Combined)*

Treatment	*N*	Initial Mean (Std. Dev.)	Adj. Mean	*F*	Sig.
IO-DE	60	3.51 (.34)	3.50	2.03	.16
TRAD-DE	7[illegible]	3.43 (.32)	3.40		

Table 3. *Mean Scores of IO-DE and TRAD-DE Groups on Routine Tests (All Sites Combined)*

Treatment	*N*	Mean (0-5)	Std. Dev.	*t*	Sig.
IO-DE	65	3.43	.94	.60	.55
TRAD-DE	83	3.52	.89		

Table 4. *Mean Scores of IO-DE and TRAD-DE Groups on Conceptual Tests*

Treatment	*N*	Mean (0-5)	Std. Dev.	*t*	Sig.
IO-DE	30	2.88	1.09	9.70	.00
TRAD-DE	42	.70	.82		

difference, computing two steps of Euler's method, the IO-DE students outperformed the TRAD-DE students (mean score of 3.29 versus 1.14). A plausible explanation for this result is that IO-DE project students actually reinvented Euler's method, whereas we suspect that this method was given to TRAD-DE students as an algorithm to follow. Reinventing this algorithm, versus appropriating it, could therefore account for the fact that IO-DE students were better at carrying out the procedure than students in the TRAD-DE classes. IO-DE project students did not, however, reinvent analytic techniques, such as separation of variables, and thus the fact that there were no statistically significant differences on these problems is consistent with this interpretation.

Conceptual Assessment

The conceptual assessment was aimed at evaluating students' relational understandings of important ideas and concepts. Two problems focused on the meaning or relationship between exact and approximate solutions, two problems focused on aspects related to modeling, and four problems dealt with the behavior or structure of solutions and/or the solution space. Because the IO-DE project was designed to foster conceptual development, we predicted that the IO-DE students would perform better on the conceptual assessment than the TRAD-DE students. As shown in Table 4, the IO-DE group did score significantly higher than the TRAD-DE group.

In light of the result that IO-DE project students' mean score was more that 2 points (out of a possible 5) higher than students in the TRAD-DE classes, one might expect that the IO-DE students would perform better than the TRAD-DE students on each item. Indeed, an item-by-item analysis reveals that IO-DE project students scored better than the TRAD-DE students on all eight items, with significant differences (at the 0.05 level) on seven of the eight problems.

Concluding Remarks

The effect of the IO-DE project in maintaining desirable mathematical views and in developing students' skills and relational understandings as judged by the three assessment instruments was largely positive, despite the fact that IO-DE instructors in general had less experience than the TRAD-DE instructors. These findings support our conjecture that, when coupled with careful attention to developments within mathematics itself, theoretical advances that initially started in elementary school classrooms (and which are beginning to spread to the rest of K-12) can be profitably leveraged and adapted to the university setting. As such, our work in differential equations may serve as a model for others interested in exploring the prospects and possibilities of improving undergraduate mathematics education in ways that connect with innovations at the K-12 level. Too often we hear teachers say that such and such an idea and approach to teaching and learning is fine at this or that level, but would not be appropriate for their level. We hope that this study calls into question such statements.

In our work we found that the heuristics from the instructional design theory of Realistic Mathematics Education and the constructs of social norms and sociomathematical norms constitute a useful collection of theoretical ideas that can inform and guide undergraduate mathematics education. Certainly these are not the only possibly useful theoretical ideas. Our position is, however, that innovations are more likely to result in positive outcomes if they are theoretically-driven rather than driven by the use of technology or collaborative learning, for example.

Given the promising main results, we are currently engaged in further, more focused analysis of these data. For example, we are analyzing the different strategies used by IO-DE and TRAD-DE students. We suspect that IO-DE students used graphical, numerical, and analytic techniques more flexibly, but further analysis is needed to support or refute this conjecture. We are also looking for patterns and themes that cut across the routine and conceptual assessment problems. Some of the conceptual problems had a "counterpart" routine problem that will aide in this data snooping effort. Furthermore, Kwon, Rasmussen and Allen (2005) conducted a follow-up study examining retention of knowledge one year after instruction for a subset of the students from this comparison study. They reported that students in the IO-DE group scored significantly higher on the conceptually-oriented items. There was no significant difference on student performance on the procedurally-oriented items, despite the fact the analytic solutions were the main focus of the comparison classes.

Additional significance of this work is evident in the extent to which the IO-DE research program is contributing to advancing the work of teachers and the professional development of mathematicians. For example, Rasmussen and Marrongelle (in press) develop teaching strategies or "tools" that are teaching counterparts to the instructional design theory of RME. Using the IO-DE project as a case example, Wagner, Speer, and Rossa (2006) detail the role of teachers' knowledge in the implementation of curricular reforms.

Finally, we point to some of the limitations of this research. Certainly sample size is an issue. A larger sample size would add power to our findings and allow us to make comparisons across sites. Although we conducted interviews with a number of the IO-DE project students, we were not able to do so with TRAD-DE students. Therefore, an expanded study, in terms of both numbers of students involved and methodologies employed, is warranted from the current study. Such a larger study would also provide opportunities to investigate the processes by which mathematicians adapt such innovations and work toward change in their instructional practice.

Notes

1 There were two exceptions: At the South Korean site a subset of the routine items appeared on IO-DE and TRAD-DE students' midterm since their finals were not cumulative. At one of the two comparison classes at the Midwest site fifteen students responded to three of the routine questions with their post-VAMS survey.

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