CLASSROOM COMPUTER ALGEBRA

Some issues and approaches

STEPHEN ARNOLD

Computer algebra is here to stay!

Not so long ago, the common response of teachers to high school use of computer algebra systems (CAS) was invariably along the lines of, 'What will be left to teach if our students have access to devices which solve, factorise, do calculus and more?' and 'If students use such tools, then they will never learn how to do their mathematics, and their algebra skills will deteriorate,'; and the classic cry, 'What will be left for us to put into our examinations?'.

Perhaps it is the ongoing research into the classroom use of computer algebra tools, both here and overseas, which strongly points, not only to greater understanding and better attitudes as a result of access to such facilities, but indeed to no loss whatever of algebraic manipulation capabilities on the part of students.

Perhaps it is the growing wisdom of practice, as more and more classroom teachers experiment with computer algebra systems, first as tools for teaching and, later as tools for learning, and observe for themselves the benefits that follow from giving their students more and more control over the mathematics they are learning, and the ways in which they may learn it.

Perhaps it is the softening effect of increased access to 'normal' graphic calculators over the past few years, as teachers and students have come to see the powerful benefits of access to technology designed specifically for the teaching and learning of mathematics, tools which make our students active participants rather than passive spectators in the learning process. We begin to appreciate the possibility that you can never put 'too much power' in the hands of learners.

Perhaps it is simply the slightly embarrassed realisation that the comments above were the same comments we made some five or so years ago, concerning graphic calculators and, indeed, were precisely the same comments some of us made twenty or more years ago, concerning the use of scientific and four-function calculators in our classrooms. There are few among us now who would be happy to teach our subject without access to some form of supporting technology. As always, it is not the tool itself but how it may best be used that lies at the heart of teaching and learning issues: good teachers, as always, make use of technology in different ways at different times, and there will always remain times when a good teacher instructs students to put aside their calculators (of whatever variety) and work unaided.

So where do we stand after some fifteen years of exploring the possibilities for the use of computer algebra systems in high school mathematics classrooms? At the heart of all the issues and possibilities lies a single central thought: learning to use the new tools is really about learning to ask new questions. Many traditional questions become trivial, some traditional approaches are revealed as, at best, unnecessary and, at worst, a distraction from the mathematics to be learnt.

Consider, for example, the important question of what a solution should look like in a CAS environment: should students still be required to record every step of the process as if it were being solved by traditional means? When four-function calculators were first allowed in classrooms, there were many of us then who believed that students still needed to show all working, as if they were doing the long multiplication or the long division by hand! The instant answer available using the technology was, initially, simply a check that the correct steps had been taken. Few of us, now, however, expect all the steps of a long division to be demonstrated in a solution: we have realised that the process itself was not the point of the exercise: rather it was a means to an end — so with computer algebra tools.

We are likely, in the next few years, to see such time-honoured and treasured mathematical processes as expanding and factorising, even differentiation and integration, be relegated to the growing pile of 'historically interesting' mathematical procedures; hard to conceive for those of us brought up on almost a complete diet of such processes — and, indeed, proud of our ability to succeed with such where so many others of our peers failed and fell by the wayside of mathematics.

Herein, too, lies a critical issue regarding the acceptance and use of CAS: algebraic facility has long served a gatekeeper role for deciding who will earn entry to the hallowed halls of higher mathematics and, through these, to the many and increasing rewards an understanding of mathematical thinking and, consequently, technology, will bring.

As our society becomes ever more dependent on technology for functioning at all levels, can we afford to continue to limit so drastically those who would maintain and extend these operations? Think for a moment of those without a strong facility in their multiplication tables who, in the past, were weeded out early and denied access to senior mathematics courses. The universal availability of hand calculators has exposed this culling process as unnecessary. Many now teach capable and motivated students who, in fact, succeed in their study of mathematics despite not being good at their tables, since they can rely on technological support when necessary.

The same type of barrier is now being confronted by computer algebra tools: why should those who are motivated and able to understand the concepts and processes of higher mathematics be denied access when scaffolding tools are now so readily available? Question your assumptions concerning those who are 'worthy' to progress to the study of higher mathematics, and realise that there are now important social priorities at stake: can we afford to continue to deny access to those who are willing and capable, simply because their skills of algebraic manipulation are not as strong as we have required in the past.

In fact, it remains true that the effective use of such tools usually requires greater mathematical understanding and concept development rather than less. Knowing which operation to choose, knowing which process to follow, and interpreting the result intelligently are all higher cognitive functions than knowing the steps of a manipulative process.

I have long been impatient with those well-meaning proponents of computer algebra systems who parade a series of mathematical 'tricks' before teachers and students as a demonstration of the power and worth of such tools. To me the ability to spit out 1000 decimal places of Pi, or to instantly solve a difficult equation with exact solutions, or to factorise a degree five polynomial are little more than gimmicks, and are quickly revealed as such when placed in the crucible of the classroom. Students are far more critical users of technology than we, and are frequently unimpressed by such displays of computing power: they want to know 'Why?' and quite rightly so.

In my pursuit of 'good questions' of worthwhile applications for teaching and learning, I am usually far more interested in what such systems cannot do than what they can! A question to which a computer algebra system can produce an immediate answer (no matter how complicated) is often of little value. Rather we pursue questions which require intervention by the user, which demand input of mathematical knowledge and understanding in order to access the kernel of a solution that lies at its heart; or perhaps there is no single solution: rather an investigation which leads the traveler on a mathematical journey, rich with understanding and insight, while never reaching a well-defined conclusion?

Consider the questions which follow in this context, as exposing in some ways the 'inadequacies' or 'short-comings' of these powerful tools, but, in so doing, revealing rich and powerful mathematics along the way.

Question 1

Solve the following four equations (giving both exact and approximate solutions where possible).

(i) $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\dots}}}}$

(iii) $x = \sqrt{1 + x}$

(iii) $x = 1 + \dfrac{1}{x}$

(iv) $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$

(v) Why?

(vi) What other numbers can be represented in these ways?

While examples (ii) and (iii) are readily solved using a computer algebra system (and revealed as being closely related) it takes considerable mathematical insight to see their connection with the other two equations, even though all have the same solution(s) and, at heart, express the same quadratic form.

When provided with a powerful computer algebra tool, students (and their teachers) expect much of algebra and calculus to become trivial — they are disappointed at how often they are unable to simply enter a standard form and produce a 'correct' result. Instead, the reality is that there is usually some work required: first, in identifying an algebraic or mathematical object upon which to act, then, choosing an appropriate action; evaluating the result of that action: does it move me forward in my solution process or investigation? This last usually leads to another round of mathematical action.

Of course, the real advantage of access to computer algebra is the 'grunt' it adds to your mathematical toolkit! It represents the ultimate mathematical investigative assistant. Consider, for example, the problems above: by simply adding the questions at the end, students are sent off in search of patterns and relationships and, at last, they are really 'doing mathematics'!

Teachers frequently ask, 'Of what use is a device which merely produces a result?'. My response has always been, 'It is a very small step from verifying results to verifying conjectures.'

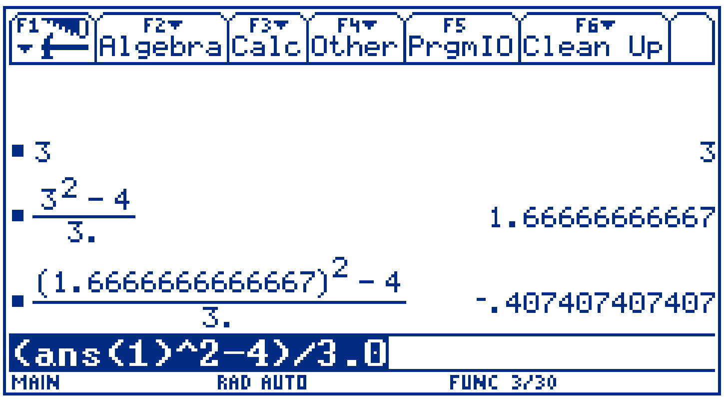
Question 2

A suitable extension activity, following on from the previous investigation, could involve a study of recursive equations. The equations given above are examples of such recursive equations since the main variable is described in terms of itself. It is surprisingly easy to create such an equation.

Consider, for example, a simple quadratic, such as $x^2 - 3x - 4 = 0$. Simply isolating the x-term from the middle of the equation results in:

$$x = \frac{x^2 - 4}{3}$$

Students may investigate such equations, especially their application to approximation methods of equation solving. Try entering a guess, and then using the ans feature of any graphic calculator in this formula to observe the effect of repeated applications. What might be advantages and disadvantages of such a method? How 'good' does your first guess need to be? Could such a method be used to solve any equation?



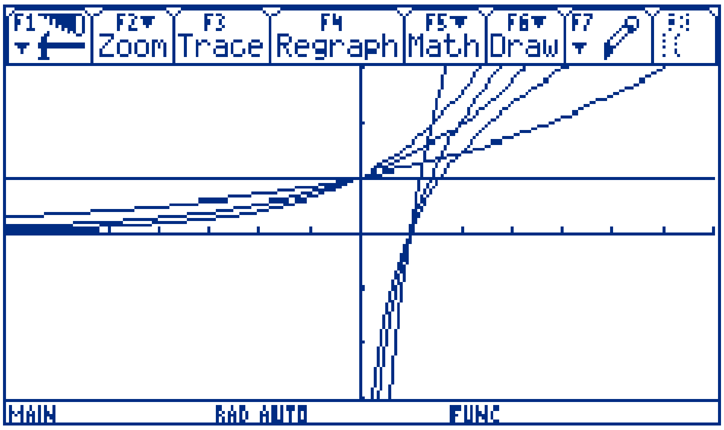
Question 3

At exactly what value of a do the graphs of $y = a^x$ and $y = \log_a(x)$ kiss?
[NOTE the use of the technical term for the tangential meeting of two curves at a point!]

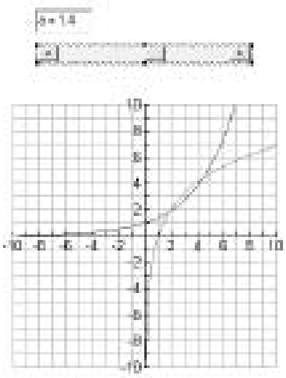
First attempts to solve the two equations quickly reveal the problems associated with having two variables! (Although an even earlier problem may arise in trying to find an appropriate way to represent the second function: in this case, sending students back to the logarithm laws, from which may be derived the change of base rule.)

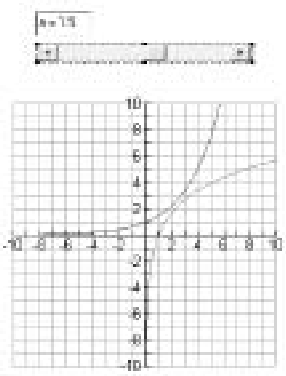
Thus, $\log_a(x)$ becomes $\dfrac{\ln(x)}{\ln(a)}$.

A variety of investigative devices are available to students in such a context, even on a 'normal' graphic calculator. Consider, for example, setting the value of the variable a as a list of numbers, such as {1, 1.2, 1.4, 1.6} as shown.



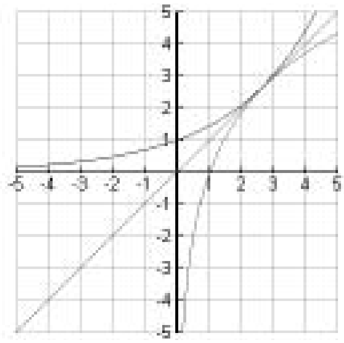
The marvelous 'slider' available within *TI Interactive!* seems perfectly suited to this task. Simply set the range of values for a, and then move the slider until the desired effect is achieved, quickly narrowing the solution down to a value between 1.4 and 1.5 (shown).





Such a question, however, requires more than trial and error, especially if an exact solution is specified. The importance of the initial trial-and-error approach, however, should not be underestimated. Students who use it are taking control of the problem: checking to see if it is a reasonable question, and where to look for their answer or answers.

It might be hoped that the next step towards a solution would involve the recognition of the two functions as *inverses* of each other, and a subsequent recognition of the role of the identity function, $y = x$, in this scenario. This should suggest, to some, that another property of the point of intersection of the two curves lies in the value of the gradient of the tangent to both curves being equal to 1.



$a = 1.45$

This gives a more promising approach, solving two equations with derivatives equal to 1, to produce two solutions in terms of a.

$$\text{solve}\left(\frac{d}{dx}\left(a^x\right) = 1, x\right)$$

$$x = \frac{\ln\left(\frac{1}{\ln(a)}\right)}{\ln(a)} \text{ and } \frac{1}{\ln(a)} > 0$$

$$\text{solve}\left(\frac{d}{dx}\frac{\ln(x)}{\ln(a)} = 1, x\right)$$

$$x = \frac{1}{\ln(a)}$$

Equating these two solutions leads to an exact answer, as required. The level of mathematical knowledge and insight required, however, puts paid to any suggestion that use of CAS will lead to a degrading of mathematical capabilities.

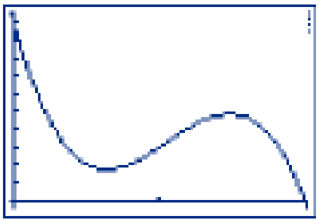
$$\text{solve}\left(\frac{\ln\left(\frac{1}{\ln(a)}\right)}{\ln(a)} = \frac{1}{\ln(a)}, a\right)$$

$$a = e^{e^{-1}}$$

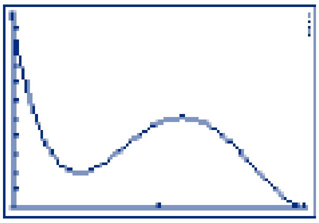
Question 4

Designing roller coasters and water slides provide nice opportunities for students to increase their familiarity with graphical forms, and the effects of various transformations upon them. Consider the following example.

(i) Design a roller coaster which will fit into a 10 metre section of the fair-ground, to be no more than 10 metres high. Use a cubic equation as the model for your ride, such as that shown. Evaluate the effectiveness of your design.



(ii) Describe the ways in which a quartic design would improve your ride. Find a suitable quartic function which will serve as the model for your roller coaster.



(iii) CHALLENGE: Use two different function types (quadratic/cubic, quartic/sine, etc.) to design an improved roller coaster which will fit into the space allowed. Ensure that the ride is smooth, and explain clearly any improvements which your design will have over the previous two models.

The first two components of this question may be suitably attempted on any graphic calculator (in fact, the simplest solution involves entering points into statistical lists and doing a regression on these!), but the last question would benefit from the support offered by computer algebra. In particular, ensuring that the curves join smoothly requires that the gradients of both are equal at their point of intersection, and this is indeed an interesting requirement, mathematically.

Conclusion

Such questions offer ideal illustrations of the power of computer algebra as a classroom learning tool. These tools clearly support students, not in mindlessly producing results, but in purposeful and strategic investigation of problems, which are likely to be beyond the reach of many, working unaided. This is, after all, the purpose of all scaffolding: it allows users to see beyond and to reach further than they could without such assistance. And finally, when no longer required, perhaps to be put aside, leaving an independent and free-standing structure.

Computer algebra tools are sometimes referred to as 'symbolic manipulators' as, indeed, they are: devices which support the manipulation of symbols. Our students in such an environment, however, are much more than mere manipulators of symbols. They potentially become active, insightful explorers of mathematical concepts and relationships, ably supported in this process by powerful technological aids, which not only serve to relieve the syntactical burden. They make public the mathematical thinking of the user, helping to expose tacit knowledge in the domain of algebraic thinking which is so often difficult to articulate. They offer explicit signposts along the path of mathematical discovery (when unsure of the next step, one can always browse the menus and see what options are available). They are a regular source of mathematical surprise, even for experienced users. So powerful are these tools, in fact, that some might say they provide mathematical wings to soar when others are forced to walk! Whatever the rhetoric, it is time for teachers of mathematics to become better acquainted with the possibilities as the tools become increasingly appropriate and affordable, the guidelines for their effective use become better defined and, as a result, the widespread classroom use of computer algebra tools draws ever closer.

Stephen Arnold
Australian Catholic University (Signadou)
s.arnold@signadou.acu.edu.au