TEACHING MATHEMATICS TO COLLEGE STUDENTS WITH MATHEMATICS-RELATED LEARNING DISABILITIES: REPORT FROM THE CLASSROOM

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***Abstract.* This article reports on action research that took place in one section of a college general education mathematics course in which all three students who were enrolled had diagnosed learning disabilities related to mathematics. The project emerged in response to a question about performance in a mathematics course in which making sense of mathematics would be a primary focus, explaining one's work would be expected, and discourse among members would be a routine occurrence. Implications for teaching similar courses to students who have a mathematics-related learning disability are discussed.**

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Literature related to postsecondary students who have mathematics-related learning disabilities (LD) is scarce. As a result, there are few content-related teaching suggestions to guide student-centered college mathematics faculty who have students with diagnosed LD in their classes. Some faculty rely on campus learning centers to assist these students, others help the students themselves. Generally, college-level students with LD do not have access to the degree of support that existed for them at lower grade levels. Some students with LD become discouraged when they cannot keep up in their mathematics class, and withdraw. Others persevere; they expend great amounts of effort and time and take advantage of college tutoring services and faculty office hours, yet, fail the course. Some institutions have course waiver or substitution policies or offer special sections of their required mathematics courses, but many do not.

This article reports on an action research project situated in a section of a general education mathematics course that enrolled three students with diagnosed LD related to mathematics. All students had a history of multiple attempts to satisfy the college's mathematics requirements and, with the exception of mathematics and science, all had performed at or above average levels in their courses.

The author sought suggestions from the literature for teaching course topics to the enrolled students. While unsuccessful in locating teaching suggestions, the author noted that many LD specialists do not favor current reform efforts in mathematics (Jones & Wilson, 1997; Maccini & Ruhl, 2000; Miller & Mercer, 1997), preferring the more traditional "present, practice, and test" approach. Mathematics reform efforts, based on the premise that students must make sense of mathematics, have been central to the author's professional practice. She considered traditional forms of instruction guided by behaviorist psychology, the norm in previous studies, as necessary but not sufficient for mathematics instruction.

The question naturally arose: How would college students who have mathematics-related LD perform in a course where making sense of mathematics is a critical component, where explaining one's work is expected, and where discourse among members is common practice? This question provided the impetus for the project. In teaching the course, the author intended to utilize reform methodology, including use of manipulatives, journal writing, and multiple forms of assessment. In order to contribute to the literature on teaching college-level mathematics to students who have mathematics-related LD, she planned to document her course modifications and chronicle students' efforts in making sense of the mathematics in an environment based on constructivist principles.

This article reports the results of the project, which utilized qualitative methodology. After reviewing the literature related to characteristics of students who have mathematics-related LD and teaching strategies suggested therein, the institution and its mathematics course requirement are described. In the methodology section, the author relates the impact of prior research on planning the course, completed prior to knowledge of student profiles. Next, she describes the students, based on data they provided during the first class, and presents one unit of the course, the mathematics of finance, in some detail. She describes modifications of original plans and includes examples of student work as evidence of instruction broadly based on constructivist principles. In the final section, the author discusses problems that might have been averted with additional information, as well as implications for practice.

Mathematics-Related Learning Disabilities and Teaching Suggestions

Some college students who possess average to above-average intelligence but are less successful in particular academic areas are described as having LD (Miles & Forcht, 1995). The characteristics of LD related to mathematics are diverse and can be connected to issues related to language, information processing and cognition (Daley, 1994; Strawser & Miller, 2001).

Vocabulary and reading issues impact mathematics performance. For example, words in English whose meaning in mathematical contexts differs can cause confusion. In algebra, the terms "reduce" or "cancel" are used when the goal is to simplify expressions, but the value does not change. In statistics, the term "mean" differs from either common context in English. Small words ignored by some students while reading can drastically alter meanings. Interpreting "x is less than y" as "x less than y" and using "y – x" instead of "x < y" will likely result in an incorrect solution that may be unrelated to understanding of mathematical concepts.

Students who have information-processing difficulties (Miller & Mercer, 1997) may not understand what the professor is saying or may not be able to listen and take notes at the same time. Others may copy notes from the overhead or blackboard incorrectly or they may leave out numbers when copying answers from calculators to paper. For example, they might interpret the number 98 as 89, or 86 as 68 in processing, even if the number is written correctly on the paper. Students who have motor difficulty may have poor or slow handwriting. They often have "holes" in their notes, resulting in gaps that interfere with content understanding. Further, attention deficits affect processing of mathematics problems that require multistep solutions: students lose the problem focus partway through a solution. Memory issues appear in students who do well on daily tasks but fail exams. Others can memorize and retrieve information on demand, but may not be able to connect mathematics concepts or know where to begin or end a task.

A specific LD subtype that primarily affects mathematics, dyscalculia or nonverbal learning disability (Strawser & Miller, 2001), is not language based and can be traced to the right hemisphere of the brain. Characteristics include selective impairment in mathematics, visual-spatial disturbances, and difficulties with social perception and development of social skills (Fleischner & Manheimer, 1997). Generalizations and abstract rules that characterize secondary and postsecondary mathematics courses are difficult for students with this diagnosis. While they can memorize definitions and state them when asked on tests, they are usually unsuccessful when asked to explain their understanding of the concepts. Similarly, they can perform a calculation on a test when it is similar to others completed in class and on assignments, but are unable to verbalize their reasoning.

Recent reviews of studies involving teaching mathematics to students with LD reveal that the amount of research in this content area has increased (Miller, Butler, & Lee, 1998), but is still underrepresented (Bryant & Dix, 1999). Both reviews build on the work of Mastropieri, Scruggs, and Shiah (1991), whose review of 30 studies found that mathematics interventions primarily addressed arithmetic computation. In the 23 studies that Bryant and Dix cited in their review spanning 1988-1997, only 2 were at the algebra level. Miller et al. cited all but three of the Bryant and Dix studies, adding 32 more; of these, only 4 were at the high school level. The three studies reviewed by Hughes and Smith (1990) that described mathematics beyond test score results provided descriptive, not empirical, research results. No studies were found that discussed mathematics content at the college level.

In light of the paucity of research on teaching college-level mathematics content to students with LD, faculty must consider research results from studies conducted at lower grade levels. Whether strategies found to be effective at lower levels transfer to older learners remains to be empirically validated. However, effective strategies cited in the above reviews go further than "present, practice, and test." They often appear in the repertoires of effective college mathematics instructors, particularly those who teach general education level courses:

1. Make the mathematics content relevant and authentic (Witzel, Smith, & Brownell, 2001).
2. Employ a concrete-to-abstract sequence (Fleischner & Manheimer, 1997; Maccini & Ruhl, 2000; Miles & Forcht, 1995; Witzel et al., 2001) that starts with a demonstration or activities using manipulatives, moves to a representational phase with specific examples and diagrams, and ends with an abstract generalization, rule, or proven theorem.
3. Provide opportunities for guided practice in solving problems prior to independent practice (Witzel et al., 2001), perhaps with another student in the classroom, so that students have a clear understanding of the process.
4. Provide opportunities for students to verbalize their process to other students and practice writing solutions (Miles & Forcht, 1995).

Daley (1994) offers suggestions for teachers who are planning mathematics curriculum for students with mathematics-related LD. Specifically, he recommends that instruction (a) include concepts as well as when and how to apply them; (b) include age-appropriate materials; (c) utilize visual, auditory, and kinesthetic methods of learning; and (d) specify mastery criteria for each skill based on students' conceptual and cognitive level.

He also recommends that the curriculum include instructions for teaching based on assessment. Two useful forms of informal assessment include analysis of error patterns and a diagnostic interview in which students verbalize their thought processes while they solve problems.

Miles and Forcht (1995) describe a multistep strategy suitable for upper-level mathematics that goes beyond verbalization. First, the student reads the presented mathematics problem and copies it into a notebook. Then the student verbalizes and writes the steps needed to solve the problem. The dual process of verbalizing and writing aids the student in clarifying simple errors and understanding the concepts and processes involved. The mentor, usually not the classroom instructor, guides the student through the verbalization process using appropriate questions and rephrasing student statements. Once the problem is solved, the student is instructed to recall the verbalization at each step, writing down and numbering the verbal statements in order at the bottom of the page. The statements are numbered and the statement number is placed at the point in the solution where a given step occurred. The authors report success using this technique with five students enrolled in high school and college-level algebra and calculus courses. The students met weekly for two hour-long sessions one-on-one with a mentor.

The Institution and Its General Education Mathematics Requirement

The institution at which this study took place is a four-year comprehensive state college in the northeast that enrolls approximately 7,100 undergraduate students (5,500 full time). The college has two requirements related to mathematics. The first, a competency requirement, addresses basic skills. Students have several options for satisfying the requirement, including SAT mathematics score, competency test, or a non-credit course.

After meeting the competency requirement, undergraduate students satisfy the general education mathematics requirement by successfully completing one of several course options. With a goal of promoting informed citizenship in general education courses, the aim is for students to be able to recognize and understand the role of mathematics in the world and make sound judgments relative to mathematics in their own lives. Most students take the course, Contemporary Topics in Mathematics, and this project occurred in a section of that course. Four main topics are covered: the mathematics of finance, the mathematics of social choice, elementary graph theory, and basic probability. Students are expected to gain understanding in all areas, but not with the breadth and depth expected in advanced courses in these topics.

METHOD

Planning the Course

Based on their previous unsuccessful attempts to satisfy the college mathematics requirement, the author anticipated that the enrollees would have more deficiencies in secondary mathematics content than students in other sections. Further, she expected that students' perceptions of self as learners would lack confidence, and therefore planned for them to use their strengths to compensate for their deficiencies. For example, students would communicate understanding using their self-identified strongest communication method – written, oral, or a combination. Since no diagnostic testing results would be available before the course began, the author included two short learning style inventories in the first class in order to learn about their style preferences and be able to apply the findings when planning content organization and delivery. For example, if students showed a preference for a holistic processing style, an overview of the material would precede individual mathematical components. If, on the other hand, they were more detail-oriented, instruction would start with components and build the big picture. The goal was to make appropriate accommodations while maintaining course integrity.

Since the students' majors were in social work and communication, probability was replaced with descriptive statistics. Extensive use of hand-held calculator technology was planned. Since test anxiety was likely, plans for assessment included short in-class assessments with open resources, extended at-home tasks, journal writing, oral presentation of problems, focused writing that described process and understanding, and small projects with presentations. Students would have opportunities to explain orally the written work they submitted for evaluation in order to identify errors resulting from number reversals, calculator keystrokes, and copying that were not connected to the concepts being studied. Since the course text was not reader-friendly, portions from several texts were combined and handouts prepared. Finally, anticipating that sequencing might be an issue, organizers for multistep situations were developed.

Strawser and Miller (2001) suggested that student success requires an interpersonal connection with the in-structor. Therefore, it was decided that students would need to know that the instructor would be patient and understanding of their situation (multiple prior failures while attempting to meet the requirements, faculty that made them feel small, etc.). During class, students would convey their understanding orally and have an opportunity to complete practice exercises under supervision. In order to have sufficient class time for these functions, students had 6 hours of class time for the 3 semester-hour course, twice the usual amount. Because they were upper-class students, the author planned to share her expectations with them, specifically, attendance at all classes and a reasonable attempt at assigned work

Course Participants

One male, whom we will call Tim, and two female students, identified here as Laurel and Tina, all Caucasian, enrolled in the course. Both females were second-semester seniors and had a history of multiple attempts to pass the college's mathematics competency requirement and the general education requirement in mathematics. Laurel and Tim, both mature students in their forties, were preparing for degrees in social work; Tina, a traditional-age student, was a media communication major. All students signed releases for their diagnostic testing results; however, the information was not received until five weeks into the course.

Information from a general background questionnaire, individual interviews, and several brief learning style inventories (Barsch, 1980; Gregorc, 1982) revealed that all had learning issues dating back to the elementary grades. All struggled with mathematics and science courses, and all recounted enormous difficulty learning long division. High school mathematics preparation included general mathematics and business mathematics courses; none had taken geometry, and the most advanced course any student had taken was Algebra I.

Tim had earned his GED several years after dropping out of high school, and eventually earned an associate's degree at a community college. He described himself as a loner. Laurel had finished her high school requirements in an evening program, and after a series of low-paying jobs, a failed marriage, and difficulties raising her child with LD, she had completed her associate's degree part time at a community college. Finally, Tina had graduated from high school with her class and attended a private two-year college before transferring to the college. She described social difficulties with her peer group. Both females had enrolled in the non-credit course that satisfies the mathematics competency requirement three times before passing, and Tim required extensive individual tutoring prior to taking the competency examination. Tina had withdrawn from another section of this course in the previous semester due to her failing status.

The Barsch Learning Style Inventory (1980) gave an indication of learning preferences in the visual, auditory, and kinesthetic areas, based upon "always," "sometimes," and "never" responses to questions. None of the students approached learning from a kinesthetic perspective, suggesting they might not find concrete manipulatives useful for learning. Tim and Tina preferred to receive information visually, so it was assumed that they would depend on reading the material and seeing clear diagrams. Laurel, who described herself as dyslexic, preferred an auditory approach, so she would depend on listening to take in her information.

Teaching strategies based on students' visual preferences included using the board and overhead projector to list the essential points of a lecture and providing outlines/organizers for use during lecture. Since visual learners depend on textbooks and class notes, it was essential that instructor-supplied information be clearly written and make sense to the students. Laurel needed to listen to learn, which suggested that she would benefit from group discussions, organized lectures, and tape recording the class so that she could listen as often as she necessary to clarify her understanding.

The Gregorc Style Delineator (Gregorc, 1982), which reveals preferences for perceiving and processing information showed that while Tim and Laurel preferred taking in information using reason and their emotions, Tina preferred to use her five senses. None of the students indicated a preference for using a linear, step-by-step, methodical manner, assembling and linking data in a chain-like fashion to process information. Instead, all indicated a preference for using nonlinear, leaping methods, imprinting large chunks of data on the mind in fractions of a second to be kept in readiness until demanded.

Tina's profile suggested a highly independent, creative individual who would "march to a different drummer" and be ready to fight the system. Her manner of dress and outward appearance and her saying that she wanted to "get back" at various individuals corresponded with this style. Tim and Laurel's profiles suggested they would talk through their ideas in a "talk all around the issue" fashion before conveying the kernel they wanted to express. They would enjoy cooperative learning activities and would need a relaxed, warm atmosphere to feel comfortable. Tim was very articulate when he spoke, but showed little facial affect. His social interactions appeared stilted, and he rarely made direct eye contact. Laurel, on the other hand, was very expressive, smiled often, and openly shared her insecurities about the course. Students' choices of major seemed to fit with their learning style.

The students were very verbal. They openly shared previous unsuccessful attempts with college mathematics, including the role that faculty played. They praised faculty who were supportive and criticized those who wrote messages like "See me!" on returned papers. They spoke of experiences with the College Learning Center while trying to pass the mathematics competency. They personalized situations and encounters (e.g., both females withdrew from mathematics courses in which they perceived a personal affront). They described themselves as individuals who struggled with mathematics, who needed patience and understanding, and an instructor who believed in them. The author shared that her learning preferences were very different from theirs and that, therefore, ways of thinking that made sense to her might not make sense to them. She requested feedback when the instruction was confusing. Two-way communication was adopted.

Based on conversation and learning style profiles, the author anticipated that the financial section of the course, with its formulas and multistep equations, would be difficult for the students, so she began to create organizers to support their process. She thought that students would manage the calculations in descriptive statistics, voting methods, and determining fair division, but might have difficulty in analyzing results. She expected that the visual nature of graph theory would provide an enjoyable change of pace. Their nonlinear approach to processing information suggested that motivating interests in topics contextually before introducing the mathematics would be useful.

The planning process used during the course employed a backward design model for unit planning (Wiggins & McTighe, 1998) and task selection from familiar, real-world contexts. Credit cards, savings accounts, and loans were natural for financial situations; experiences in resolving schedule conflicts provided an entry into graph theory; a case of employee layoff due to age discrimination motivated the statistics section; and settling an estate led into the mathematics of social choice. In the next section, the mathematics of finance section of the course is illustrated.

The Mathematics of Finance

In backward design (Wiggins & McTighe, 1998), the endpoint decisions are made before action occurs. In this case, goals for students included developing the ability to determine whether advertisements for loans, mortgages, and annuities were correct, and which provided the best deal for them. In the vocabulary that appeared in texts, different English words were used to describe the same mathematical concept when the context changed slightly (e.g., amount and future value in compound interest calculations, lump sum deposit and principal in loan situations). Anticipating confusion, this was adjusted so that the focus would remain on the main concepts and corresponding formulas. Texts tend to change symbolism when they change English words, so a common notation was adopted throughout the unit. In working backward from loan repayment calculations through annuities and compound interest to simple interest, one question guided planning, "What mathematics concepts are essential to understanding this section?"

Motivating discussions that began this and other units were essential to creating a shared experience and a community atmosphere. Sometimes the instructor's questions or comments about mathematics and real-world connections sparked thoughts among students that were not connected to the mathematics at hand, a characteristic of a random processing style. It required delicate balancing to acknowledge the importance of the contributions from an interpersonal viewpoint while keeping the mathematics content at the forefront.

Figure 1. Student boardwork for “35 is 24% of what number.”

Tim	Laurel	Tina
see long division below	$\frac{24}{100}$ $\frac{35}{x}$	$35 = .24x$

Tim:

```
      145
24 )3500
    24
    110
     96
     140
     120
      20
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Our discussion of financial mathematics began with a review of loan ads from the weekend newspaper and the question, “Which loan would you choose and why?” Responses varied, but revealed students’ contextual understanding. Tim and Laurel shared experiences with student loans, and Tina described her mother’s credit card debt issues. They relayed situations of student loans, bank accounts, and credit cards with full awareness of what was happening, but admitted they had no idea whether the numbers were correct. Empowerment, in this case the ability to check figures, was a powerful motivator.

Concepts essential to simple interest include percent and time, so we reviewed these first and made frequent references to the students’ personal experiences. When they worked basic problems involving percent, Tim did every calculation with pencil and paper, including long division. He said that he had never used a calculator. Both Laurel and Tina used scientific calculators as they completed the tasks. Laurel performed each calculation sequence many times, whereas Tina finished the problems quickly, completing aspects of each problem mentally. During these early observations and interactions, characteristics described as common among students with LD were evident:

1. Writing and/or copying number of figures incorrectly
2. Difficulty with sequences of mathematical steps
3. Difficulty with naming mathematical concepts, terms or operations
4. Decoding mathematical context into mathematical symbols incorrectly

Figure 2. Using the distributive law to create one-step expressions.

Original approach to computing selling price:

$MU = C \bullet \%$

$SP = C + MU$

Distributive law approach combines both steps from the original approach

$SP = C + C \bullet \%$

$SP = C\,(1 + \%)$

Note: C: cost; *MU*: amount of markup; %: percent of markup; *SP*: selling price.

5. Incorrect interpretation and use of numerical symbols and/or arithmetic signs
6. Incorrect computations
7. Trial-and-error sequence of calculator keystrokes
8. Immature appearance of work on paper.

Student responses to "35 is 24% of what number" appear in Figure 1. It was not clear how Tim handled the change from 24% to 24 and whether the shift from 35 to 3500 was conscious. When asked to explain, Tim could not articulate the process he had used. Laurel used

Figure 3a. Textbook conceptual approach to loan calculations.

Text Approach

"One way to think of paying off a loan is to imagine two accounts, one for the money borrow and the interest that is owed, the other for our payments (also getting interest). To pay off the loan we deposit the payments in an account at the same interest rate as the loan. This account grows as we make payments. Meanwhile the account with the borrowed money in it and the accrued interest also grows. When our payment account has the same amount of money as the borrowed account we can pay off the loan" (Hathaway, 2000, p. 109).

Note.

A: amount after interest has been computed; R: annual rate of interest;
RP: regular deposit or loan payment; t: number of payments;
P: lump sum amount for deposit or loan, the principal.

The text, in using R and t, expects the student to make adjustments for interest periods. For example, if $R = 6\%$ annually and the loan is for three years with interest compounded monthly, the monthly interest would be $\frac{6\%}{12}$. Also, the "t" would be 36, corresponding to 12 payments per year for three years.

$$A = P(1+R)^t \qquad\qquad A = RP\left[\frac{(1+R)^t - 1}{R}\right]$$

Interest paid on money borrowed *Regular payment calculation*

Next, the right-hand sides of the two formulas for A are equated.

$$P(1+R)^t = RP\left[\frac{(1+R)^t - 1}{R}\right]$$

In solving for P, the amount of the loan, the text proceeds as illustrated below.

$$P = RP\left[\frac{(1+R)^t - 1}{R\cdot(1+R)^t}\right] = RP\left[\frac{1-(1+R)^{-t}}{R}\right]$$

(1) (2)

In (1), the connection to the equated formulas can be identified. However, (2), a simplified mathematical expression, is more removed from the concepts involved and more complex due to the negative exponent in the numerator.

Figure 3b. Course adjustments for conceptual approach to loan calculations.

Adaptation

In the course, we adapted the conceptual explanation slightly:
(a) Bank view: the bank makes a "lump sum" deposit with us that is the amount of the loan. The bank wants a compound interest return on its investment.
(b) Our view: we make a regular deposit each month with the bank, as with an annuity, until the total we accumulate matches the amount the bank wants.
Rather than require students to complete the extra step for multiple interest periods each year, we used formula representations that retained the concept connection, even though its appearance was less tidy.

A: amount after interest has been computed;
RP: regular deposit or loan payment;
P: lump sum amount for deposit or loan, the principal;
r: annual rate of interest for both viewpoints;
n: number of payments per year;
t: number of years for deposit or loan.

As before,

$$A = P\left(1+\frac{r}{n}\right)^{nt} \qquad A = RP\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

Interest paid on money borrowed *Regular payment calculation*

Again, the right-hand sides of the two formulas for A are equated.

$$P\left(1+\frac{r}{n}\right)^{nt} = RP\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$$

We refer back to a simple equation, $x \bullet 3 = 12$, and review that $x = 12 \div 3$.
We apply the same process to $x \bullet \frac{3}{4} = 12$, and write $x = 12 \div \frac{3}{4}$. We extend these ideas to the current situation.

With $P\left(1+\frac{r}{n}\right)^{nt} = RP\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]$, we write

$$P = RP\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right] \div \left(1+\frac{r}{n}\right)^{nt} \text{ or } \frac{RP\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right]}{\left(1+\frac{r}{n}\right)^{nt}}, \text{ as needed.}$$

The calculation formula is neither elegant nor parsimonious; however it retains both concept components.

a proportion approach, but her boardwork lacked the "=" sign. In similar tasks, she often mixed up the location of the missing number, especially if it represented the entire amount. She used a trial-and-error approach with the calculator by solving most problems in several ways and choosing the solution that seemed right. Due to her reasonable number sense, she produced correct answers much of the time. However, she could not explain her process, one of cross-multiplication and solving the equation. Tina mentally changed the percent to its decimal equivalent and wrote only the equation on the board—the method she used correctly for all types of percent problems. She used her calculator to do the arithmetic. Her explanation of the process was sketchy, "I just made an equation and solved it."

While all three students knew a method that led to correct answers, none could explain their method, and all illustrated different thinking processes. None of them admitted to understanding what the others did, and none saw any reason for learning another method than the one they were using. Facilitating discourse to create shared understanding, an important component in constructivist pedagogy, began then and became part of the classroom routine.

In preparation for financial calculations, ideas such as markup and discount have to be expressed with a single mathematical equation (e.g., Selling price = Cost [1 + % markup]). Students calculated correct responses using a two-step process (i.e., find the amount of markup [multiply cost by percent of markup] and add the amount of markup to the cost to arrive at the selling price). However, this method does not extend easily to calculating the amount in a bank account after 15 years when interest is paid monthly. Creating the more efficient one-step equation uses the distributive law of multiplication over addition, a law that was not familiar to students. To support sense making of this abstract principle, we built it with manipulatives, recorded our manipulative results with pictures on the board, and attached numbers to the pictures. After several numerical examples, we generalized the rule for real numbers, a(b + c) = ab + ac. Students practiced using the distributive law to simplify expressions and solve equations. Then we returned to the markup situation, as shown in Figure 2. We reviewed the distributive law frequently throughout the unit.

Mathematicians prefer expressions that are parsimonious and elegant. In creating such expressions through simplification, however, the concept connections can become obscured. Students often have difficulty with the series of steps that produces the simplified expression, and the end product prized by mathematicians may have little meaning for students. One adaptation made in this unit was to focus on the two big ideas that are the basis of financial calculations, which include savings, annuities, and loans, and to use the expressions for those ideas to calculate loan components without further simplification. These are (a) compound growth with a single deposit, and (b) compound growth with regular deposits, also known as ordinary annuities. The expressions used were neither elegant nor parsimonious, but they made sense to the students because the connection to the big ideas was obvious. Figure 3a and b presents the text's approach (Hathaway, 2000) for loan calculations and the adaptation used in this course.

Many difficulties involved mathematical language, such as symbols for division, words that indicate division, how to enter division situations correctly on the

Figure 4. Sample of Tim's work.

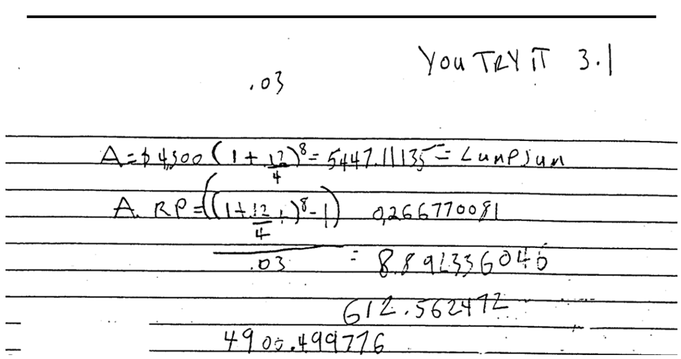


Figure 5. Financial decision making: Buying a new car.

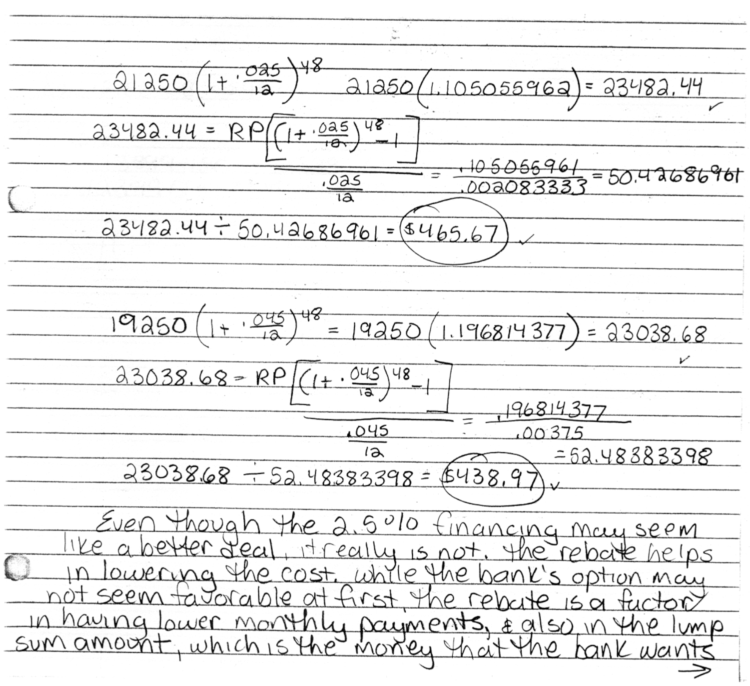
President's Day is noted as the holiday for good deals on new car purchases. With the cold weather, most of the holiday sales have been extended. You have decided to buy a new car. Part of the reason for purchasing the model you have dreamed about is that the auto manufacturer will give you 2.5% financing over a four-year period. This compares favorably with the 4.5% best rate you can get at your local bank. The salesperson explains that the auto manufacturer offers you a choice – either 2.5% financing or a cash rebate of $2000.
If you accept the cash rebate, you must go to the bank for financing. Both the auto manufacturer and the bank compound monthly and expect monthly payments.

Suppose that the car has a sticker price of $ 21,500 and the dealer offers you $250 for your trade-in. You have two choices.
(a) Finance the $21,250 balance at 2.5% over four years with the auto manufacturer
(b) Finance the $19,250 ($21,500 - $2000 cash rebate - $250 trade-in) at 4.5% over four years with your local bank

Which should you choose – the lower rate or the lower purchase price?

Write a report that shows your consideration of this loan situation. Include the calculations you used to support your decision.

Tina's work:


$21250\left(1+\frac{.025}{12}\right)^{48}$ $21250(1.105055962) = 23482.44$ ✓

$23482.44 = RP\left[\frac{\left(1+\frac{.025}{12}\right)^{48}-1}{\frac{.025}{12}}\right] = \frac{.105055961}{.002083333} = 50.42686961$

$23482.44 \div 50.42686961 = \$465.67$ ✓

$19250\left(1+\frac{.045}{12}\right)^{48} = 19250(1.196814377) = 23038.68$ ✓

$23038.68 = RP\left[\frac{\left(1+\frac{.045}{12}\right)^{48}-1}{\frac{.045}{12}}\right] = \frac{.196814377}{.00375} = 52.48383398$

$23038.68 \div 52.48383398 = \$438.97$ ✓

Even though the 2.5% financing may seem like a better deal, it really is not. the rebate helps in lowering the cost. while the bank's option may not seem favorable at first, the rebate is a factor in having lower monthly payments, & also in the lump sum amount, which is the money that the bank wants →

calculator, and use of the "=" sign. Mathematicians use "=" to mean several things, and they do so without conscious attention. However, those who struggle in mathematics often use the equal sign incorrectly. The written communication can be hard to decipher, and correct thinking may be judged as incorrect. Because loan-related calculations require that a number of steps be completed, an organizer, originally prepared by the author and revised with student input, was available to support the sequence of calculations. Laurel completed every exercise on an organizer sheet, Tina used it as a model in front of her, and Tim gave no indication that he ever used it.

A piece of Tim's work submitted toward the end of the section on financial function calculations illustrates the communication issue (see Figure 4). When asked about this work, Tim stated, "The equal sign must be placed next to what you are trying to find." He seemed to use that visual reminder to maintain focus on his goal. Tim said he memorized the steps for calculating compound interest and annuities because we used the same format throughout. He learned to use a scientific calculator for operations that made sense to him, such as exponentiation and messy divisions. Once he mastered the steps for properly using the calculator, he liked using it and saw no need to record intermediate steps for his work. He recorded the results from the calculator exactly as they appeared on the screen. While it would have been ideal for him to report his final answer with only two decimal places, his failure to round in intermediate steps demonstrates more understanding of money, not less. As the reader might suspect, writing was laborious for Tim.

Assessment During the Unit

A brief quiz took place early in the third week. Immediately, the atmosphere in the room changed and tension was very apparent. In an unsuccessful attempt to diffuse anxiety, the instructor spoke quietly with students about their processing. In several instances what they spoke differed from what they wrote, and their anxiety interfered with their ability to see the conflict. Laurel, in particular, described how annuities differed from lump-sum compound interest situations, but she struggled to do the calculations, even with the organizer to guide her. She had calculated correctly many similar examples prior to the quiz. As the time for the quiz completion neared, her anxiety soared. She began to cry and abruptly left the room. As Tina, who had finished her quiz some 15 minutes earlier, left to check on Laurel, it became obvious that in-class assessments were not going to contribute to the learning process for this group.

To compensate, the number of oral student presentations was increased and questions of the work by the other students were introduced. As students defended their work, differences emerged between what they thought and what they wrote. Usually their thought processes were correct but they missed a step, reversed digits, or transferred an incorrect number from the calculator to the blackboard. Saying to the presenter, "I don't follow what you did in your second line," and "Why did you do the calculation in the fourth line?" supported their finding the errors. The instructor's questioning process evolved to include non-presenting students. For example, she might address a non-presenting student, "What questions do you want to ask?" and "What writing needs clarification?" Eventually her role decreased as the learning community became estab-

Figure 6. Excerpt from Laurel's journal.

2/21 2/28 I was an absolute disaster inside! As soon as I heard the words test graded assignment, it was all over! That is how I felt! I don't even want to think of what would happen if I did not get new sheets for the test. I spent hours and hours and hours working on this - checking and rechecking work that was just fine - but I have come too far to get a bad grade on the test! Plus I did find really foolish mistakes and fixed them. I also wrote a very sloppy report by hand, and retyped it that morning. I knew I just had to fix it, I worked too hard for that! I am very proud of my work and I just hope that I got a decent grade - it looks like I did it all correctly, but I don't have faith in my math skills to be TOO certain! I admit that I did not put very much effort in to the home work and personally felt it was over kill in addition to the take home test, so I decided to just do the best I can and not make myself crazy over it, as it was new material and all the other stuff was so stuck in my head I couldn't put new stuff in!

lished and interchanges primarily involved students. During oral presentations, students revealed their understanding of the content with reduced anxiety. They saw how their disabilities interfered with communication of their understanding.

The end-of-unit unit assessment consisted of tasks involving Internet financial sites, loan offers included in credit card statements, and others requiring decision making about finances. Students had to validate the correctness of the information and decide whether to avail themselves of the financial opportunities being offered, accompanying their calculations and decision with written explanations. One task and Tina's response appear in Figure 5.

Students maintained journals during the unit, which they submitted after completing the unit assessment. They recorded thoughts about the unit as it progressed (e.g., where they were confused, where they were having difficulty, the questions they wanted to ask the next class). They also described their thoughts about financial mathematics at the conclusion of the unit. While the journals provided insight into process, individual writing style controlled how much they wrote and the clarity with which they expressed their thoughts. Tim, who hand-wrote his journal, wrote very brief entries that yielded minimal additional information, whereas Tina's entries were detailed and lengthy. Laurel's anxiety is evident in the excerpt in Figure 6.

RESULTS

Were students successful in making sense of the mathematics of finance in an environment based on constructivist principles? Did reform methodology, including multiple forms of assessment, use of manipulatives, and integration of writing, contribute to student learning? Excerpts of student work from this unit illustrate their sense-making of the material and suggest that they will be more effective consumers in money-related matters. Their insights into loans and buying a car exceeded that of many students in other sections of the course taught by the author. Although not tactile learners, the students appreciated the demonstration of the distributive law combining manipulatives and pictorial representations. Laurel said, "I won't have to memorize a rule I don't understand because this method makes sense." Use of multiple forms of assessment, especially oral presentations, discussion, and writing, provided insight into students' understanding of the content that would not have been possible using traditional testing alone because mathematical language issues interfered with students' communication of their understanding. With respect to financial mathematics decisions, these students demonstrated that they are able to compare financial situations such as credit card loans and investments and make sound judgments. They met the requirements of general education courses in mathematics.

A few brief comments about the other units follow. In the third unit, the goals for basic descriptive statistics, which included interpreting data from graphs and a project in which each students selected an existing data set, asked questions about it, and presented an analysis, were accomplished. Interpretation of statistical information presented graphically was difficult for Tim and Laurel, but they managed this visual content because they could rely on the context for meaning and could discuss their emerging understanding with each other until it made sense. The final unit on fair division and social choice was interesting to students and, like the financial unit, was easily supported with organizers and frequent presentations. However, the author's initial thinking that the graph theory section would be enjoyable to students because of its visual nature was totally incorrect. Instead, the graphs turned out to be visually overwhelming to all students. Tim and Laurel struggled to count the number of paths from a vertex. Fortunately, however, after two days of frustration for all of us, diagnostic testing results finally were available. Aspects pertinent to general performance and mathematics are summarized.

Student Profiles and Course Modifications

Tina. According to her report, Tina has superior verbal skills and weak visual spatial skills. Her reported SAT Verbal score was 790, in contrast to her performance on the WIAT-II mathematics reasoning and numerical operations, which was at the second and seventh percentile, respectively. The report stated that her troubles with mathematics began in fourth grade, specifically with memorization of the multiplication tables and long division. Tina never received resource or remediation services during her elementary and secondary education, but she had lots of private tutoring in mathematics. She reported no difficulty with reading. She successfully completed three semesters at a private two-year college in a neighboring state before transferring. According to the report, Tina said that the transfer was a good move for her, but the work is much more challenging.

The report summary stated that Tina exhibits a superior to very-superior range of verbal intellectual ability with average to low-average visual spatial skills (WAIS-III: Verbal, 93rd percentile; Performance, 45th percentile), indicative of a nonverbal learning disability and severe mathematics disorder; her full profile is consistent with developmental right-hemisphere vulnerability. Recommendations included accommodations for students who have LD, including substantial modification or waiver of mathematics courses.

Laurel. Laurel's report revealed a woman of average intelligence, with significant differences between her verbal and performance scores (WAIS-III: Verbal 104; Performance, 74; Full Scale, 90). She did not receive resource help in her pre-college education; she reported that she took a business math course in high school. She satisfied her mathematics requirement at the community college with accounting courses. Her WRAT-III results indicated an arithmetic level consonant with beginning high school and spelling and reading levels comparable to those of high school graduates. Sequential problems that interfere with her visually organizing and reasoning mathematical concepts were noted. She attempted the non-credit college mathematics requirement three times before she passed, and reported that she needed lots of individual tutoring to do so. However, because of her persistence, she gained basic early algebraic concepts and mathematics reasoning.

The report summary indicated that she does not have an LD in the mathematics area per se, but has visual organization difficulties that create problems in mathematics reasoning courses, necessitating individual instruction and course modification to achieve in college-level mathematics courses. Comments stated that Laurel has the basic reading and writing abilities to succeed in the social service field; that she is aware of her difficulty following and figuring sequential events; and that with time and persistence, she can achieve her goals.

Tim. Tim had dropped out of high school 30 years prior to taking this course. Dates reported for his GED (1996) and community college degree (1985) conflict with Tim's responses in an interview with the author. The report described a history of substance abuse and mental health concerns. It also noted that Tim has always had difficulty with mathematics, but is comfortable to the pre-algebra level. He received no resource support or tutoring in mathematics in prior education. The WAIS-R revealed that in spite of overall well-developed general intelligence, Tim presented serious deficits in mental arithmetic, basic arithmetic processing, visual sequencing, and common sense (Verbal, 117; Performance, 80; Full Scale, 98). He also exhibited difficulties in both functional and educational reading, achieving a grade equivalent of 4.3 on the Adult Basic Learning Examination. Comprehension, drawing inferences, and making conclusions based on printed text were described as problematic. Tim had considerable difficulty with the physical act of writing and fundamental language grammar and usage. On the Arithmetic Subtest of the Wide Range Achievement Test, 3rd ed., Tim achieved a fifth-grade equivalent, which indicated that he has basic understanding of numbers, operations, place value, and numeration, but lacks skill for application. Tim was described as having a mathematics disorder in mathematical ability. His mathematical calculation and reasoning fell substantially below that expected for his chronological age, measured intelligence, and age-appropriate education.

All three psychological profiles identified disabilities in the visual-spatial dimension. While the results confirmed much of the information gained from the learning style inventories and personal observation, they also revealed the limitations of these instruments and the instructor's lack of knowledge about specific mathematics disabilities.

It is no wonder that graph theory was hard! Fortunately, in mathematics there are multiple ways to represent the same ideas. We were able to use matrix representations in place of the visual approach and addressed basic questions. However, the author realized that typical problems in graph theory topics, namely, Euler paths, Hamiltonian circuits, and the Traveling Salesman problem, would be very cumbersome in matrix format. Therefore, it was necessary to modify content in this section of the course. We limited work to concept formation for paths and circuits in elementary problems and then considered practical applications. We used videos of snow plow and delivery routes to illustrate Euler paths and Hamiltonian circuits. We considered problems that graph theory can be used to solve from diverse areas, such as airplane schedules and placement of cell phone towers, and discussed ways to approach these problems.

The discussion was lengthy, and had many tangents. The instructor's role was to support students staying focused and to be sure that they considered all reasonable aspects of the problem. They often used colored chips and squares to make a visual image of their thoughts and worked hard to get their classmates to understand. At times it was difficult to follow their reasoning, but the effort paid off in (a) confidence for them and (b) insight into students' thinking for the author. In spite of their frustration with using visual methods, through discussion, the students were able to recognize types of problems that graph theory could be used to solve.

For their final task in this section, Laurel and Tim used matrix methods to plan small breakout groups of gang members so as to avoid conflicts; Tina performed a similar analysis with respect to television stations and airing of commercials. Students worked hard with moderate success to explain their methods verbally, but they were less successful in transferring thoughts to a written format.

Figure 7. Loan payments calculation organizer.

Loan payment calculations combine both lump sum deposit and annuity calculation processes. With the formula in this format, the numerator is the "A" in lump sum deposits and the denominator is the value that, when multiplied by the regular payment, results in the "A" for annuities. With this organizer, we are calculating the regular payment, or how much we will pay back to the bank each month on our loan. *P* represents the bank's lump sum deposit amount and the customer's full amount of the loan.

$$\frac{P(1+\frac{r}{n})^{nt}}{\frac{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}} = RP$$

1. Calculate $\frac{r}{n}$: [ ]

2. Calculate n•t: [ ]

3. Compute: $(1+\frac{r}{n})^{nt}$ [ ]

4. Multiply this value by P. This is the numerator. It is the lump sum amount the bank wants back. [ ]

5. Calculate the denominator now. Subtract 1 from the value in step 3, $(1+\frac{r}{n})^{nt}-1$. Now, divide this answer by $\frac{r}{n}$. [ ]

6. Take the value in step 4 and divide by the value in step 5. The loan payment, RP, is: [ ]

Implications for Practice

The course was based on constructivist principles – students built their understanding together. The course met twice a week for 150 minutes. With few exceptions, direct instruction was limited to 30 minutes per class. The instructor's role for the remainder was that of questioner, facilitator of discussion, and guide for students' process of making sense of the content. Approximately 60% of the questions posed required higher-order thinking. The extended meeting time, which included two short breaks, allowed students to verbalize initial understanding, complete some individual work that revealed misunderstanding, revisit the content and fill in the gaps, and build a mental structure.

In the mode of a student-centered, sense-making classroom, the author honored students' learning style and trusted them to explain whether the calculation organizers she had prepared were useful. Students discussed where they found them confusing, and the question "What would need to change to make them clearer?" resulted in a joint product that was more useful than anything the student or the author, with opposite styles, could have developed alone (see Figure 7).

Practice homework problems on material that made sense were essential to student learning, and the unit assessment provided evidence of understanding. However, distinguishing between issues of mathematical understanding and mathematical communication in their work was challenging and time consuming. Efforts to resolve differences between oral presentations, written explanations, and calculations usually revealed correct conceptions of the mathematics and incorrect button-pushing on the calculator or incorrect transfer of results from the calculator to the paper or copying errors from one section of the paper to another. The real-world contexts and their understanding of them, as occurred in spirited discussions throughout all units, motivated their interest. Students' thought processes and explanations were rarely sequential. Like those of many mathematicians, the author's processes are sequential. With patience on all sides, with asking for a repeat explanation where statements were not clear and students' willingness to repeat, students ended up revealing mathematical reasoning adequate to meet course requirements. The extended meeting time was a critical component in their success.

DISCUSSION

The studies reviewed addressed little of the mathematics content that students with LD study in high school. No empirical studies reporting mathematics content for college students who have LD exist (Hughes & Smith, 1990). Ellis and Wortham (1999) reported that in the last 20 years, many techniques have been developed that promote self-efficacy, self-advocacy, self-control, and self-monitoring of the learning process. However, these authors acknowledge that most instruction for students who have LD does not emphasize these techniques, but is based on accommodations in assessment and grading, content, and nature of assigned tasks due in part to inclusion classroom instruction. They argue that accommodations can "water down" the content and limit students' growth. Elements of watered-down content from their perspective include memorization of loosely connected facts, few opportunities to engage in higher-order thinking, and a simplified curriculum that inhibits students' ability to make connections. They argue for setting the goal for a "watered-up" curriculum that aims for more depth, student construction of knowledge, making connections within and across disciplines, and developing effective mental habits, including higher-order thinking skills.

Their vision meshes with essential components of current mathematics reform and reflects constructivist principles upon which this course was conducted. However, this course was not typical. That it existed was due in large measure to lessons Laurel had learned about self-advocacy. The students were not typical college students either; no one had gone beyond the first year of algebra in their previous learning. Yet they were successful in learning most of the content in this course because they had an interest in the topics, they were required to think at higher levels and connect broad concepts, and they were not required to memorize or use elegant formulas or procedures. Finally, the substantial amount of instruction time and small class size were major contributing factors to success.

Reported observations and examples of student work illustrate the diversity of LD among these students and highlight strategies that contributed to their success. However, validation of these and other learning strategies for college-level students is needed. The assessment process described here was not traditional and was very time consuming. Will other college mathematics faculty judge it valid? Other implications for practice remain. How might colleges utilize experiences from this project in their general education mathematics courses? How can they support the success of students who have LD in mathematics courses when no special section exists, when additional time is not possible, and when instructors expect parsimony and elegance?

Strawser and Miller (2001), in their discussion of further research, address limitations due to the complex nature of mathematics-related LD. However, mathematics faculty need support to be effective when they teach students who have mathematics-related learning disabilities. The literature must address the issues. With the increasing number of students with LD in higher education, there is a critical need for mathematics and learning specialist faculty to share experiences, debate learning philosophy and pedagogical strategies, and develop a theoretical framework for empirical research with this population.

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