CREDIT CARD MATHEMATICS

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The history of mathematics is full of rich examples that can help students to see the place of the discipline within our cultural heritage. Valuable as this can be, it also has the unfortunate side-effect of making students think that all the math has already been done and they do not get a sense that the subject is dynamic and growing. Furthermore, most modern mathematics is very complex and not accessible to students in secondary school. This article presents a mathematical topic which is historical in the sense that it was first presented almost fifty years ago but modern because it is still used billions of times each and every day all over the world — and one that students in upper primary and lower secondary school can appreciate quite readily!

Hans Peter Luhn was born in Germany in 1896. Although he originally learned the printing trade in order to join the family business, he later moved to the United States and worked as an engineer. A prolific inventor, he held many patents for items as diverse as raincoats to a thread counting machine used in the textile industry. He joined IBM in 1941 as a senior research engineer and became well known for his development of the KWIC (keyword in context) indexing system, now a commonplace method in information and library science.

In 1954, Luhn filed a patent application for a mechanical device that would check if account numbers had been correctly keyed into a machine. As the mechanical processing of data had become important, it was also important to ensure that people's account numbers were correctly keyed in so that bills could be assigned to the correct account. In his description of a 'computer for verifying numbers' he explains the mathematical system that the device would use. The idea is simple enough: given an account number, an extra digit is produced by a mathematical combination of the original digits. This new digit, often called a *check digit*, is then appended to the original account number. When the entire number is keyed into the machine, the machine uses all but the last digit to calculate what the check digit should be and then compares it to the check digit that is actually part of the number. If they agree then the number is correct, otherwise an input error has been made. Luhn realised that simply adding the digits of the number together and then making the check digit the number that needed to be added to this sum to make a multiple of 10 would trap single errors. However, he also knew that although single errors are common, many human input errors are due to the transposition of digits: for example, 34 gets keyed in as 43. Since these two types of errors make up the vast majority of input errors, he felt that if he could develop a system to trap both of them, then it would be highly effective.

In order to trap transposition errors, there must be a different effect on the sum of having numbers in different positions. Luhn's scheme achieves this in an extremely simple fashion. Start with the digit at the extreme right of the number and multiply every other digit in turn by 2. If the result of this multiplication is a two digit number then add the digits together to get a 1 digit result, since his machine could only deal with single digits. For example, if the digit was 5, doubling produces 10 and the result would then be 1 + 0 = 1. All of these results are summed and the check digit is the digit required to make this sum a multiple of 10. This digit is then appended to the original number and this new number becomes the account number with its own built in checking system. Even today, credit cards still use this system to ensure that numbers are correctly keyed in or scanned. If we imagine how often credit cards are used all over the world in any given day, it is easy to understand the claim that this bit of mathematics is used billions of times per day!

Let us look at a specific example. A typical credit card might have the number 3761 974058 67557. (Editor's note: the majority of credit cards actually have 16 digits, except for American Express cards which have 15, as in the example here.) The last digit (check digit) is 7 so we do not consider it as part of the actual number to start with. Now let us see how Luhn's method would establish what the check digit ought to be; numbers to be doubled are underlined.

Number:	$3\ \underline{7}\ 6\ \underline{1}\ 9\ \underline{7}\ 4\ \underline{0}\ 5\ \underline{8}\ 6\ \underline{7}\ 5\ \underline{5}$
Results:	3 5 6 2 9 5 4 0 5 7 6 5 5 1

The sum of these results is 63 and so the check digit should be 7. (What needs to be added to 63 to bring it up to 70, making the answer a multiple of 10?) In the given number, this is precisely what we find. This method is described in ANSI (American National Standards Institute) X4.13 which specifies standards for credit cards; sometimes this is referred to as the *IBM check*, or the *Modulo 10 check*. Luhn's patent application makes clear that he is the original developer of the idea. Students might be interested in looking at the original patent application which can be found on the Internet at http://patft.uspto.gov/netahtml/srchnum.htm and searching for patent number 2 950 048 (click on the images button).

One of the strengths of Luhn's method is its simplicity and another is the fact that it can be used on numbers of any length. It is also clear that since any error in a single digit can never make a difference of 10 in the sum, single errors will always be detected. Now, what about transpositions?

Although an algebraic approach can be used it is just as valid to argue from an arithmetic perspective. Transpositions will remain undetected if a digit contributes the same result to the sum in either a doubled or undoubled position. We can easily see that 0 is one such digit since $1 \times 0 = 0$ and $2 \times 0 = 0$. If there were another digit then it could be paired with 0 to produce a pair of digits that could be transposed without any effect. Is there such a digit? A quick check will show that 9 is the only other such digit since $1 \times 9 = 9$ and $2 \times 9 = 18$ which gives a result of $1 + 8 = 9$. Since the result from a digit of 9 is identical regardless of whether it is in a doubled or undoubled position, its position can be changed without being detected. Consequently, 90 keyed in as 09 will not be detected and 09 keyed in as 90 will not be detected. Still, 98% of all transpositions are detected, so as a compromise between simplicity and efficiency this system is still very good! Although much more sophisticated (and better) systems now exist for checking numbers, Luhn's method is still used extensively, mainly I suspect because it would be a huge endeavour to change the system on all credit cards now in existence, re-issue numbers and reprogram all of the systems that use this method for checking.

The first activity sheet (Checking credit card numbers) provides students with the background to Luhn's method and provides the opportunity to practice the algorithm. This makes a good exercise in mental computation. After completing it they could be challenged to do three things:

- check the numbers on 2 or 3 of their parents credit cards;
- find a credit card advertisement (they are all over the place) and check the number on the card in the ad;
- check the account number found on their parents' utility bill, gas bill or telephone bill.

The number on the card in the ad will likely

Checking credit card numbers

In 1960, Hans Peter Luhn, an engineer with IBM, was granted a patent for a mechanical device that would check if numbers had been entered correctly. The method that he describes in his application is still used today to check credit card numbers. Luhn's method calculates an extra digit (called a check digit) and puts it at the end of the original number.

1. To calculate the check digit (?) first multiply every other digit by 2 starting from the right most digit (5 in this case) and working towards the first digit on the left. If the result is a 1 digit number write it in the chart, if it is a 2 digit number then add the digits first before writing it in the chart. For example, $2 \times 5 = 10$, add the digits to get $1 + 0 = 1$. $1 \times 3 = 3$, write it in. $2 \times 6 = 12$, add the digits to get $1 + 2 = 3$ and write this in. All the other values are calculated in the same way.

2. Use this method to fill in the missing digits in the chart below.

Multiple	1	2	1	2	1	2	1	2	1	2
Digit	3	7	2	1	5	8	3	0	1	6
Result						7		3	3	1

3. Add the results to get $3 + 5 + 2 + 2 + 5 + 7 + 3 + 0 + 1 + 3 + 3 + 1 = 35$ and the check digit is the number that has to be added to 35 to produce a multiple of 10. In this case the check digit would be 5 since $35 + 5 = 40$ which is a multiple of 10. When the credit card company issues the card it would have the number 3721 583 016 355 on it. Now whenever this number is keyed in, the computer uses the first 12 digits to calculate the check digit (5) and compares it with the last digit of the number (also 5) to establish that the number is correct. If the computer were to calculate the check digit to be anything other than 5 this would mean that the number was incorrectly entered.

Once you get used to it, just write down the number (without the check digit) and underline every other number starting with the right most digit and then you know these are the doubled ones. Write down the results and find their sum. For example, to find the check digit for the number 3721 456 098 12? start by writing 3 7 2 1 4 5 6 0 9 8 1 2, then underline the digits to be doubled: $3\ \underline{7}\ 2\ \underline{1}\ 4\ \underline{5}\ 6\ \underline{0}\ 9\ \underline{8}\ 1\ \underline{2}$. Then write the sum as $3 + 5 + 2 + 2 + 4 + 1 + 6 + 0 + 9 + 7 + 1 + 4 = 44$. This would make the check digit 6.

Exercise

1. Find the check digit (?) for each of the following:
   (a) 3721 345 678 56?
   (b) 3721 112 232 78?
   (c) 3721 009 670 55?
   (d) 3721 023 981 06?

2. Verify that each of the following could be a valid credit card number:
   (a) 5191 9257 1676
   (b) 5191 6936 8981
   (c) 5191 3371 3015
   (d) 5191 0905 4576

be invalid. Students could be asked to suggest reasons why this would be the case and obviously fraud is one consideration. This discussion leads nicely into the second activity sheet (Which ones are fakes?). There is fundamentally no difference between this sheet and the previous one except that it is presented in a way to be engaging for the students – perhaps they will notice that the bird on the card is a loon (pun intended)!

Clearly, credit card numbers are sensitive information so teachers must be careful about either revealing their own or asking students to bring in samples. In order to allow for the discussion with some aura of realism, however, Table 1 contains a series of possible credit card numbers that can be used in the classroom. They are accurate in that they obey Luhn's algorithm but they have NO validity as real card numbers. Emphasise to the students that these are not actual credit card numbers and have not been issued to anyone: they are just possible examples, so do not try charging anything with them!

Table 1.
Numbers that satisfy the Luhn Algorithm.

5 3 8 7 8 8 8 4 3 8 4 5 9 9 2
5 3 0 0 9 2 3 3 7 5 5 7 5 3 8
5 3 1 5 6 9 3 1 6 4 5 1 2 4 6
5 3 4 8 7 3 7 9 0 0 6 3 8 8 2
5 3 0 6 3 8 0 5 3 5 7 2 8 0 2
5 3 3 8 4 2 5 3 7 1 8 4 4 3 5
5 3 7 6 2 3 9 0 0 7 9 1 7 9 0
5 3 3 9 9 1 6 0 5 0 8 0 2 0 5
5 3 1 9 0 8 7 0 5 1 9 7 5 4 1
5 3 9 0 8 7 5 4 2 2 1 8 1 9 0
5 3 9 2 0 5 7 6 4 9 4 3 9 4 5
5 3 1 2 9 8 5 5 3 8 4 6 5 0 0
5 3 4 5 0 3 0 4 8 4 5 8 1 4 3
5 3 7 0 0 6 0 6 4 6 8 0 7 4 6
5 3 1 3 7 3 1 8 5 2 0 6 6 9 4
5 3 9 2 8 0 3 9 5 3 1 3 9 3 3
5 3 0 8 5 6 2 6 4 2 1 7 6 4 1
5 3 7 8 6 3 4 7 5 3 1 5 0 8 4
5 3 8 4 3 9 3 5 3 1 1 9 7 0 5
5 3 1 6 4 3 8 3 8 8 2 1 3 4 1
5 3 2 2 1 0 8 1 6 7 2 5 0 5 7
5 3 0 6 2 2 6 8 5 9 4 2 8 9 8

Extension

This material also lends itself to more challenging problem solving exercises. For example students could be asked questions such as:

- Take any valid credit card number and change any digit to something else. Then calculate the check digit. Does the system tell you that a mistake has been made?
- Take any valid credit card number and transpose any pair of digits. Then calculate the check digit. Does the system tell you that a mistake has been made? Transpose a different pair of digits. Is the result the same?

These questions should start students thinking on why the algorithm traps transposition errors. They could fill in a chart such as the one below to further establish the ideas.

Digit	0	1	2	3	4	5	6	7	8	9
Contribution to sum if not doubled			2							
Contribution to sum if doubled		2								

Will Luhn's method trap all transposition errors? Explain.

Have the students establish a way to find any missing digit in a credit card number.

This last task is not trivial, as they need to take into account the position of the missing digit (doubled or undoubled position) and work backwards from the result to the original digit.

I have always believed that mathematics is where you find it. In addition to providing a number of places to practise mental calculation, organisation and problem solving skills, this topic should help students to realise that new mathematics is developed all the time and that it often endures and permeates the world around us, even when we cannot directly see its results!

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Which ones are fakes?

Credit card fraud is a serious problem for banks. Some of these cards are definitely fakes. Use the Luhn method to establish which cards could not possibly be valid.

$CR 3721 583 014 350 JOHN H DOE	$CR 3721 261 128 464 JOHN H DOE
$CR 3721 117 111 784 JOHN H DOE	$CR 3721 040 387 928 JOHN H DOE
$CR 3721 394 291 885 JOHN H DOE	$CR 3721 281 430 312 JOHN H DOE
$CR 3721 092 404 477 JOHN H DOE	$CR 3721 002 297 206 JOHN H DOE
$CR 3721 481 233 484 JOHN H DOE	$CR 3721 001 390 288 JOHN H DOE