Problem-posing STRATEGIES used by Years 8 and 9 students

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Introduction

According to Kilpatrick (1987), in the mathematics classrooms problem posing can be applied as a *goal* or as a means of instruction. Using problem posing as a goal of instruction involves asking students to respond to a range of problem-posing prompts. The main goal of this paper is a classification of mathematics questions created by Years 8 and 9 students in response to a problem-posing prompt based on a specific question presented.

The task

At the beginning of the school year, two classes of Years 8 and 9 students, without any specific training in problem posing, were invited to pose problems on the basis of the following problem-posing prompt:

> Make up as many problems as you can using the following calculation:
> $3 \times 25 + 15 \div 5 - 4$.

The problem-posing prompt presented above requires students to pose questions that relate, somehow, to a specific mathematical expression. The prompt draws on key knowledge — understanding and using the hierarchy of the four operations. By asking students to pose questions relating to the given calculation, the expectation was that students would reflect with problems that mirrored the level of their mathematical experience (Kilpatrick, 1987).

Initial procedure for analysis of the problem-posing products

Students' problem-posing products were initially divided into three groups: correct responses, correct intermediate responses and responses that should be excluded from further analysis.

Correct responses

Students responded to the problem-posing prompt in various ways. Some students presented their questions precisely, in the form of well-structured problems. Below is one of Gloria's responses which was classified a correct response:

> What would the answer be if the "–" was a "+" and "+" was a "–"?

Correct intermediate responses

Some students listed possibilities for different arrangements of the elements in a problem, or constructed problem situations, assuming that the goal was the value of the mathematical expression or the values of numbers replaced with asterisks. These responses were described as correct intermediate results. Below is an illustrative example taken from Ani's responses; the goal statement in it is transparent:

> $3 + 25 \div 15 \times 5 - 4$

Another class of problem-posing products was also referred to as correct intermediate. These were the problems that contained surplus or insufficient information. Although some problems were not written precisely, they contained important information about the problem-posing strategies developed by students. Problems with surplus information posed by Christine are presented below. In this case, some of the brackets, for example around 3×25 and 15, are not needed:

Example 1: $[(3 \times 25) + (15)] \div (5 - 4)$

Example 2: $(3 \times 25) + [(15) \div (5 - 4)]$

Problem-posing products excluded from further analysis

For a small number of responses, the decision was made that they should be excluded from further analysis. These included examples in which students did not attempt a response or problems which did not provide enough written evidence to allow valid judgements about the students' actions to be made. Several students posed problems for which they admitted that they "remembered from the book" or that they "didn't create because they read it somewhere"; for example:

The sides of a triangle are 3, 4 and 5. What is the area?

After analysing students' written responses, the strategies students used for problem posing were classified into three categories: (a) reformulation; (b) reconstruction; and (c) imitation.

Reformulation strategy

When the problem-posing actions of students resulted in a rearrangement of the elements in the problem structure in ways which did not change the nature of the problem, the problem-posing strategy was defined as *reformulation*. In other words, the problem-posing products are the same or identical to the given problem and differ from the initial problem only in the presentation of the information in the problem statement. Students reformulated in different ways, as shown in the following examples.

1. Rearrangement of numerical information

Students rearranged the numerical information in the initial problem in such a way that, although the problem-posing product seemed different, in fact, it was a problem that was identical with the initial problem.

Example 1: $3 \times 25 - 4 + 15 \div 5$

Example 2: $-4 + 15 \div 5 + 3 \times 25$

The examples presented here illustrate how students applied the commutative law to obtain problems identical with the given problem. The problem-posing products can be obtained by changing the positions of some groups of numbers in the initial problem. Applying the commutative law for the addition operation is, in fact, an action which does not lead to a different problem.

Changing the places of groups of numbers and variables in a specific problem and justifying (when appropriate) that the problem obtained was identical with the given one, was an action which was an inseparable part of students' work when they were involved in solving equations or inequalities, proving identities, analysing the problem statements of word or geometry problems, and so on. It was also observed that rearranging the information in a problem statement was used by students when they were asked to present a specific problem in their own words.

2. Adding irrelevant structure

Students also generated problems by introducing additional elements to the problem structure, such as one, two or more pairs of brackets. For example, some pupils used brackets to pose problems identical with the initial one. These examples show students' problem-posing products incorporating one or two pairs of brackets that are irrelevant to the problem structure. In these cases the brackets are used in inappropriate ways, suggesting that students who posed these problems have a limited understanding of the hierarchy of mathematical operations.

Example 1: $(3 \times 25) + (15 \div 5) - 4$

Example 2: $(3 \times 25) + [(15 \div 5) - 4]$

3. Replacing mathematical operations with equivalent forms

A few students retained the identity of the problem by presenting some of the mathematical operations in an equivalent form.

Example 1: 3(25) + 15/5 – 4

Example 2: 3(25) + 3 – 4

In this example, students' work was based on the presentation of multiplication and division in equivalent forms. Example 2 in fact represents an intermediate result when the value of $3 \times 25 + 15 \div 5 - 4$ is calculated.

4. Replacing numerical information with equivalent expressions

A few students tried to pose problems identical with the given problem by replacing some of the numbers with the result of two arithmetic operations. In such cases, students tried to present the problem content in a more complex form by preserving the problem identity.

$(2 + 1) \times (16 + 9) + (3 \times 5) \div (25 \div 5) - 4$

5. Combinations of two or more sub-categories

Students also tended to apply two or more problem-posing actions in their formulation of the given mathematical problem. Examples of students' problem-posing products defined under a reformulation strategy, which produced a problem identical with the given problem by combining two or more problem-posing actions, are presented below:

$-4 + (2 + 1) \times 25 + (10 + 5) \div 5$

6. Interpreting the calculation in a real-life context

The final group of problems defined under reformulation can be described as problems in which students made connections between a mathematical expression and a real-life situation. These have been categorised as reformulation because the product differs from the initial problem only in the presentation of its structure. The example below provides examples of students' interpretations of the basic calculation in real-life contexts. In the first two cases the students had expressed to the teacher their frustration in trying to find a suitable context in which to pose problems.

The problem-posing products presented by the students who had expressed difficulty in finding an appropriate context suggest that they were attempting to interpret the structure of the whole calculation as a sequence of interrelated real-life situations.

Example 1:
I bought three $25 items of clothing and gave my 5 brothers and sisters $15 between them and lost $4. How much money:
a) did I start with?
b) did my brothers and sisters get each?

Example 2:
Cameron had 3 guitars which had 25 strings on each, but as a birthday present he was given 15 spare strings. So, he decided to sell the spare strings to 5 other people. While selling the strings he lost 4. How many strings does he have left including the ones on the guitars?

Changes which led to changes in the *nature* of the problem are not regarded as reformulations. Some of the strategies used by students in the reconstruction of the problem are presented in the next section.

Reconstruction strategies

A problem-posing strategy is referred to as *reconstruction* when the problem-posing product is obtained by modifications made to the initial problem and when these modifications change the nature of the problem. Thus the problem-posing products relate, in some way, to the given problem but differ from it in content.

Examples of students' work classified into the reconstruction sub-category are given in the following examples.

1. Changing the order of the numerical information

Students applied a reconstruction strategy to obtain problems from the initial problem when they changed the order of the numbers but keeping the order and the types of the mathematical operations. Below are some examples of students' responses of this type. In fact, all examples presented illustrate problem-posing products which are similar to the given problem but which differ from the initial problem in their content.

Example 1: $3 \times 25 + 15 \div 4 - 5$

Example 2: $5 \times 4 + 3 \div 25 - 15$

2. Changing the order of the operations

In other problem-posing products, the order of the operations was changed while the numbers and their order were kept the same.

Example 1: $3 + 25 \div 15 - 5 \times 4$

Example 2: $3 \times 25 + 15 \div 4 - 5$

The examples here show that the student had tried to pose other examples that resembled the initial problem but differed from it in the way the operations and the numbers were combined.

3. Changing the numbers

Students also posed new problems by changing the numerical information and retaining the same operations and their order:

$2 \div 1 - 15 \times 7 + 40$

This example shows the application of a reconstruction strategy in which both the numbers and the order of the operations are changed.

4. Regrouping the problem information by using brackets

Students also made changes to the initial problem structure by imitating some traditional classroom activities — solving problems with brackets — creating possibilities by using one, two or more pairs of brackets to obtain different problems. The examples below illustrate some typical examples of problems posed when students inserted additional structure (brackets). All examples shown here were posed by Blair.

Example 1: $3 \times \{25 + [(15 \div 5) - 4]\}$

Example 2: $3 \times [(25 + 15) \div 5] - 4$

Example 3: $3 \times \{(25 + [15 \div (5 - 4)]\}$

5. Presenting a mathematical operation in an equivalent form

Some students combined the use of brackets with the representation of division and multiplication in an equivalent form;

Example 1: $\frac{3(25+15)}{5} - 4$

Example 2: $\frac{3 \times 25 + 15}{5 - 4}$

Example 3: $\frac{3(25+15)}{5 - 4}$

6. Taking sub-structures

Problems were also obtained by selecting substructures of the given calculation. For example, some students posed simple calculation problems by taking some of the numbers and one or two of the given operations. These examples were drawn from Peter's work:

Example 1: $3 \times 25 + 15$

Example 2: $3 - 4$

Example 3: $3 \div 5$

7. Combinations of two or more strategies

Some students combined two or more consecutive strategies and obtained new problems. For example, in some cases both the order of the operations and the order of the numbers were changed.

Example 1: $5 \div 15 + 4 - 3 \times 25$

Example 2: $15 - 4 \div 5 + 3 \times 25$

All problems included here differ from the initial problem in their content and they also include additional information (the brackets) that is relevant and changes the nature of the given problem.

Example 1: $3((-4 + 15)\ 25) \div 5$

Example 2: $((25 + 15) \div 5 - (-4 \times 3)$

Example 3: $(-4 + 25) \times 3 + (15 \div 5)$

The next group of problem-posing products represent a combination of three basic sub-categories. In these cases, students obtained new problems by changing the order of the numbers and the order of the operations, and by presenting the division or multiplication in equivalent forms.

Example 1: $\dfrac{15 \times 3 - 25}{5 + 4}$

Example 2: $\dfrac{25 \times 4 - 3}{15} + 5$

Example 3: $3\left(-\dfrac{4}{5}\right) + 15 \times 25$

Imitation strategy

A problem-posing strategy will be referred to as *imitation* when the problem-posing product is obtained from the given problem-posing prompt by the addition of a structure which is *relevant* to the problem, and the problem-posing product resembles a *previously encountered or solved* problem. In other words, the imitation strategy takes into account two important issues: the problem-posing product has an extended structure and the student has encountered these types of problems before.

The examples of imitation sub-categories follow.

1. Interpreting the division operation as a ratio

Some students interpreted division as a ratio and then they posed word problems based on the use of this new interpretation in a real-life context. The example shown below was posed by Brad, one of the best students in the class.

If the above ratio [3 × 25 + 15 : 5 – 4] is used to make a miniature of a famous painting, which has an original size of 50 cm × 60 cm, what size will the miniature be?

2. Extending the problem structure by changing the goal

A few students extended the structure of the given problem by constructing a new goal statement. Students changed the structure of the given problem by extending the goal statement in such a way that the initial problem became a step of the solution process of the new problem. The problem-posing products resemble types of problems that were solved in some of the previous lessons. Students incorporated in the questions well known terms and concepts such as "prime factors", "number of factors", "last digit", and so on.

Example 1:
Which are the prime factors of this [3 × 25 + 15 ÷ 5 – 4] calculation?

Example 2:
Around which two digits could you place brackets so that the answer [of the calculation 3 × 25 + 15 ÷ 5 – 4] is minimal?

Example 3:
Write the prime factorisation of the result of this [3 × 25 + 15 ÷ 5 – 4] calculation.

Example 4:
How many factors does the result [of the calculation 3 × 25 + 15 ÷ 5 – 4] have?

Example 5:
What is the last digit of 3 × 25 + 15 ÷ 5 – 4?

Implications for teaching and learning

The issue of the extent to which problem posing can be considered as an index of a student's problem-solving ability was first raised by Kilpatrick (1987). At the beginning of the school year, when asked, most students posed problems which they knew how to solve. In other words, the problem-posing products did not represent problems for the authors. As the school year progressed, students started to feel free to pose more complex questions. In some cases the students admitted that they had not solved the problem yet, but indicated that, if a solution was provided, then they would be able to understand it. On a number of occasions, some students recognised that they understood what the problem was about, but that they could not solve it, "because it is very difficult."

Data from the classroom, such as tests and homework indicate, that problem-posing skills, as with all other skills, could be developed and nurtured. At the end of the school year, students exposed to a range of problem-posing activities were observed to pay more attention to the quality of problems posed and to problem difficulty. There was a strong tendency for students to pose problems by using the imitation strategy, and to pose problems from different categories rather than to pose problems by reformulation or reconstruction or to pose more problems from the same category.

Reference

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 123–147). Hillsdale, NJ: Lawrence Erlbaum.

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editorial

I recently attended a talk by Merrilyn Goos, a mathematics educator at the University of Queensland, and the university Teacher of the Year in the social sciences. In discussing what makes for quality teaching, Merrilyn emphasised the importance of enthusiasm. Quality teachers not only have a passion for their subject, but also eagerness to share this with their students. It is worth noting that this idea is embedded in the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools*.

Too often we forget that teachers who demonstrate excitement about mathematics and a belief that all students can be equally involved with the subject are the teachers that are remembered long after school days are finished.

This issue demonstrates the range of interesting aspects of mathematics in which we can engage our students. Games, new and interesting applications of technology, or poetry — truly there is something for everyone in maths. Let's not forget that, as teachers, our first duty is to engage our students so as to introduce them to the subject that we all find so interesting in so many different ways. Maintain the passion!

Rosemary Callingham