A Constructivist Lesson to Introduce ARITHMETIC SEQUENCES with PATTERNS

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Pattern recognition is a critical component of success in mathematics. Students at all levels should be provided with opportunities to investigate and uncover patterns throughout their mathematical careers. Further, they should be allowed to explore situations in which pattern recognition plays a vital role in the construction of important mathematical knowledge. This article will discuss a lesson that introduces arithmetic sequences through a simple, yet rich exploration of a pattern. I have taught this lesson in a methods course for secondary mathematics teachers to provide them with a model of a constructivist lesson that they will be able to implement in their own classrooms when they begin teaching. It is, also, beneficial to this group as it affords them an opportunity to participate in a cooperative learning activity that lends itself to a wide variety of solution methods.

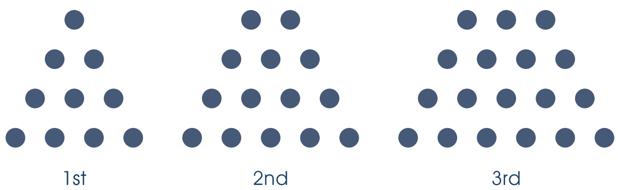

Figure 1. The sequence.

Overview

The lesson is comprised of the following components:

(a) individually, students determine the number of marbles in the 4th term of the sequence (see Figure 1, source unknown);

(b) in groups, students develop a variety of formulas for the nth term;

(c) the whole class shares and discusses the means by which the initial individual solutions to the 4th term were found;

(d) formulas for the nth term solution are considered leading to the construction of the traditional formula for the nth term of an arithmetic sequence;

(e) the whole class discusses the relative advantages and disadvantages of general versus recursive solutions;

(f) the whole class discusses this approach versus a more traditional lesson to introduce sequences;

(g) the whole class reflects on their participation in this constructivist lesson.

Early stages of the lesson

I begin the lesson by asking the students to determine individually as quickly as they can how many marbles would be in the fourth term of the sequence shown in Figure 1. My purpose is not to put them under time pressure but rather to have them find a solution in the manner that occurs most naturally to them. They are asked to keep this result to themselves so that everyone can determine the next term without being influenced by the thoughts of others. After about 30 seconds, I ask the students to record this initial thought process so that it is preserved for a discussion that will take place later in the lesson.

Next, I ask the students to work in groups of four to find the nth term of this sequence. Unfortunately, for some of these preservice teachers, this is the first time they have been asked to work on a mathematical problem in groups. For this reason, despite having been told to work cooperatively, students sometimes continue to work individually with little discussion heard in the classroom. I jokingly announce that I should be hearing more interaction since they are supposed to be working on this problem together. My comments are usually enough to get them jump started into sharing their ideas within the group regarding the determination of an nth term formula for this sequence. At this point, I circulate around the classroom to make sure that all groups are working cooperatively and making progress. Usually, a few of the students want to try to impress the professor rather than make sure that all members of the group understand the solution they have found. Through my interaction with them, I help them to realise that this is not the goal of the activity.

After a group has found a correct solution, I surprise them by asking them to find another solution. If they struggle with this, I suggest they think about how less sophisticated mathematics students might approach the problem. This takes the pressure off students who may feel that their method of solution is not as elegant as the first one that was shown in their group. When a second correct solution has been achieved, I tell them to find even more solutions. The students soon get the message that I will not be satisfied until they have considered this problem from many viewpoints and have correctly solved it in many ways. Depending on what solutions a group has attained, I provide hints of other directions they might consider in looking for yet another solution.

Variety of solutions

When I feel a sufficient variety of solutions have been developed, I ask the students to think back to the way in which each of them initially determined how many marbles were in the 4th term. The preservice teachers who participate in this lesson are amazed at the variety of approaches that were used by their classmates. Despite the simplicity of this pattern, widely different solution methods can be employed. During this discussion, we find that some students found the 4th term by drawing or thinking about what the next term would look like pictorially (see Figure 2). While others counted the number of marbles in each of the first three terms, 10, 14, 18, and then extended this numeric pattern to attain 22. Many recognised a visual pattern that in each successive term an additional “diagonal” of four marbles is added (see Figure 3). Others noticed that the first term is comprised of rows with 1, 2, 3, 4 marbles, the second term with rows of 2, 3, 4, 5 marbles, therefore they surmised that the 4th term will contain rows of 4, 5, 6, 7 marbles, for a total of 22.

The search for the *n*th term, also, provides a rich environment for investigation. Again some students approach the problem from a geometric perspective while others are more abstract in their approach. At this stage, I have each group present and explain a solution on the board. I continue this process until all of the wide variety of solutions have been

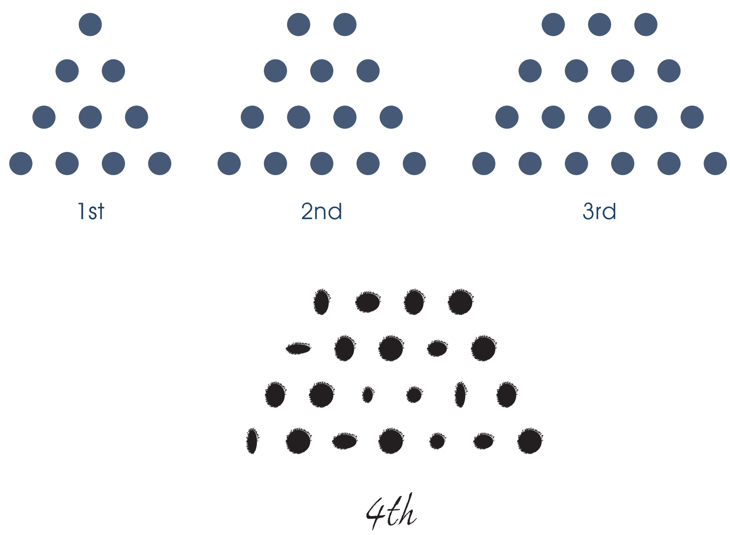

Figure 2. Drawing the next term.

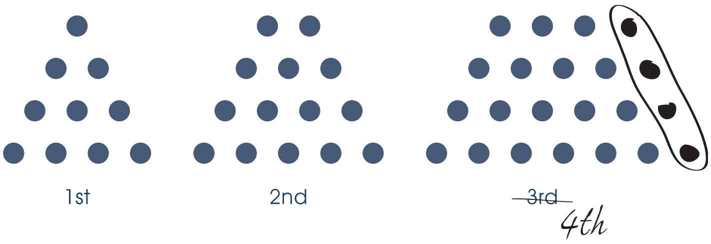

Figure 3. Adding a diagonal.

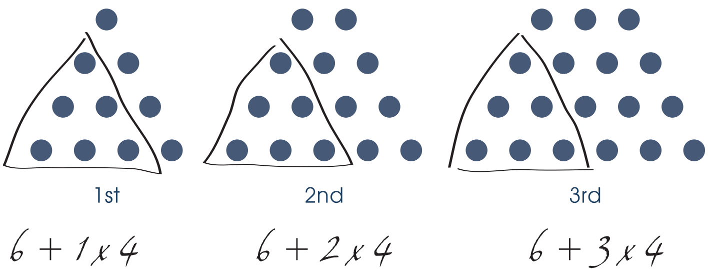

Figure 4. Identifying a base of six marbles.

shown and explained. Many of these solutions parallel approaches that had been used to find the 4th term. Those who visualised an additional diagonal being added often arrive at the formula $4N + 6$, by isolating the 6 marbles in the corner as a base with the number of diagonals added to each term being the same as the number of the term (see Figure 4). Those who found the 4th term by adding

$$4 + 5 + 6 + 7$$

find the same formula by adding

$$(N) + (N + 1) + (N + 2) + (N + 3).$$

Still others start with the first term of 10 and add one less than n diagonals attaining:

$$10 + (N - 1) \times 4.$$

Some simply state that the nth term is found by adding 4 to the 9th term, thus yielding the recursive formula:

$$a(n) = a(n - 1) + 4.$$

After the presentations have been completed, I encourage the students to verify that all of these solutions are equivalent. They are quick to recognise that with the exception of the recursive formula, all of the solutions are mathematically equivalent. At this point, I ask them to name the type of sequence represented by this pattern. Of course, they respond that it is an arithmetic sequence. I suggest that if they were teaching this lesson to a class that had not previously seen arithmetic sequences, this would be the time when appropriate terminology would be introduced. In particular, the terms arithmetic sequence, common difference, and first term would be discussed in light of the conceptual knowledge that has been gained during the lesson. Thus, I recommend that the terminology be developed after the students have begun to understand the concepts of arithmetic sequences and *not* before.

I continue the discussion by asking the class to think about which of the formulas that have been developed would be most useful. I suggest that a formula in which the values of the newly learned terminology are prominent would be best. This leads them to see that $10 + (n - 1) \times 4$ is the most logical since the first term, 10, and the common difference, 4, can be identified easily in this formula. Of course these students have seen this formula before, making this stage of the lesson somewhat contrived. The same connection can easily be attained with a group of high school students who have not encountered arithmetic sequences prior to this lesson. The culmination of this stage of the lesson is the development of the generalised formula,

$$a(n) = a(1) + (n - 1)d.$$

General versus recursive solutions

The inclusion of a recursive means of finding the nth term provides for a good opportunity to discuss the relative advantages of general versus recursive formulas. Until recently, the development of a general term formula was generally accepted as the ultimate or best solution to a problem of this nature. Indeed, the typical high school curriculum is virtually devoid of any mention of recursive formulas. With the increase in speed and power of computers, however, iterative algorithms are regaining importance and need to be considered in the high school classroom when an appropriate opportunity arises. The relative importance of iterative versus general term formulas provides interesting fodder for discussion. There are always those who staunchly defend the general term formula as the ultimate goal. Others suggest that since not all patterns can be explained by a general formula, it is, also, important for students to have experience with iterative formulas. In other cases, iterative formulas are simpler or can be programmed into an algorithm and, therefore, the computer more easily. Thus, without diminishing the importance of determining a general term formula, we should not completely abandon the discussion of recursive formulas. An additional benefit of the recursive formula in this instance is that it reinforces the essence of an arithmetic sequence as one in which the same amount is added to each term to attain the next term.

Comparing lessons

Participating in a lesson of this type may be a new experience for some of the members of this class. I conclude by having these preservice teachers discuss the lesson in contrast to a more traditional approach to teaching arithmetic sequences. Many students suggest that typically a lesson on sequences begins with a formula that should be memorised. Then students are asked to find the nth term given values for the other variables in the formula. Later, other variables in the formula are found through formulaic or numerical manipulation. Finally, after proficiency has been achieved in the previous stages, interesting word problems might be considered. Most agree that the traditional approach may not convey the depth of conceptual understanding developed in the constructivist lesson they have just experienced.

An important supplemental conclusion for preservice teachers to draw from this lesson is that constructivism yields mathematically correct solutions that hold students responsible for the same level of mathematical rigor that they have always been expected to attain. It is important to address early in the methods class any misconceptions preservice teachers might have regarding fuzzy mathematics or the theory that anything goes in a constructivist lesson. Through the discussion I emphasise that the biggest difference between this lesson and a more traditional one is in the way in which a formula or knowledge base is built *not* in the rigor of that knowledge. Most students agree that the constructivist approach allows for a better development of conceptual understanding regarding arithmetic sequences. Some students, though, contend that the traditional method is preferable since it can be completed faster. Naturally, the members of this class are among the success stories in mathematics education. I remind them that as teachers they need to think not only about the way in which they learned mathematics but that they must think about making mathematical concepts accessible to *all* of their students. While this discussion will be ongoing throughout the semester, most members of the class acknowledge that for some students the formula including the subscript notation can be intimidating to students. The idea of showing the inherent simplicity of the concept, constructing the formula, and then discussing the requisite terminology and notation is appealing to most members of the class.

Conclusion

For several years, I have used this lesson to introduce preservice teachers to a constructivist means of introducing arithmetic sequences. I teach this lesson during the first class of the semester since it provides several benefits beyond its primary intent of modelling a constructivist lesson. In particular, it provides an opportunity for preservice teachers to explore and solve a problem while working cooperatively in groups. An experience that for many of them has been rare since for the most part, they have only been expected to reach a single correct solution to a problem, individually, in their previous mathematics classes. Further, this lesson forces these preservice teachers to view a problem from the perspectives by which others see it, an important skill for a new teacher to develop. Thus in addition to experiencing a constructivist lesson first-hand, they realise the importance of communicating with others to solve a problem in a variety of ways.

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