Frontline Learning Research
An official journal of EARLI

Frontline Learning Research special Issue: Vol. 13 No. 2 (2025) 67 - 101
ISSN 2295-3159

Exploring Math Moments: Middle-schoolers' Phases of Problem-solving, Executive Functions in Practice, and Collaborative Problem Solving

K. Ann Renninger[1], Gertraud Benke[2], Ricardo Böheim[3], Julien Corven[4], Maria Consuelo De Dios[1], Maeve R. Hogan[1], Moe Htet Kyaw[1], Ana G. Michels[1], Marina Nakayama[1], Pablo E. Torres[5], Helena Werneck[1], & Feven Yared[1]

[1]Swarthmore College, USA
[2]University of Klagenfurt, Austria
[3]Technical University of Munich, Germany
[4]Illinois State University, USA
[5]Facultad de Educación, Pontificia Universidad Católica de Chile

Article received 27 September 2023 / Article revised 18 October 2024 / Accepted 24 October 2024/ Available online 14 March 2025

Abstract

Collaborative problem solving (CPS) has been shown to both engage and benefit students' learning of mathematics. However, there is evidence that group work is not always easy to facilitate, in part because educators lack details about learners' engagement during group work: the processes of problem solving involved, and how these are engaged. In this exploratory study, we focused on these processes in the moments of related math activity, or math moments, engaged by two groups of interested, urban, middle-school aged students during four sessions of work in the Virtual Math Teams (VMT) environment. We examined three phases of their problem solving: Exploring, Constructing, and Checking. In addition, to further describe the students' cognitive and behavioral engagement, we considered both the process of students' use of executive functions (EF), during problem solving, termed executive functions in practice (EFP), as well as the stage of CPS (Participation, Cooperation, and Collaboration), during phases of problem solving. We learned that the relation between each phase of problem solving, categories of EFP, and stages of CPS vary; for example, the problem-solving phase of Exploring was found to have a more positive effect on EFP and CPS than either Constructing or Checking. Implications for educational practice, and next steps for related research are described.

Keywords: momentary engagement, problem solving, executive functions, collaborative problem solving

Corresponding author: K. Ann Renninger, 500 College Ave. Swarthmore, PA 19081, USA, krennin1@swarthmore.edu. *DOI: https://doi.org/10.14786/flr.v13i2.1371*

1. Introduction

Studies have shown that, compared to individual activity, students' collaboration during mathematical problem-solving increases their opportunities for engagement (e.g., Webb et al., 2019) and can support students to develop their identities as learners of mathematics (e.g., Featherstone et al., 2011). However, following a review of 66 qualitative and quantitative studies, van Leeuwen and Janssen (2019) concluded that teachers' abilities to adjust the assistance they offer their students during collaborative activities vary, and their effectiveness, in turn, significantly impacts whether student learning occurs. To understand how collaborative problem solving (CPS) might be optimized in classroom instruction, detail is needed about students' engagement in phases of problem solving during moments when they are working with mathematics.

Momentary engagement refers to individuals' activity, or participation, during a brief period of time, and has primarily been discussed as situation-specific (Nolen et al., 2015; Symonds et al., 2021). Dietrich et al. (2022) pointed to the importance of further examining such experiences to consider complex change. Investigation of momentary engagement in group-based contexts, in particular, could extend understanding of the role of group work in supporting students' attention to tasks (e.g., Hmelo-Silver et al. 2018; Pollastri, et al., 2013), enhancing reasoning (e.g., Barron, 2003), and promoting the development of a collective working memory that may increase their capacity for problem solving (Kirschner et al., 2018; van den Bossche, et al. 2011; Zambrano et al., 2019).

In this article, we report on findings from an exploratory study of two groups of urban middle school students' work with online collaborative problem solving in the Virtual Math Teams (VMT, https://vmt.mathematicalthinking.org/) environment (Stahl, 2013). These data are publicly available and de-identified. The two groups each worked with the same four sessions of open-ended dynamic geometry problems, and their sustained engagement across the sessions indicated that all participants had a developed interest in the problem solving they were doing (Renninger et al., manuscript in preparation; Renninger & Hidi, 2016). To study cognitive and behavioral engagement during phases of the students' problem solving (Exploring, Constructing, and Checking), we investigated the students' use of core categories of executive functions (EF; Working Memory, Cognitive Flexibility, Inhibitory Control) which we describe as executive functions in practice (EFP), their cognitive engagement, as well as their behavioral engagement in each stage of CPS (Participation, Cooperation, Collaboration).

1.1 Momentary engagement and mathematics

Studies of engagement in academic tasks have primarily focused on three types of engagement: cognitive, behavioral, and affective (e.g., Skinner & Pitzer, 2012; see Fredricks et al., 2004; Fredricks & McColskey, 2012 for reviews). Research has shown that these types of engagement co-occur and are malleable. As Pohl (2020) explained, students who are completing math tasks benefit from being cognitively engaged as this positions them to recall math concepts and to use these in planning strategies, work on solving the problem, monitor their progress, and evaluate their correctness. Moreover, research has shown that when students are sufficiently behaviorally engaged to complete tasks and participate in their mathematics class, their cognitive engagement is effectively supported (Dong, et al., 2020), and they are able to maintain engagement (Cook, et. al., 2020). Within-person fluctuations in engagement levels have also been observed, leading a number of researchers to call for more situated and moment-specific analyses (e.g., Dietrich et al., 2022; Nolen, 2020; Rogat et al., 2022; Salmela-Aro et al., 2021; Symonds et al., 2019, 2021). Given these findings, the study of momentary engagement in the context *of collaborative activity in mathematics may be particularly important.*

1.1.1 Mathematical sense-making and practices

Mathematical sense-making is central to the process of problem solving in mathematics, as it describes individuals' developing understanding and flexible use of mathematical concepts (e.g., Harel & Sowder, 2005; Schoenfeld, 1992/2016). The development of mathematical sense-making is a cognitive process that requires students to work with new information and make connections to what they already know (e.g., Kasmer & Kim, 2011; Rau & Matthews, 2017); it also is a skill that is developed and enhanced through collaboration in specific contexts, such as problem solving (Bonotto, 2005; Gerson 2008; Kelton et al., 2018; Stahl, 2013).

Problem solving has been variously described as a set of steps, or phases, that begins with understanding the problem and progresses through a sequence that includes making a plan, carrying it out, and checking (Polya, 1945; Schoenfeld, 1992/2016). Stahl (2013) clarified that the process is not so linear when individuals are working collaboratively with open-ended, inquiry problems that allow exploration. In the collaborative context, the process involves discovery which includes explorative dragging (exploring), experimental construction (constructing), and determination of dependencies (checking)—phases of problem solving that have been shown to benefit conceptual understanding.

Boaler and Selling (2017), for example, reported that students in mathematics classrooms that have explicitly prioritized the exploration of strategies as a component of problem solving have a deeper understanding of mathematics content, as well as more positive feelings about mathematics, than students in classrooms that do not encourage exploration. Studies also have provided evidence that checking work following problem solving is associated with higher levels of student performance and conceptual understanding (e.g., Eshuis et al., 2019; Kuhn et al., 2020; Zhang et al. 2021).

1.1.2 How mathematics practices are engaged, EFP and CPS

Study of students' use of cognitive and behavioral engagement with phases of problem solving has the potential to provide detail about how students engage in collaborative mathematics activities. Research has addressed the outcomes of EFs, as individual categories of behavior, and as a composite description of engagement (e.g., Mann et al., 2017; Younger et al., 2023) and CPS (e.g., Andrews-Todd & Forsyth, 2020) individually. To the best of our knowledge, however, no one has either investigated the use of executive functions (EFs) and/or stages of CPS during students' behavioral engagement in phases of problem solving.

In the present study, we use EFP to describe the process of students' use of EFs in naturally occurring settings such as collaborative group work. Whereas EFs are usually studied in controlled settings with standardized tasks (Bailey et al., 2018, Chan et al., 2008; McCoy, 2019), in our investigation we are focused on the process of students' cognitive and behavioral engagement with problem solving; therefore, we review EFs as they are activated and practiced in an ecologically valid context.

In general, three core EFs are used to describe individuals' cognitive engagement with tasks: working memory, cognitive flexibility, and inhibitory control (Diamond, 2013). Working memory refers to the ability to recall and manipulate relevant information to work with tasks (Bailey et al., 2018; Diamond, 2013; Radvansky & Copeland, 2006). Cognitive flexibility describes the ability to adjust one's behavior or thoughts to changed circumstances such as unexpected failure, or opportunity (Diamond, 2013; Jacques & Zelazo, 2001). Inhibitory control pertains to the ability to resist the impulse to respond to a situational demand and to instead engage in a more appropriate but subdominant response (Letang et al., 2021; Veraska et al., 2020).

Studies of EFs have shown that they are critical for both achievement and learning (Caviola et al., 2020; Jose et al., 2020; Long et al., 2011; Mann et al., 2017; Skaguerlund et al., 2019). In mathematics, research has pointed to positive associations between math achievement and core EFs (e.g., Brookman-Byrne et al., 2018; Clark et al., 2010; Swanson & Beebe-Frankenberger, 2004; see also Cragg & Gilmore, 2014 for a review of the literature on EF and mathematics). Specifically, in solving a mathematical problem, working memory allows students to recall and apply previously learned knowledge (Karpike, 2012; Peng et al., 2016); cognitive flexibility allows students to sort through different solutions and choose the most efficient strategy (Huizinga et al., 2014; Yeniad, et al., 2013);

and inhibitory control allows students to focus their attention on the task and ignore distractions (Bishara & Kaplan, 2022; Ponitz et al., 2009).

Different EFs also may be uniquely utilized in one or another part of problem solving. For example, Viterbori et al. (2017) found that when students are working on multi-step problems, working memory significantly predicted problem solving accuracy, presumably because it allowed students to make use of correct mathematical information. Viterbori et al. also reported that students may need to call upon cognitive flexibility to shift between multiple representations of a problem. In other work, Lee et al. (2009) suggested that inhibitory control may be particularly necessary for students' understanding when a problem includes irrelevant information that needs to be ignored.

By contrast, collaborative problem solving (CPS) describes stages in the process of two or more individuals working together to achieve a shared goal: participation, cooperation, and collaboration. Drawing upon the A3C framework, referring to attendance, coordination, cooperation, and collaboration (Jeong et al., 2017) and self-regulated learning theory (Hadwin et al., 2011), we describe CPS stages as reflecting the extent to which students' behavioral engagement is influenced by other students in their group. Participation refers to individual behaviors in which a student may act according to their own goals and methods, but their goals and methods are dependent on and shaped by other people; Cooperation refers to group behaviors in which a common goal is explicitly established, but the methods to achieve it are not joint; Collaboration refers to group behaviors in which goals and methods are shared and jointly enacted

Peer-to-peer interactions during group problem solving provide a basis for knowledge construction that results in shared group cognition that is not present when an individual works alone (Stahl, 2013; van den Bossche, 2011; Zambrano, et al., 2019). Students' behavioral engagement when working with a group has been shown to differ from individual problem solving. For example, Sun et al. (2022) found that fifth grade students' behavioral engagement was highest during collaborative work and lowest during independent work associated with direct instruction. Furthermore, outcomes of group problem solving often differ from individual problem solving as shown by Mohammadhasani and Asadi's (2020) study of students' completion of mathematics problems in online collaborative groups. They reported that those who completed the problems collaboratively experienced greater learning gains than those who worked on them individually. These findings are consistent with other studies demonstrating that students working in groups on academic tasks outperform those working individually on the same tasks (e.g., Eshuis et al., 2019; Sankaranarayanan et al., 2021).

1.2 Current study

In this study, we explored the process of two groups' momentary engagement during four sessions of online collaborative problem solving in the VMT environment. Our goal was to detail the process of the students' problem solving during group work, considering variations in cognitive and behavioral engagement among students who were interested and on task. We selected the groups for study based on the following criteria as assessed in a prior study (Renninger et al., manuscript in preparation): high levels of interest in mathematics (all participants in each group studied, remained on task throughout each of the problem-solving sessions; Renninger & Hidi, 2016), similar number of math moments (see section 2.3.1), but differences in their demonstrated levels of collaboration (for the methodological determination, see 2.3.4). Specifically, Group 1 was less collaborative than Group 2. Although there were different numbers of students in the two groups selected for analysis, data from prior study suggested that differences in the number of participants in student groups did not influence the observable dynamics of groups.

Study of two groups that had high levels of interest allowed study of high levels of cognitive and behavioral engagement during group work; study of groups that had approximately the same number of moments of math-related activity, or math moments ensured that the structure of the sessions were similar for both groups. These two criteria allowed us to focus on potential differences between the groups related to the quality of the students' collaboration. We expected that these data could provide a preliminary mapping of the engagement of students in this age groups' online collaborative problem solving and could help to address what teachers need to know to effectively support their students to work collaboratively.

Many studies of engagement measure two time points, usually before and after task completion; however, as Siegler (1998) pointed out, study of any type of change ideally assesses and measures change while it is occurring. Given findings such as Symonds et al.'s (2021) indicating that momentary engagement varies within individuals, methods of measurement should ideally assess student engagement across the entire process of problem solving, and, in the group context, should account for co-negotiated engagement processes (Rogat et al., 2022). Here, the chat and replayer functions of VMT provided us with moment-to-moment records of the students' cognitive and behavioral engagement. We focused on three features: (a) the phases of problem solving engaged – Exploring, Constructing, and Checking; (b) the students' use of core EFP – Working Memory, Cognitive Flexibility, and Inhibitory Control during phases of problem solving providing information about cognitive engagement; and (c) each stage of CPS – Participation, Cooperation, Collaboration– during phases of problem solving, which provided information about their behavioral engagement.

Two research questions were addressed:

RQ 1: What is the relative proportion of each phase of problem solving overall, and is the distribution of phases similar by session? Do these proportions vary for two groups with different levels of collaboration?

RQ 2: How do groups engage with different phases of problem solving? Are there differences in the cognitive and behavioral engagement of two groups with different levels of collaboration? Specifically, what is the relation between each phase of problem solving and the EFP and CPS of each group? Is there change across sessions?

2. Methods

2.1 Participants

Participants were two groups of middle school students from the same urban school who were enrolled in an after-school program. Group 1 had three members and Group 2 had four members; their participation in the group was anonymous and their data were de-identified.

2.2 VMT learning environment

The VMT environment is an online, multi-user version of GeoGebra that includes a shared workspace and a chat tool to the side of the screen (see Figure 1). Each group of students worked in a separate VMT room. Group members communicated with each other through the chat, which allowed them to type and submit messages that were displayed to all of them. Only one student was able to interact with the shared space at a time, and did so by clicking on a "take control" button. While one student was "in control" others made suggestions through the chat. The same groups of students worked together on each of the four sessions of geometry topics (see Table 1). The first three sessions each lasted for an hour. During the fourth session, the students worked for more than two hours, mid-way through the session, they took a break. The problems on which the students worked and the instructions were the same for each group in each session, and prompted student use of the chat to discuss their work.

2.3 Problems

The problems for each of the sessions were rich, open-ended geometry problems that were specifically designed to scaffold support for developing skills in "collaborative and mathematical discourse, exploring dependencies, geometric construction, analytic explanation and domain content" (p. 163, Stahl, 2013). Problem content for each session was distinct and was anchored in the preliminary standards for high-school geometry described in the Common Core State Standards Initiative (2011). Thus, the students were asked to engage in working with alternate representations and dependencies, including congruence, symmetry, and rigid transformations. Problems for each session were sequenced to support the students' developing levels of skill for understanding and working with proof, consistent with the van Hiele levels for geometric reasoning (see deVilliers, 2003, as cited in Stahl, 2013). Problems first focused the students on noticing and wondering (Ray-Riek, 2013), and followed this with encouragement to describe what they were doing and their justification for this in their chat-based explanations of their work.

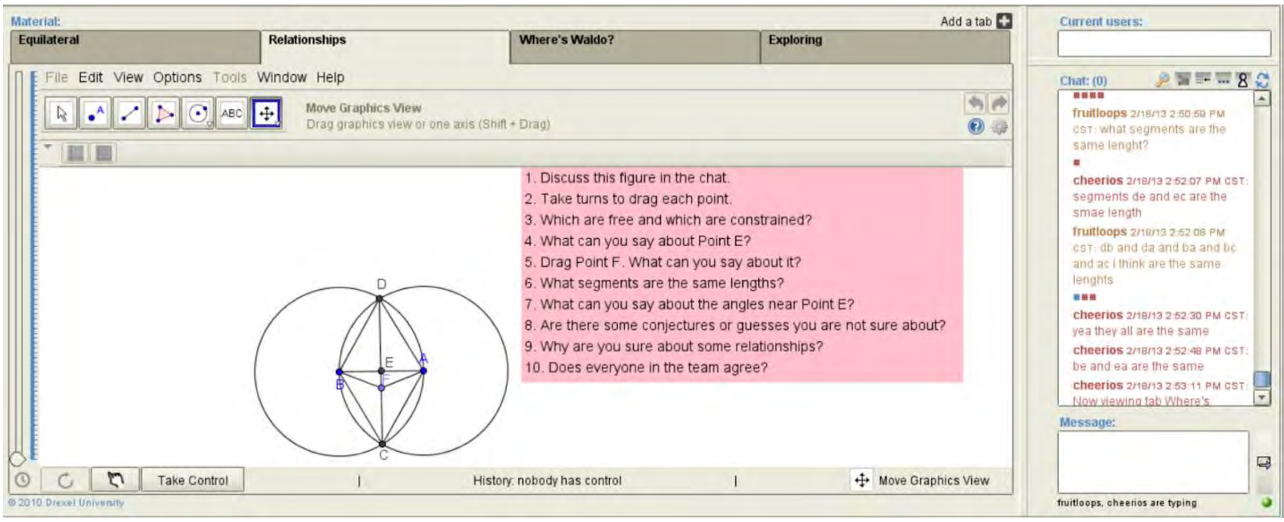
2.4 Coding and data reduction

2.4.1 Math moments

Math moments refer to chunks of related and sequential mathematics activity during open-ended problem solving. In the present study, identifying the moments when students were working with mathematics (as opposed to socializing or asking procedural questions) during each session of problem solving provided the context for studying the student groups' cognitive and behavioral engagement during phases of problem solving. As such, math moments were not identified based on screen or tab in this VMT context. Rather, the group members set the direction of the math moments that they engaged, and we aggregated these based on the group members' steps in their discovery process as they worked on the open-ended problem solving of each session. Thus, while the sessions of problem solving were comparable to each other in the opportunities they afforded, individual math moments might not be due to their having different foci, lengths, etc. For this reason, we aggregated these data for analyses at the session level.

Math moments were consensually identified by two researchers for each group (Hill, 2012; see Tables 2 and 3 for a description of the content and duration of the math moments engaged by each group' across sessions). Most of the moments for each session for each group were math moments. While it was expected that the groups would vary in their work with each of the problems, the math moments identified for each included similar content (e.g., constructing an equilateral triangle, or discussing the meaning of constrained points), and/or often reflected the prompts that the problem provided. The consensual reliability check was employed to ensure that a consistent set of considerations informed the identification of math moments for each group, and between groups (e.g., the decision that multiple attempts at the same construction were described as multiple moments).

Figure 1.
Screenshot of VMT Activity.



Note. The screenshot is of Topic 2, Equilateral Triangles.

Table 1

Session Topic Descriptions

Topic 1 Introduction to VMT	*Objective*: Understand how to construct objects and create dependencies *Session Goals*: • Use construction tools (i.e. point tool, compass tool, etc.) to construct points, segments, and figures • Construct appropriately independent and dependent points • Perform "drag tests" (moving points of a given figure) to test dependencies
Topic 2 Equilateral Triangles	*Objective*: Understand the properties of triangles *Session Goals*: • Construct an equilateral triangle using two circles with a common radius • Generate sets of dependent relationships of a given figure with an inscribed triangle • Converse about the different properties of triangles
Topic 3 Perpendicular Bisectors	*Objective*: Understand the properties of perpendicular lines and perpendicular bisectors *Session Goals:* • Construct a perpendicular bisector of the radius of a circle • Construct a perpendicular bisector through an arbitrary point on a line • Construct a parallel line tool
Topic 4 Inscribed Polygons	*Objective:* Understand geometric proportions *Session Goals:* • Construct an equilateral triangle inscribed within another proportional equilateral triangle • Construct a square inscribed within another proportional square • Construct a hexagon inscribed within another proportional hexagon

2.4.2 Phases of problem solving

As described in Table 4, three phases of problem solving may characterize math moments: Exploring, Constructing, or Checking (e.g., Polya, 1945; Salminen-Saari et al., 2021; Schoenfeld, 1992/2016; Stahl, 2013). Although more formal conceptualizations of problem solving may point to these phases occurring sequentially, as Stahl (2013) noted, groups working with problems in VMT sessions may engage in these phases in any order. For example, a group may enter the Constructing phase and begin to construct figures before the group has had a chance to explore the problem, or the group may skip the Checking phase entirely after they have completed their constructions. Two researchers coded the phases of problem solving associated with each math moment, following which an independent researcher coded 20% of the data drawn at random and conducted a reliability check; reliability was substantial, k= .67 (Landis & Koch, 1977).

2.4.3 EFP

Group members' EFP were coded at the individual student level for each math moment. All students were coded for categories of behaviors associated with Inhibitory Control, Working Memory, and Cognitive Flexibility (see Table 5). As shown in Table 5, the coding rubric for assessing each EFP was derived from the existing literature on the relevant EF and consisted of multiple items. By definition, the coding of EFP involved (a) reviewing the students' process during problem solving, (b) required identifying the opportunities to use EFP created by the group and afforded by the activity's design, and was followed by (c) consideration of to what extent individuals took advantage of opportunities.

Coding of EFP was undertaken in three steps. First, two researchers reviewed each group's work using the replayer focusing on the identified math moments for the groups. Second, the researchers had an initial meeting to discuss and agree on opportunities that were identified for group participation (e.g., to code Working Memory, the researchers needed to agree about whether there is information that the students need to recall to complete a task). We only scored items describing behaviors for which students had opportunities to engage in. When students did not have an opportunity to use EFP, we coded this as nonapplicable.

Third, the researchers conducted an independent assessment of each student's EFP during math moments using the questions described in Table 5. For each question, students were scored on a four-point scale consisting of the values 1 (fully exhibited described behaviors), 0.5 (had some but not all behaviors), -0.5 (had minimal number of associated behaviors), and -1 (did not exhibit the behaviors). Scores ranged from -1 to 1, with a 1 indicating that the student took full advantage of the opportunities afforded in that moment to exhibit a given EF, and a -1 indicating that a participant took no advantage of those opportunities. As described above, an exception was made for instances in which no opportunity was present (n/a). This affected Cognitive Flexibility scores the most, possibly because it is a higher order EF (Diamond & Ling, 2016) more than other EFP, leading to fewer moments in which Cognitive Flexibility was coded.

Finally, an average score for each type of EFP (Working Memory, Cognitive Flexibility, and Inhibitory Control) was calculated for each individual and for the group. The average EFP score was an aggregate of Working Memory, Cognitive Flexibility, and Inhibitory Control average scores for the individual and the group. Amount of each type of EFP was calculated for each student. To confirm reliability, two raters independently rated students' use of EFP. All coding was established through discussion; scores were reviewed and revised following Hill (2012) until 100% agreement was achieved.

Table 2
Group 1's Math Moment Content and Duration, Sessions 1-4

Session 1: Introduction	Session 2: Equilateral Triangles	Session 3: Perpendicular Bisectors	Session 4: Inscribed Polygons
1. Segment A construction (03:18)	1. Segment DE construction (06:29)	1. Segment construction (03:06)	1. Polygon drag and discussion (17:31)
2. Dragging points A and B (03:13)	2. Triangle DEF construction (04:47)	2. Circle constructions (02:57)	2. Outer triangle construction (07:16)
3. Various segment construction (02:34)	3. Firsts attempt at construction of circles A and B (06:27)	3. Segment construction (02:51)	3. First attempt at inner triangle construction (12:07)
4. Discussion of figure (01:48)	4. Second attempt at construction of circles A and B (06:39)	4. First attempt at line J construction (04:18)	4. Inner triangle drag test (09:32)
5. Experimenting with text (11:23)	5. Point C construction s(03:23)	5. Second attempt at line J construction (04:02)	5. Tab hints (00:22)
6. Various segment construction (01:38)	6. Triangle ABC construction (02:44)	6. Line J drag test (06:01)	6. Inner triangle drag test (03:48)
7. Circle constructions (02:09)	7. Discussion of impressions (03:25)	7. Line construction (02:52)	7. Exploring compass tool (16:31)[a]
8. Discussion of figure (01:21)	8. Discussion of constrainment (03:11)	8. Circle constructions (01:28)	8. Second attempt at inner triangle construction (25:43)
9. Angle Constructions (03:25)	9. Discussion of segment lengths (01:58)	9. Line constructions (03:45)	9. Reflection (09:42)
10. Experimenting with grid (03:14)	10. Discussion of angles (00:15)	10. Perpendicular line drag test (00:49)	10. Discussion of dependencies (06:19)
11. Polygon constructions (02:34)	11. Discussion of unsure relationships (03:26)	11. Circle constructions (03:08)	11. Testing dependencies (02:57)
12. Intersection constructions (09:02)	12. Discussion of triangle types (02:30)	12. Line constructions (05:03)	12. Planning outer square (04:00)
13. Segment and dependencies (02:48)	13. Discussion of unsure relationships (03:41)	13. Dragging lines (02:07)	13. Outer square construction (17:46)
	14. Dragging triangles (03:38)	14. Creating tools (08:49)	14. Planning inner square (03:48)
		15. Testing tools (05:59)	15. Inner square construction (06:01)
			16. Reflection (03:27)

Note. Duration is reported in minutes and seconds based on time stamp.
[a] Students took a break and then resumed work.

Table 3
Group 2's Math Moment Content and Duration, Sessions 1-4

Session 1: Introduction	Session 2: Equilateral Triangles	Session 3: Perpendicular Bisectors	Session 4: Inscribed Polygons
1. Planning problem (05:57)	1. Triangle DEF construction (07:29)	1. Segment IJ construction (03:06)	1. Discussion of impressions (07:15)
2. Exploring angles (04:06)	2. Circle G construction (02:03)	2. Circle I and J constructions (03:30)	2. First attempt at outer triangle construction (02:37)
3. Shape constructions (02:38)	3. First attempt at circle H construction (03:06)	3. Segment KL construction without perpendicularity (06:20)	3. First attempt at inner triangle construction (01:39)
4. Experimenting with display (09:21)	4. 2nd attempt at circle H construction (05:12)	4. Segment KL construction with perpendicularity (03:06)	4. Testing construction (06:47)
5. Segment constructions (04:19)	5. Circle J and L construction (05:26)	5. MN line construction (10:09)	5. Second attempt at outer triangle construction (32:47)[a]
6. Dragging figures (05:51)	6. Triangle JKL construction (02:08)	6. Discussion of perpendicular bisectors (04:49)	6. Discussion of compass tool (23:37)
7. Experimenting with circles (03:54)	7. Circle K and J construction (02:37)	7. Testing perpendicularity (16:44)	7. Second attempt at inner triangle construction (13:34)
8. Experimenting with compass tool (07:08)	8. Triangle JKL construction (01:14)	8. Creating new tool (04:21)	8. Discussion of impressions (05:42)
9. Discussion of figures (02:41)	9. Discussion of figures (05:29)	9. Testing new tool (11:09)	9. First attempt at outer square construction (07:03)
10. Line segment constructions (04:31)	10. Discussion of impressions (05:00)		10. Second attempt at outer square construction (04:22)
	11. Discussion of constrainment (03:02)		11. Third attempt at outer square construction (04:37)

	12. Discussion of segment lengths (03:18)	12. Fourth attempt at outer square construction (14:08)
	13. Discussion of angles (08:08)	
	14. Discussion of triangle types (02:55)	
	15. Discussion of triangle changes (03:47)	

Note. Duration is reported in minutes and seconds based on time stamp.
[a] Students took a break and then resumed work

Table 4
Phases of Mathematical Problem Solving During Math Moments, Definitions and Examples

Phase	Definition	Example
Exploring	Examining characteristics and dependencies among the geometric elements either to understand the problem or plan how to solve it	*Multiple students drag a point in a pre-constructed triangle; others observe their peers' dragging and comment on how the proportions and size of the triangle is affected*
Constructing	Purposefully adding or modifying geometric elements in order advance problem solving by creating new figures or tools	*The group constructs a triangle in a way that its sides are equilateral*
Checking	Exploring, moving, turning, flipping, or resizing existing elements in order to assert their relationship	*The group drags all vertices of an equilateral triangle to see if it stays equilateral*

Note. Math moments may also include hybrids of the different phases of problem solving or discussion.

2.4.4 CPS

Group members' CPS was coded using a three-step process that paralleled that described for coding and scoring EFP. CPS was coded at the individual level for behaviors associated with participation, and at the group level for behaviors associated with cooperation and collaboration (see Table 6). Although the groups were drawn for analysis based on differences in levels of collaboration, here we studied each stage of each group's CPS to understand their collaboration during phases of problem solving, and in relation to EFP during phases of problem solving. As depicted in Table 6, the coding rubric for each stage of CPS was derived from the existing literature and included multiple items. Reliability of all coding was established through discussion; scores were reviewed and revised following Hill (2012) until 100% agreement was achieved.

Table 5

Rubric for Scoring Categories of Core Executive Functions in Practice (EFP)

Category	Item	Examples
Working Memory	Does the student bring up and use mathematical concepts, ideas, practices, or tools that haven't been mentioned in the session? *(e.g., Karpike, 2012; Radvansky & Copeland, 2006)*	*The student brings up a new term that is relevant in a discussion, such as using the term "right triangle" to identify a triangle that has a right angle.*
	Does the student bring up or maintain attention to and use mathematical concepts, ideas, practices, or tools that have been mentioned in the session? *(e.g., Bailey et al., 2018; Gathercole, et al., 2006)*	*The student is chatting with another student and utilizes ideas mentioned in the directions, such as talking about the length of different segments of a triangle.*
	Does the student remember how to use relevant features of the tool? *(e.g., Diamond & Ling, 2016; Gathercole, et al., 2006)*	*The student uses the circle-making tool correctly in order to complete a step or solve a problem, or the student guides another student on how to use the tool.*
Cognitive Flexibility	Does the student recognize that a shift in approach or perspective might be needed? (*e.g., Laureiro-Martínez & Brusoni, 2018; Huizinga et al., 2014)*	*When a method of construction does not yield the expected results, such as the construction of an acute triangle when the goal was a right triangle, the student asks their group members if there is another way to construct the figure.*
	Does the student switch perspectives, frames, or strategies due to an external perspective? *(e.g., Laureiro-Martínez & Brusoni, 2018; Huizinga et al., 2014)*	*The student changes their method of construction after another student points out that the current method is not suitable, such as learning from a peer to anchor points to lines due to a peer's suggestion after initially making improper intersections.*
	Does the student move beyond a strategy that is not working? (*e.g., Diamond & Ling, 2016*; *Huizinga et al., 2014;* Jacques & Zelazo, 2001*)*	*When the current method of solving the problem doesn't yield the expected construction, the student suggests and/or tries another method, such as using the radii of circles to measure length after visual line estimations were unsuccessful.*
Inhibitory Control	Does the student exhibit signs of semantic inhibition by inhibiting past conceptions of an idea to engage in problem solving? *(e.g., Cervera-Crespo & Gonzalez-Alvarez, 2017; Chan et al., 2008)*	*The student purposely ignores or rejects the urge to create a construction based on their visual understanding of geometry (e.g., by stopping a group member from copying the example), which is an indication that they realize that their construction may not be correct even if it "looks" right.*
	Does the student follow and/or revisit norms for group practice? *(e.g., Diamond & Ling, 2016)*	*The student responds when questions or statements are addressed to them specifically, such as releasing control when another student asks to take control.*

	Does the student exhibit signs of response inhibition by controlling a motor impulse to instead act more appropriately? *(e.g., Chan et al., 2008; Verbruggen & Logan, 2008)*	*When a construction is not working as expected, the student tries different, unfamiliar tools to resolve the problem with the construction rather than using the same, familiar tool again.*
	Throughout the session, does the student attend to the task? *(e.g., Diamond, 2013; Chan et al., 2008)*	*All of the student's activity is relevant to the content.*

Table 6

Rubric for Scoring Stages of Collaborative Problem Solving (CPS)

Level of Analysis, Stage	Item and Sample Sources	Examples
Individual Level Participation	Does the student make use of the following resources: directions, the group members, past problems/parts of the current problem, math knowledge, tools? *(Melzner et al., 2020; Su et al., 2018)*	*To construct a figure, the student looks to the directions for steps, asks group members for confirmation, uses their math knowledge to choose the appropriate tool, and uses the tool to construct the figure by employing a previously established method.*
	Does the student attend to and notice things that need further investigation or explanation? *(Su et al., 2018; Zhang et al., 2021)*	*The student attends to the construction being made by another student and communicates in the chat what they notice.*
	Does the student work through their goals? Options: No follow through; Follow through, no consideration for goals; Follow through, incomplete/failed; Follow through, complete; Modified goal/new attempt *(Straub & Rummel, 2021; Zhang et al., 2021)*	*The student follows through with steps they or their group have identified.*
	Does the student recognize (is the student aware) that another perspective or approach is being us ed/suggested? *(Straub & Rummel, 2021; Zhang et al., 2021)*	*The student, while in control, implements suggestions from other students and changes their original construction to fit their peers' suggestions.*
	Throughout the entire session, is the student's engagement with the program balanced? *(e.g,. Melzner et al, 2020; Straub & Rummel, 2021)*	*The student participates in most math moments and has an equal balance of chat and tool use overall.*
Group Level Cooperation	Does the group define their goals? *(e.g., Jeong, et al., 2017; Kuhn et al., 2020)*	*Students all agree on a specific step to do before a student actually attempts the step.*
	Does the group share new ideas, including brainstorms they may be uncertain about? *(e.g., Marek et al, 2015; Mercer & Sams, 2006)*	*Students bring up different ways or methods to solve a problem.*

	Do students maintain positive communication and turn-taking? *(e.g., Eshuis et al., 2019; Mercer & Sams, 2006)*	*Students make sure that everyone has had a chance to take control and, if not, encourages others to participate.*
	Do students engage in discussion during and/or after problem solving beyond naming the directions? (e.g., Phelps & Damon, 1989)	*A student uses the chat function to ask another student a question, and that student responds and answers the question.*
Collaboration	Does the group execute their goals together? *(e.g., Andrews-Todd & Forsyth, 2020; Jeong, et al. 2017; Marek et al., 2015)*	*When constructing a figure, each person takes control to construct some part of the figure.*
	Is there evidence of shared understanding? (*e.g., Kuhn et al., 2020; Mercer & Sams, 2006)*	*When reflecting on the problem after an attempt, students discuss and agree on their satisfaction with their attempt.*
	If challenges and alternatives are raised, are they pursued and negotiated? *(e.g., Andrews-Todd & Forsyth, 2020, Marek et al, 2015)*	*A student disagrees with a statement made by another student, and there is a discussion about the disagreement.*
	Does the group incorporate viewpoints from each other when necessary? *(e.g., Kuhn et al., 2020; Mercer & Sams, 2006)*	*A student makes a suggestion for a specific construction, and the student in control makes the suggested construction.*
	Is there evidence of extension of thinking? *(e.g., Eshuis et al, 2019; Jeong, et al., 2017; Mercer & Sams, 2006)*	*Students build off on each other's ideas by introducing new perspectives.*
	Does the group provide justification or give mathematical reasoning for each other? *(e.g., Phelps & Damon, 1989; Vandenberg et al., 2021)*	*When a question is posed by a student, other students answer the question and attempt to explain their answers.*
	Throughout the activity, can continuity be identified among the group? *(e.g., Eshuis et al, 2019; Mercer & Sams, 2006)*	*Throughout the entire session, students consistently build on and add to each other's constructions and discussions.*

2.5 Analysis strategy

Data analyzed for each group's cognitive and behavioral engagement with each session of problem solving includes information about the students' activity in the shared workspace, as well as their written contributions in the chat. Observed information about student participation is reported first, followed by a description of analyses of the problem solving of each group. To answer RQ1, we first used Wald Z tests to compare the proportion of phases in each group's total number of math moments. We used Wald Z to directly compare each proportion as some moments were coded in more than one phase. To consider the distribution of phases between sessions, bar graphs were developed to show patterns in the data.

In RQ2, we explored how each group engaged during math moments; we studied both their cognitive (EFP) and behavioral (CPS) engagement. We divided our analyses into three parts. The first set of analyses focused on the scores for aggregated sessions to observe general behavior. To understand the importance of each phase of problem solving on participants' engagement, we used bar graphs to compare EFP and CPS scores in each of the three phases.

For the second and third parts, we ran multivariable regression analyses. Each analysis had a particular EF (Average EFP, Working Memory, Inhibitory Control, or Cognitive Flexibility) or CPS (Participation, Cooperation, or Collaboration) indicator score, averaged across students in a group for each moment, as the outcome variable. In analyzing phases, comparison always assessed one phase against the other two phases. The initial model (Model 1) included the phase and the group as binary predictors, whereas Model 2 (when appropriate) also included a phase-group interaction term. We looked for statistically significant coefficients for the phase predictor, the group predictor, and the phase-group interaction predictor. A significant phase coefficient suggested that the outcome differed between moments in that phase compared to moments not in that phase. A significant group coefficient indicated that the two groups differed on that outcome on average. A significant phase-group interaction suggested that the effect of phase on that outcome differed between the two groups. Because the within-group correlations of observations may result in downwardly biased standard errors in an ordinary linear regression (Moulton, 1986), we employed sandwich estimators for the standard errors to correct for this possibility. We conducted one follow-up analysis on the effect of phase on Collaboration using just the Group 1 data, given that Group 2 was selected because they evidenced more collaborative behaviors than Group 1, to determine whether any particular phase had a more significant effect on this outcome.

Finally, for the fourth part, we used line graphs to further consider change across sessions for each group's EFP and CPS scores during each phase of problem solving. We looked for patterns of increase and/or decrease in each outcome across sessions.

3. Results and Discussion

A total of 8-16 math moments were identified in each session of problem solving (see Tables 2 and 3; see Table 4 for an explanation and illustration of each phase of problem solving). Moments in which students were disengaged or were not actively working on problem solving (e.g., learning to use one of the VMT tools) were not analyzed. Although both groups worked for the same amount of time on each session, the number of math moments within problem solving sessions varied by group and by session, but not to a statistically significant degree.

We report and discuss results by research question.

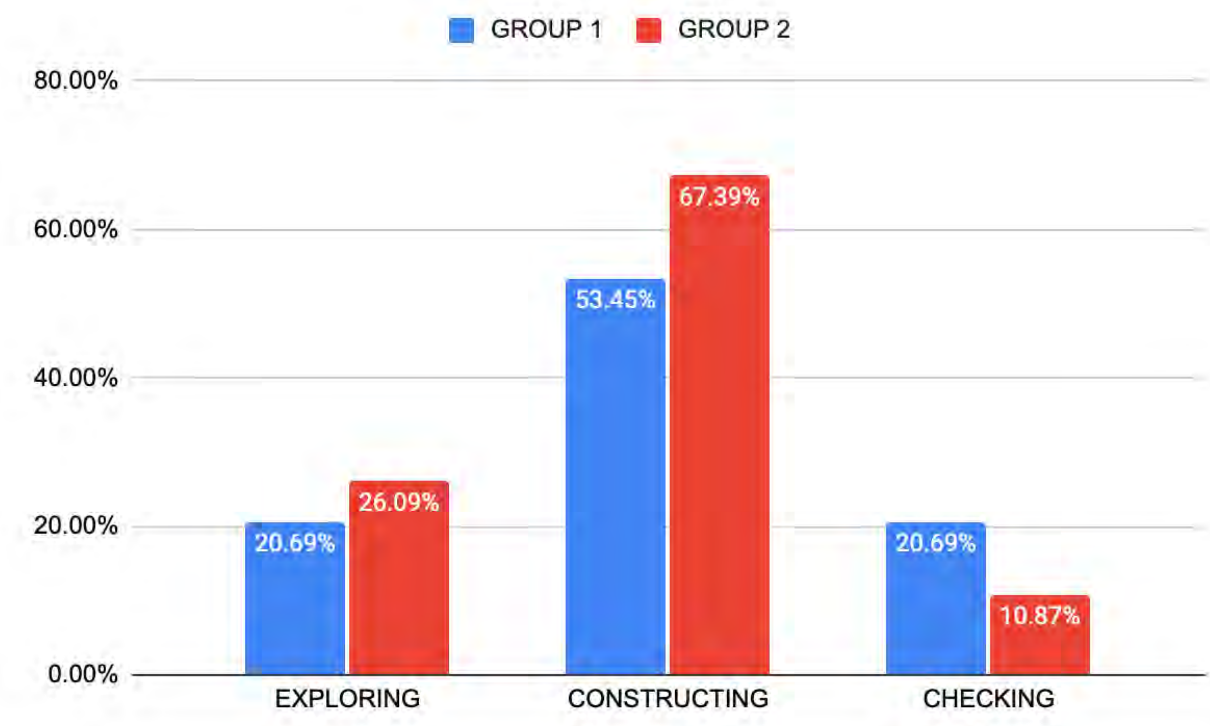
3.1 RQ 1: What is the relative proportion of each phase of problem solving overall, and is the distribution of phases similar by session? Do these proportions vary for two groups with different levels of collaboration?

We begin addressing RQ1 by overviewing findings from analyses of aggregated data from all four sessions and both groups. We examined the relative proportion of the three phases of problem solving, as well as the between-group differences in these proportions.

3.1.1 Overall findings

As shown in Figure 2, both groups spent most of their time constructing. Although we identified very few moments that included more than one phase of problem solving, these moments were counted in the analyses of all relevant phases.

Figure 2. *Relative Proportions of Phases of Problem Solving*



3.1.2 Between-group comparisons

For both groups, we identified similar numbers of math moments corresponding to each phase of problem solving (see Table 7). As such, it appeared that across the two groups, the frequency of engagement in each phase of problem solving was approximately the same. In other words, phase and group were not interacting. We suggest, however, this does not necessarily mean that the two groups were engaged in problem solving in the same way. They could potentially vary in how they engaged and the strategies they employed, given that the groups were purposefully selected for study based on differences in their collaboration. We addressed this question in RQ2, by analyzing students' EFP and CPS.

Table 7

Groups' Math Moments by Phase

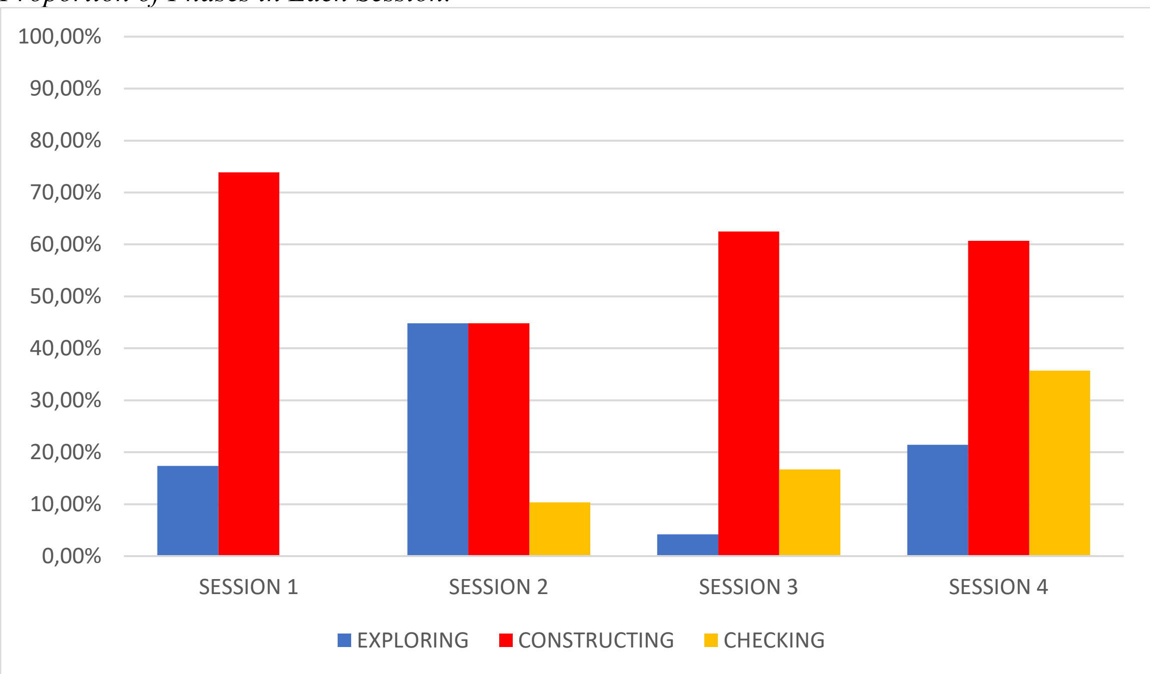
Phase	Group 1		Group 2		z	*p*
	n	% of moments	n	% of moments		

Exploring	12	20.7	12	26.1	-0.649	0.516
Constructing	31	53.4	31	67.4	-1.439	0.150
Checking	12	20.7	5	10.9	1.345	0.177

3.1.3 Analyses by session

Overall, for both groups, the relative proportions of all phases of problem solving fluctuated considerably (Figure 3). We included the data from the first session in our analyses as it was the students' orientation to engaging with the VMT. Similar to Symonds et al.'s (2021) results, this meant that students' engagement in different sessions varied. We conjectured that this might also indicate that the phases engaged were influenced by the context (the topic, the prompts of the activity; opportunities created during the groups' engagement). Indeed, collaborative contexts have been found to offer learners support for engagement that enables the development of cognitive, as well as social skills (Romero-López et al., 2020; Sun et al., 2022).

Figure 3.
Proportion of Phases in Each Session.



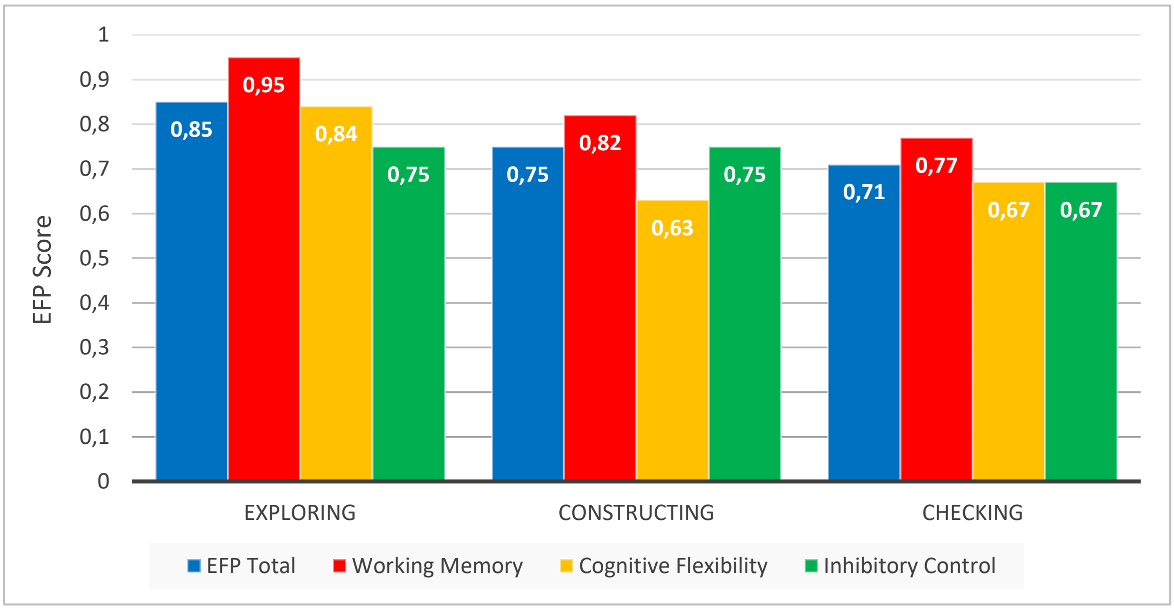
3.2 RQ 2: How do the two groups engage with the different phases of problem solving? Specifically, what is the relation between each phase of problem solving and the EFP and CPS of each group? Do these relations differ for the two groups? Is there change across sessions?

Having examined the frequencies of each group's engagement in different phases of problem solving in RQ1, we then turned to considering how each of these phases related to students' cognitive and behavioral engagement both as individuals and as a group. RQ2 examined how Exploring, Constructing, and Checking were associated with participants' EFP and CPS scores. We began this investigation with analyses that used aggregated data from all sessions and both groups. An Average EFP score, representing the mean of the three EFP, was calculated. Then, both overall and across sessions, we addressed how each group engaged EFP and CPS during each phase of problem solving.

3.2.1 Phases

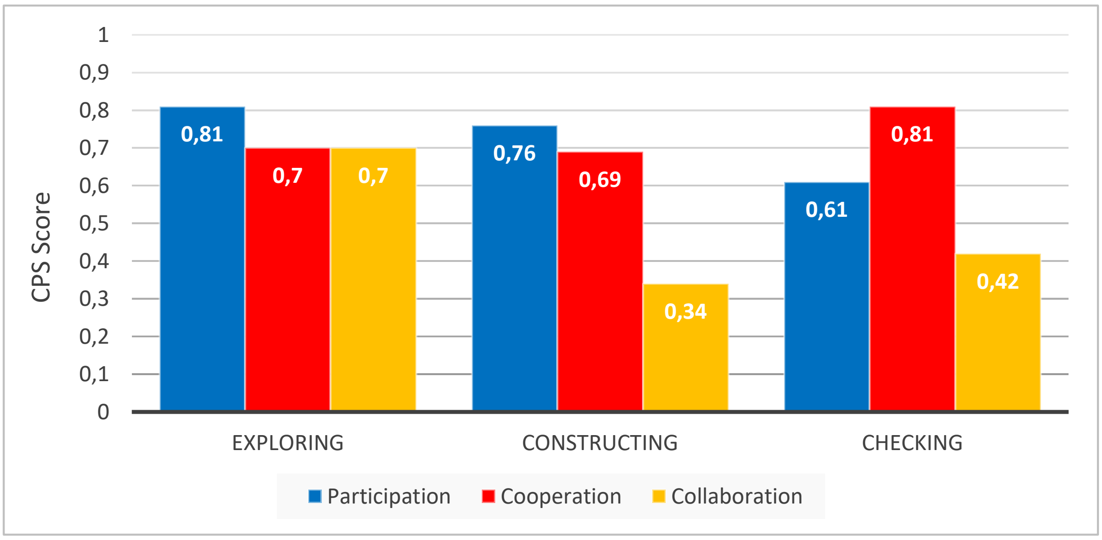
As shown in Figure 4, different patterns were identified in students' EFP scores during each of the three phases of problem solving. We found that Average EFP, which combined the three categories of EFP, was highest during Exploring (0.85). The same pattern was true with each of the core EFP (Working Memory, Cognitive Flexibility, Inhibitory Control): scores were higher in moments of Exploring. Scores were similar during Constructing and Checking, but on average, as indicated by Average EFP, Constructing was associated with slightly higher average EFP than Checking, and specifically with higher Working Memory and Inhibitory Control than Checking. Nevertheless, Cognitive Flexibility was higher during Checking than during Constructing. We also observed that, across all phases, Working Memory was always the highest-scoring EFP. Inhibitory Control was the second highest overall, and Cognitive Flexibility was the lowest.

Figure 4.
EFP Scores by Phase of Problem Solving.



CPS scores also varied by phase, as shown in Figure 5. Overall, we observed that the three CPS scores were high and did not differ substantially during moments of Exploring; Collaboration – the most developed stage of CPS – was especially high in this phase. Math moments that included Exploring had the highest CPS scores; Exploring also was the phase in which the three stages of CPS were more balanced.

Figure 5.
CPS Scores by Phase of Problem Solving.



Analyses of each individual stage of CPS revealed additional patterns. Participation and Cooperation scores were relatively similar for each of the three phases of problem solving, whereas Collaboration was higher during Exploring and much lower during Construction and Checking. These results appeared to indicate that the phase of students' Exploring afforded more opportunities for full behavioral engagement in CPS practices than either of the two other phases of problem solving.

3.2.2 Effect of phases

In the next analyses, we examined whether the differences observed were significant. We assessed the effect of phases on EFP and CPS using regression (Table 8). As shown in Table 8, moments of Exploring had significantly higher EFP and CPS than non-Exploring moments, specifically for the indicators of Average EFP, Working Memory, Participation, and Collaboration. These findings complemented the observations of the previous section, suggesting that EFP and CPS scores were highest in moments of Exploring.

Table 8

Regressions, Between Group Analyses, EFP and CPS

		Model 1		Model 2	
Effect	Predictors	Estimate	SE	Estimate	SE
Exploring					
Average EFP					
	Intercept	.669***	.033	.647***	.036
	Exploring	.129***	.038	.235***	.060
	Group	.085*	.038	.129**	.045
	Interaction			-.192*	.076
R^2		.039		.051!	
Working Memory					
	Intercept	.698***	.050	.675***	.055
	Exploring	.159***	.047	.269**	.078
	Group	.167**	.053	.213***	.066
	Interaction			-.199*	.095
R^2		.046		.053	
Cognitive Flexibility					
	Intercept	.566***	.075	.546***	.080
	Exploring	.280**	.097	.435***	.082
	Group	-.013	.093	.029	.109
	Interaction			-.241	.164
R^2		.026†		.030	
Inhibitory Control	n.s.				
Participation					

	Intercept	.709***	.029	.704***	.032
	Exploring	.078*	.033	.106	.055
	Group	.100**	.032	.111**	.039
	Interaction			-.050	.067
R^2		.039		.040	
Cooperation					
	Intercept	.676***	.027	.681***	.030
	Exploring	.043	.028	.017	.055
	Group	.211***	.029	.200***	.035
	Interaction			-.047	.035
R^2		.134		.135	
Collaboration					
	Intercept	.302***	.032	.282***	.036
	Exploring	.316***	.036	.414***	.048
	Group	.255***	.043	.296***	.054
	Interaction			-.177*	.070
R^2		.183		.190	
Constructing					
Average EFP					
	Intercept	.696***	.038	.713***	.046
	Constructing	-.001	.039	-.031	.062
	Group	.092*	.038	.053	.060
	Interaction			.064	.078
R^2		.016†		.018	
Working Memory					
	Intercept	.742***	.054	.728***	.067
	Constructing	-.02	.054	.005	.092
	Group	.178***	.054	.209***	.076
	Interaction			-.051	.106
R^2		.029		.030	
Cognitive Flexibility	n.s.				
Inhibitory Control	n.s				

Participation					
	Intercept	.721***	.036	.690***	.046
	Constructing	.009	.033	.066	.056
	Group	.103**	.032	.174***	.052
	Interaction			-.118	.066
R^2		.028		.037	
Cooperation					
	Intercept	.671***	.033	.677***	.041
	Constructing	.025	.032	.013	.052
	Group	.210***	.030	.195***	.053
	Interaction			.024	.063
R^2		.132		.132	
Collaboration					
	Intercept	.394***	.044	.404***	.052
	Constructing	-.049	.047	-.068	.065
	Group	.279***	.045	.255***	.078
	Interaction			.039	.095
R^2		.096		.096	
Checking					
Average EFP					
	Intercept	.700***	.031	.715***	.033
	Checking	-.018	.056	-.093	.083
	Group	.090*	.038	.061	.041
	Interaction			.195*	.096
R^2		.016†		.025	
Working Memory					
	Intercept	.736***	.048	.748***	.050
	Checking	-.026	.074	-.081	.117
	Group	.173**	.054	.152*	.060
	Interaction			.144	.124
R^2		.029		.032	
Cognitive Flexibility	n.s.				

Inhibitory Control	n.s				
Participation					
	Intercept	.741***	.026	.757***	.028
	Checking	-.074	.054	-.150	.080
	Group	.097**	.032	.068*	.034
	Interaction			.200*	.094
R^2		.035		.048!	
Cooperation					
	Intercept	.656***	.027	.653***	.029
	Checking	.138***	.035	.152**	.056
	Group	.227***	.029	.232***	.033
	Interaction			-.038	.058
R^2		.159		.159	
Collaboration					
	Intercept	.336***	.031	.353****	.033
	Checking	.154*	.061	.070	.094
	Group	.287***	.043	.255***	.048
	Interaction			.217*	.103
R^2		.108		.115	

Note. We considered Model 2 only when the change in R^2 was significant, denoted by !.
SE = standard error. *** p < .001; ** p < .01; * p < .05. All estimated coefficients are unstandardized. In cases where Model 1 is not a significant improvement over the null model (denoted by † in R^2), but a predictor significantly differs from 0, we include the model in the table. n.s. indicates a model that does not significantly differ from the null model and has no statistically significant predictors.

We also found a group-phase interaction for the effect of Exploring on Average EFP. It appeared that because the Group-Exploring coefficient in Model 2 was negative, Exploring had less of an effect on Average EFP for Group 2 than Group 1. In fact, despite the positive Group coefficient in Model 2, Group 1 had higher Average EFP in moments of Exploring than Group 2. One possible explanation for this finding was that Group 2 just had consistently higher Average EFP, and we were seeing a ceiling effect. Regardless, the model results suggested that Exploring could have a strong and positive effect on Average EFP.

In contrast, we found that moments of Checking evidenced significantly higher CPS than non-Checking moments, specifically for Cooperation and Collaboration in Model 1. Interestingly, in Model 1 for Participation, Checking was not a significant predictor, although Group was. However, Model 2, which included the interaction term, was a significant improvement over Model 1; in that model, the Checking coefficient was significant and negative, whereas the Group-Checking interaction coefficient was significant and positive, and the Group coefficient was not significant. Model 2 suggests that both groups experienced similar levels of Participation in moments other than Checking, but Group 1 experienced less participation in Checking moments than Group 2. One possible explanation for this finding is that Group 2, which was selected for study because they evidenced stronger collaboration than Group 1, may have had broader Participation in moments of Checking, whereas Group 1 may have had uneven Participation in moments of Checking.

In summary, the presence of Exploring appears to strengthen students' EFP and CPS scores, particularly Collaboration, more than other phases. This finding points to the possible benefit of the development of Collaborative skills and of having time and opportunities to explore. Another potential takeaway is related to the finding that Checking was an important predictor of CPS stages, but not EFP scores. This finding could suggest that moments of Checking bring students together to work either in parallel or in coordination. However, the Group-Checking interaction for Participation and the significant Group coefficients for Cooperation and Collaboration suggest that groups also may operate differently in moments of Checking.

3.2.3 Comparing groups' EFP and CPS across phases

For Average EFP, Working Memory, Participation, Cooperation, and Collaboration, we observed significant and positive Group coefficients across multiple phase models, suggesting that Group 2 scored consistently higher overall on these indicators than Group 1. In general, we did not observe significant phase-group interactions in the models, suggesting that both groups were affected equally by each phase. However, for two models, Average EFP in Exploring and Participation in Checking, we observed phase-group interactions, as discussed in the previous section. For these two indicators, we did not observe the same patterns or phase effects for both groups. Specifically, Exploring had a substantially larger effect on Average EFP for Group 1 compared to Group 2, whereas Checking had a negative effect on Participation for Group 1 and a small positive effect on Participation for Group 2.

3.2.4 Relative collaboration between phases

Recall that the key distinction between the two groups in our study was that Group 2 evidenced substantially higher collaboration than Group 1 overall. As stated above, both Exploring and Checking had significant effects on Collaboration scores. However, we were curious about the relative importance of phase on Collaboration, and, based on inspection of the data, we were concerned that a ceiling effect on Group 2's Collaboration scores could obfuscate relationships by limiting the potential variance. Therefore, we conducted a regression analysis for Collaboration with just the Group 1 data and all three phases as predictors. In this model (Table 9), all three phases had statistically significant and positive coefficients. However, the 95% confidence interval for the Exploring coefficient did not overlap with the 95% confidence interval for the Constructing or Checking coefficient. We concluded that the Exploring phase was associated with significantly higher levels of Collaboration than either the Constructing or Checking phases for Group 1. Additionally, the R2 value for this model was .288, indicating that phases explained 28.8% of the variance in Group 1's Collaboration scores.

Table 9

Regression, Group 1 Collaboration by Phase

				95% CI	
Outcome	Predictors	Estimate	SE	LL	UL
Collaboration					
	Intercept	-.072	.073	-.215	.072
	Exploring	.768***	.094	.582	.953
	Constructing	.368***	.077	.216	.520
	Checking	.403***	.082	.241	.565
R^2		.228			

Note. SE = standard error; *CI* = confidence interval; *LL* = lower limit; *UL* = upper limit.
*** p < .001; ** p < .01; * p < .05. All estimate coefficients are unstandardized.

3.2.5 Across sessions comparison

To further explore patterns of change, line graphs were employed to compare both groups' EFP and CPS scores over time (see Figures 6 and 7). Given the small number of sessions, we have limited ability to make conclusive statements explaining patterns. However, we could make a few observations based on the figures and our analysis of these data.

Figure 6.

Average EFP Scores Across Sessions for Group 1 and Group 2.

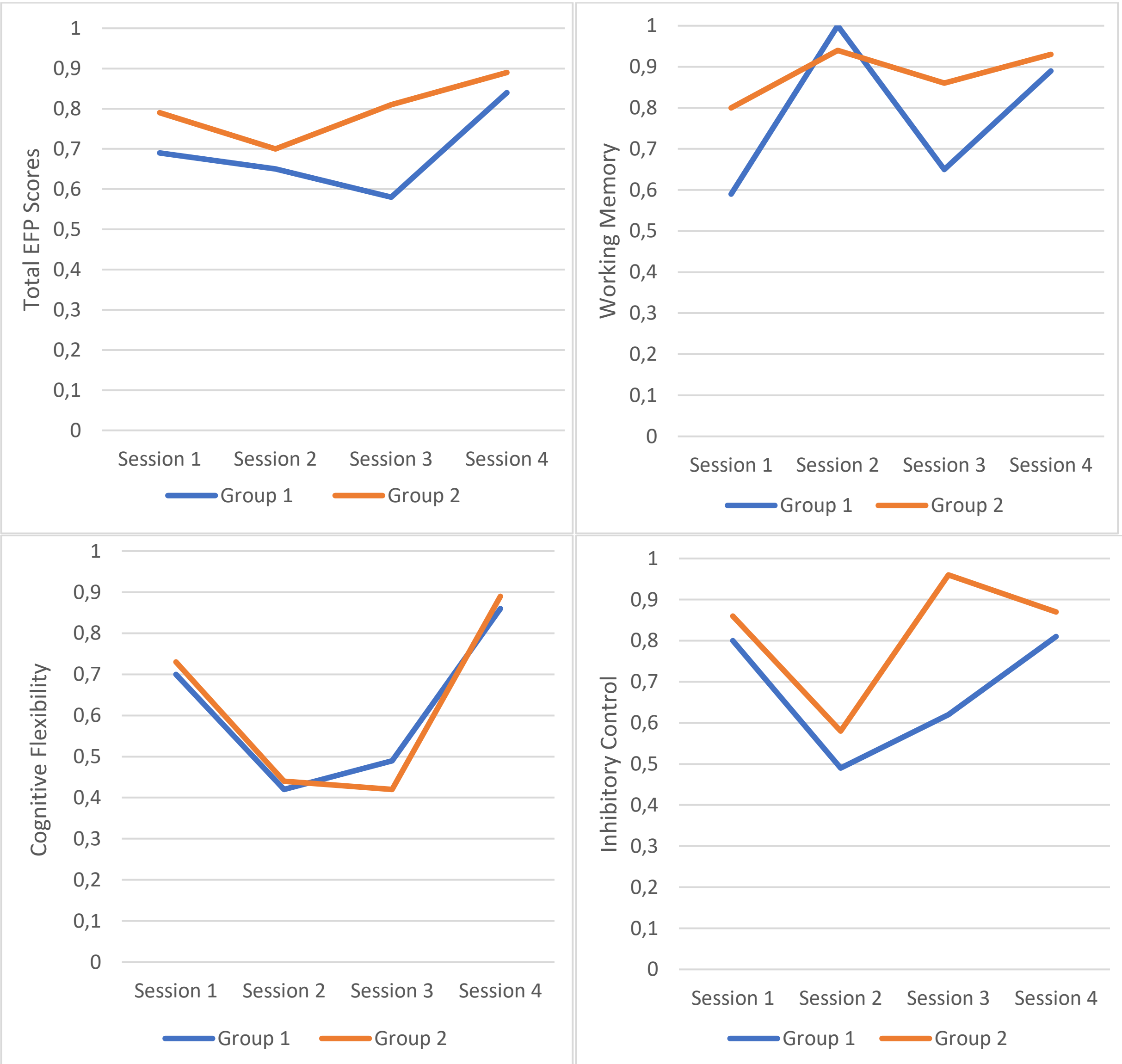
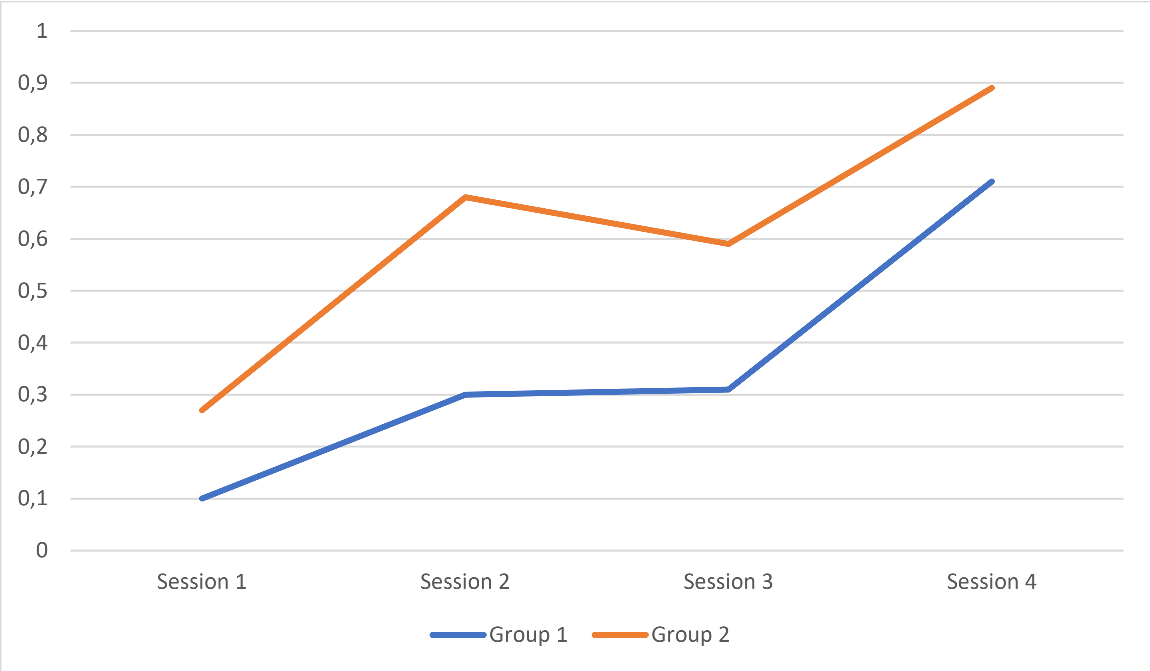


Figure 7.

Average Collaboration Scores of Group 1 and Group 2 During Constructing Moments for Four VMT Sessions.



We first note that the two groups differed in Exploring and Checking by session. Group 1 had no moments of Exploring in sessions 3 and 4, while Group 2 engaged in this phase in all sessions. No group engaged in Checking in session 1, and Group 1 engaged in Checking in session 2, but not Group 2, and both groups engaged in similar amounts of Checking in sessions 3 and 4. In what follows, we focus on Constructing, as it was not only the most common phase but also the only phase in which both groups were involved across all four sessions.

As shown in Figure 6, during Constructing moments, the EFP scores of both groups followed similar patterns of change across sessions, although the scores of Group 2 in each session were generally higher than those of Group 1. We also note that the two groups generally followed similar patterns of change in their EFP and that none of the observed patterns are linear, nor do they show a clear increase or decrease over time; both groups' EFP fluctuated from session to session.

Review of the line graphs for each of the stages of CPS during Constructing moments showed similar patterns of change; Figure 7 provides an example for Collaboration. As expected, Collaboration scores were consistently higher for Group 2 than for Group 1. We call attention to the similarity of the groups' patterns of engagement and increases in their Collaboration scores over time. These results suggest that, with time, the groups were becoming more collaborative.

This analysis does not differentiate between the effect of time and the effect of mathematical topic. Each group engaged in the problem solving sessions in the same order, so time and topic were confounded. Therefore, it was not possible to determine whether the findings were a function of change over time or characteristics specific to each session. While spending time working together may progressively increase students' EFP and CPS, the groups may also be influenced by the opportunities provided by the design of each activity. The fact that we saw similar patterns of variation may suggest that there was some shared feature or structure of the activity that was guiding the students' behavior.

4. General discussion

We undertook this study to consider the process of middle school students' momentary engagement during phases of collaborative mathematical problem solving. Although students have been repeatedly found to enjoy and benefit from opportunities to work together on problem solving (e.g., Featherstone et al., 2011; Webb et al., 2019), van Leeuwen and Janssen's (2019) review of collaborative activity and learning showed that collaboration does not always result in learning and can be difficult to facilitate. In our study design, we sought insight that could inform teachers about the cognitive and behavioral engagement of students' online collaborative mathematics activity, and purposefully examined moments during which the students were engaged in mathematics. We selected for study two groups of students who in prior study had been identified as having high levels of interest in working collaboratively online with mathematics problems—students in both groups were continuously engaged in working with their peers on math across the four sessions of problem solving. Studying interested youth enabled us to explore the potential for middle-schoolers' productive engagement during the math moments of their work together. Furthermore, because we studied students' online collaboration in the VMT environment, we were able to examine their moment-to-moment work in the workspace, as well as their interactions in the chat, which allowed study of fluctuations that differs from those possible had our analyses focused on the outcomes of cognitive or behavioral engagement, face-to-face, or even using video footage.

From RQ1, we learned that Constructing was the most frequently engaged phase of problem solving compared to the other phases of problem solving. However, RQ 2 showed that Exploring was associated with higher EFP and CPS scores, providing corroboration for Boaler and Selling's (2017) results and suggesting that students who were encouraged to explore use of strategies in their work with mathematics developed deeper understanding, as well as positive feelings, compared to students who did not receive the same support. We also noted that scores for Collaboration in particular were higher when the math moments involved Exploring.

While findings from RQ 1 suggested that the frequency of the cognitive and behavioral engagement for each group in each phase of problem solving was approximately the same, we learned from RQ 2 that how the groups engaged the phases varied. Given that we selected these two groups for study because the students had been identified as interested in the collaborative math sessions, it is not surprising that our results indicated that in all phases of problem solving, across all sessions, students in both groups maintained high and continuous levels of Participation (Renninger & Hidi, 2016). That the two groups also differed in their level of Collaboration was expected because they were selected for study based on this information. However, Group 2 also consistently scored higher than Group 1 in Cooperation in all phases of problem solving.

Although prior study has shown that collaboration develops through working with others on problem solving (Bonotto 2005; Gerson, 2008; Kelton et al., 2018; Stahl, 2013), and its importance for the development of EF has been noted (Pollastri et al., 2013), we did not know how the processes associated with EFP might unfold. Our results showed that this effect most likely originates from higher EFP engagement during Exploring. Moreover, our findings related to the stage of Collaboration are particularly interesting. In each phase of problem solving, both groups of students had similar patterns of increase in their Collaboration scores across the problem solving sessions. Thus, although Group 2 was more collaborative in general, and we do not know how details of each task affected collaboration behaviors per se, we also saw that both groups became more collaborative the more they worked together. These results are consistent with literature showing how groups become better at collaborating over time as they start to share mental models that make them more efficient and effective even when problems demand higher mental power (van den Bossche, et al., 2011; Zambrano, et al., 2019).

We also observed that despite differences in the EFP and CPS scores of the two groups, their scores fluctuated in similar ways across sessions during moments of Constructing. This finding leads us to wonder if there is some shared feature or structure of the problems (e.g., prompts for discussion, mathematics topic) that was guiding the groups' behaviors (see Lieber & Graulich, 2020), as both groups of students received the same instructions and tasks in each problem-solving session.

4.1 Implications for theory and practice

Our findings contribute to discussions of momentary engagement as both situation specific (e.g. Nolen et al., 2015; Symonds et al., 2021) and complex (Dietrich, et al., 2022); they also provide an extension of the existing literature. Focused study of the process of two groups of interested middle school students' engagement during the math moments of their phases of their problem solving reveals relatively similar fluctuations and also highlights differences in how they are engaging. Study of the students' use of EFP and stages of CPS, moreover, provides insight into what might be expected of students at this age as they engage in group work that involves open-ended problem solving. These results confirm that the collaborative context of group work promotes attention to the task (e.g., Hmelo-Silver et al., 2018; Pohl, 2020), enhances reasoning (e.g., Barron, 2003), and promotes use of working memory (e.g., Kirschner et al., 2018); they also underscore the importance of considering how students are engaging in this context. In addition, our findings suggest the potentially essential contribution of task features such as prompts to discuss math in mediating student cognitive and behavioral engagement.

These data show that students vary in their cognitive and behavioral engagement in different phases of problem solving. They further point to the benefit of student group engagement in the phase of Exploring, in particular, as Exploring was associated with increased use of Working Memory and Collaboration. Our results also suggest that students may need support to collaborate during moments that include Constructing and Checking. As such, it may be critical to support teachers to attend to what a group is doing moment to moment, and specifically to variations in students' behavior during different phases of problem solving.

4.2 Limitations and future directions

Future study with additional student groups, who vary in their level of interest in mathematics, as well as by age, and for whom demographic information is available is clearly warranted. Moreover, while for present purposes we aggregated study of moments of collaboration, a more qualitative exploration would provide a rich description of the context from which collaboration emerges. In addition to systematically examining the role of task features as determinants of fluctuations in momentary engagement during collaborative problem solving, we also suggest the utility for practitioners of additional analyses of the components of each EFP as well as those of each stage of CPS (represented by the questions used for assessment in our coding scheme rubric). Analyses such as these would provide additional detail and insight about which behaviors account for and are contributing to how student groups are engaging in problem solving.

Keypoints

- This study is the first to report on the process of students' executive functions and collaborative problem-solving during phases of problem solving.
- The Virtual Math Teams environment enabled assessments of students' moment-to-moment engagement in phases of problem-solving.
- Study findings highlight the importance of assessing both what students are doing during problem solving, as well as how they are engaging.
- The relations between each phase of problem solving, categories of EFP, and stages of CPS vary.
- The problem-solving phase of exploring has a positive effect on use of executive functions and collaborative problem solving.

Acknowledgments

The authors gratefully acknowledge Jane Huynh's editorial assistance in preparing this manuscript for publication. We also are most appreciative of support for our collaboration from an EARLI (European Association of Research on Learning and Instruction) Emerging Field Group Grant and the Jacobs Foundation, awarded to Jennifer Symonds and Ricardo Böheim, as well funding for project research from the EF+Math Program of the Advanced Education Research and Development Fund (AERDF) to K. Ann Renninger. Opinions expressed in this article are those of the authors and do not necessarily represent views of the EF+Math Program or AERDF.

References

Andrews-Todd, J., & Forsyth, C. M. (2020). Exploring social and cognitive dimensions of collaborative problem solving in an open online simulation-based task. *Computers in Human Behavior, 104,* 105759. https://doi.org/10.1016/j.chb.2018.10.025

Bailey, B. A., Andrzejewski, S. K., Greif, S. M., Svingos, A. M., & Heaton, S. C. (2018). The role of executive functioning and academic achievement in the academic self-concept of children and adolescents referred for neuropsychological assessment. *Children, 5*(7), 83. https://doi.org/10.3390/children5070083

Barron, B. (2003). When smart groups fail. *The Journal of the Learning Sciences, 12*(3), 307–359. https://doi.org/10.1207/S15327809JLS1203_1

Bishara, S., & Kaplan, S. (2022). Inhibitory control, self-efficacy, and mathematics achievements in students with learning disabilities. *International Journal of Disability, Development, and Education, 69*(3), 868–887. https://doi.org/10.1080/1034912X.2021.1925878

Boaler, J., & Selling, S. K. (2017). Psychological imprisonment or intellectual freedom? A longitudinal study of contrasting school mathematics approaches and their impact on adults' lives. *Journal for Research in Mathematics Education, 48*(1), 78–105. https://doi.org/10.5951/jresematheduc.48.1.0078

Bonotto, C. (2005). How informal out-of-school mathematics can help students make sense of formal in-school mathematics: The case of multiplying by decimal numbers. *Mathematical Thinking and Learning, 7*(4), 313–344. https://doi.org/10.1207/s15327833mtl0704_3

Brookman-Byrne, A., Mareschal, D., Tolmie, A. K., & Dumontheil, I. (2018). Inhibitory control and counterintuitive science and maths reasoning in adolescence. *PLoS One, 13*(6), e0198973-e0198973. https://doi.org/10.1371/journal.pone.0198973

Caviola, S., Colling, L. J., Mammarella, I. C., & Szűcs, D. (2020). Predictors of mathematics in primary school: Magnitude comparison, verbal and spatial working memory measures. *Developmental Science, 23*(6), e12957. https://doi.org/10.1111/desc.12957

Cervera-Crespo, T., & González-Alvarez, J. (2017). Age and semantic inhibition measured by the Hayling Task: A meta-analysis. *Archives of Clinical Neuropsychology, 32*(2), 198–214. https://doi.org/10.1093/arclin/acw088

Chan, R. C., Shum, D., Toulopoulou, T., & Chen, E. Y. (2008). Assessment of executive functions: Review of instruments and identification of critical issues. *Archives of Clinical Neuropsychology, 23*(2), 201–216. https://doi.org/10.1016/j.acn.2007.08.010

Clark, C. A., Pritchard, V. E., & Woodward, L. J. (2010). Preschool executive functioning abilities predict early mathematics achievement. *Developmental Psychology, 46*(5), 1176–1191. https://doi.org/10.1037/a0019672

Common Core State Standards Initiative (CCSSI) (2011). High school-- geometry. Common core state standards for mathematics, https://www.thecorestandards.org/Math/Content/HSG/

Cook, C. R., Thayer, A. J., Fiat, A., & Sullivan, M. (2020). Interventions to enhance affective engagement. In A. L. Reschly, A. J. Pohl, & S. L. Christenson (Eds.), *Student engagement: Effective academic,* *behavioral, cognitive, and affective interventions at school* (pp. 203–237). Springer Cham. https://doi.org/10.1007/978-3-030-37285-9_12

Cragg, L., & Gilmore, C. (2014). Skills underlying mathematics: The role of executive function in the development of mathematics proficiency. *Trends in Neuroscience and Education, 3*(2), 63–68. https://doi.org/10.1016/j.tine.2013.12.00

deVilliers, M. (2003). *Rethinking proof with the Geometer's Sketchpad*. Emeryville, CA: Key Curriculum Press.

Diamond, A. (2013). Executive functions. *Annual Review of Psychology, 64*, 135–168. https://doi.org/10.1146/annurev-psych-113011-143750

Diamond, A., & Ling, D. S. (2016). Conclusions about interventions, programs, and approaches for improving executive functions that appear justified and those that, despite much hype, do not. *Developmental Cognitive Neuroscience, 18,* 34–48. https://doi.org/10.1016/j.dcn.2015.11.005

Dietrich, J., Schmiedek, F., & Moeller, J. (2022). Academic motivation and emotions are experienced in learning situations, so let's study them. Introduction to the special issue. *Learning and Instruction, 81*, 101623. https://doi.org/10.1016/j.learninstruc.2022.101623

Dong, A., Jong, M. S. Y., & King, R. B. (2020). How does prior knowledge influence learning engagement? The mediating roles of cognitive load and help-seeking. *Frontiers in Psychology, 11*, 591203-591203. https://doi.org/10.3389/fpsyg.2020.591203

Eshuis, E. H., ter Vrugte, J., Anjewierden, A., Bollen, L., Sikken, J., & de Jong, T. (2019). Improving the quality of vocational students' collaboration and knowledge acquisition through instruction and joint reflection. *International Journal of Computer-Supported Collaborative Learning, 14*, 53–76. https://doi.org/10.1007/s11412-019-09296-0

Featherstone, H., Crespo, S., Jilk, L. M., Oslund, J. A., Parks, A. N., & Wood, M. B. (2011). *Smarter together! Collaboration and equity in the elementary math classroom.* National Council of Teachers of Mathematics.

Fredricks, J. A., & McColskey, W. (2012). The measurement of student engagement: A comparative analysis of various methods and student self-report instruments. In S. L. Christenson, A. L. Reschly, & C. Wylie (Eds.), *Handbook of research on student engagement* (pp. 763–782). Springer. https://doi.org/10.1007/978-1-4614-2018-7_37

Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research, 74*(1), 59–109. https://doi.org/10.3102/00346543074001059

Gathercole, S. E., Lamont, E., & Alloway, T. P. (2006). Working memory in the classroom. In S.J. Pickering (Ed.), *Working memory and education* (pp. 219–240). Academic Press. https://doi.org/10.1016/B978-012554465-8/50010-7

Gerson, H. (2008). David's understanding of functions and periodicity. *School Science and Mathematics, 108*(1), 28–8. https://doi.org/10.1111/j.1949-8594.2008.tb17937.x

Hadwin, A. F., Järvelä, S., & Miller, M. (2011). Self-regulated, co-regulated, and socially shared regulation of learning. In B. J. Zimmerman, & D. H. Schunk (Eds.), *Handbook of self-regulation of learning and performance* (pp. 65-84). Routledge.

Harel, G., & Sowder, L. (2005). Advanced mathematical thinking at any age: Its nature and development. *Mathematical Thinking and Learning, 7*(1), 27–50. https://doi.org/10.1207/s15327833mtl0701_3

Hill, C. E. (Ed.) (2012). *Consensual qualitative research: A practical resource for investigating social science phenomena*. American Psychological Association.

Hmelo-Silver, C. E., Kapur, M., & Hamstra, M. (2018). Learning through problem solving. In F. Fischer, C. E. Hmelo-Silver, S. R. Goldman, & P. Reimann (Eds.), *International handbook of the learning sciences* (pp. 210–220). Routledge.

Huizinga, M., Smidts, D. P., & Ridderinkhof, K. R. (2014). Change of mind: Cognitive flexibility in the classroom. *Perspectives on Language and Literacy, 40*(2), 31–35.

Jacques, S., & Zelazo, P. (2001). The Flexible Item Selection Task (FIST): A measure of executive function in preschoolers. *Developmental Neuropsychology, 20*(3), 573–591. https://doi.org/10.1207/875656401753549807

Jeong, H., Cress, U., Moskaliuk, J., & Kimmerle, J. (2017). Joint interactions in large online knowledge communities: The A3C framework. *International Journal of Computer-Supported Collaborative Learning, 12*, 133–151. https://doi.org/10.1007/s11412-017-9256-8

Jose, R. G., Samuel, A. S., & Isabel, M. M. (2020). Neuropsychology of executive functions in patients with focal lesion in the prefrontal cortex: A systematic review. *Brain and Cognition, 146*, 105633. https://doi.org/10.1016/j.bandc.2020.105633

Karpicke, J. D. (2012). Retrieval-based learning: Active retrieval promotes meaningful learning. *Current Directions in Psychological Science, 21*(3), 157–163. https://doi.org/10.1007/s10648-012-9202-2

Kasmer, L., & Kim, O. K. (2011). Using prediction to promote mathematical understanding and reasoning. *School Science and Mathematics, 111*(1), 20–33. https://doi.org/10.1111/j.1949-8594.2010.00056.x

Kelton, M. L., Ma, J. Y., Rawlings, C., Rhodehamel, B., Saraniero, P., & Nemirovsky, R. (2018). Family meshworks: Children's geographies and collective ambulatory sense-making in an immersive mathematics exhibition. *Children's Geographies, 16*(5), 543–557. https://doi.org/10.1080/14733285.2018.1495314

Kirschner, P. A., Sweller, J., Kirschner, F., & Zambrano, J. R. (2018). From Cognitive Load Theory to Collaborative Cognitive Load Theory. *International Journal of Computer-Supported Collaborative Learning, 13*(2), 213–233. https://doi.org/10.1007/s11412-018-9277-y

Kuhn, D., Capon, N., & Lai, H. (2020). Talking about group (but not individual) process aids group performance. *International Journal of Computer-Supported Collaborative Learning, 15*(2), 179–192. https://doi.org/10.1007/s11412-020-09321-7

Landis, J. R., & Koch, G. G. (1977) The measurement of observer agreement for categorical data. *Biometrics, 33*(1), 159–174. https://doi.org/10.2307/2529310

Laureiro-Martínez, D., & Brusoni, S. (2018). Cognitive flexibility and adaptive decision-making: Evidence from a laboratory study of expert decision makers. *Strategic Management Journal, 39*(4), 1031–1058. https://doi.org/10.1002/smj.2774

Lee, K., Ng, E. L., & Ng, S. F. (2009). The contributions of working memory and executive functioning to problem representation and solution generation in algebraic word problems. *Journal of Educational Psychology, 101*(2), 373–387. https://doi.org/10.1037/a0013843

Letang, M., Citron, P., Garbarg-Chenon, J., Houdé, O., & Borst, G. (2021). Bridging the gap between the lab and the classroom: An online citizen scientific research project with teachers aiming at improving inhibitory control of school-age children. *Mind, Brain and Education, 15*(1), 122–128. https://doi.org/10.1111/mbe.12272

Lieber, L., & Graulich, N. (2020). Thinking in alternatives–A task design for challenging students' problem-solving approaches in organic chemistry. *Journal of Chemical Education, 97*(10), 3731–3738. https://doi.org/10.1021/acs.jchemed.0c00248

Long, B., Spencer-Smith, M. M., Jacobs, R., Mackay, M., Leventer, R., Barnes, C., & Anderson, V. (2011). Executive function following child stroke: The impact of lesion location. *Journal of Child Neurology, 26*(3), 279-287. https://doi.org/10.1177/0883073810380049

Mann, T. D., Hund, A. M., Hesson-McInnis, M. S., & Roman, Z. J. (2017). Pathways to school readiness: Executive functioning predicts academic and social–emotional aspects of school readiness. *Mind, Brain, and Education, 11*(1), 21–31. https://doi.org/10.1111/mbe.12134

Marek, L. I., Brock, D.-J. P., & Savla, J. (2015). Evaluating collaboration for effectiveness: Conceptualization and measurement. *The American Journal of Evaluation, 36*(1), 67–85. https://doi.org/10.1177/1098214014531068

McCoy, D. C. (2019). Measuring young children's executive function and self-regulation in classrooms and other real-world settings. *Clinical Child and Family Psychology Review, 22*(1), 63–74. https://doi.org/10.1007/s10567-019-00285-1

Melzner, N., Greisel, M., Dresel, M., & Kollar, I. (2020). Regulating self-organized collaborative learning: The importance of homogeneous problem perception, immediacy and intensity of strategy use. *International Journal of Computer-Supported Collaborative Learning, 15*(2), 149–177. https://doi.org/10.1007/s11412-020-09323-5

Mercer, N., & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education, 20*(6), 507–528. https://doi.org/10.2167/le678.0

Mohammadhasani, N., & Asadi, S. (2020). The investigation of the effect of computer supported collaborative learning (CSCL) environment and dynamic mathematics software on trigonometric problem solving skill. *Technology of Education Journal, 14*(4), 867–875. https://doi.org/10.22061/tej.2020.5964.2312
Moulton, B. R. (1986). Random group effects and the precision of regression estimates. *Journal of Econometrics, 32*(3), 385–397. https://doi.org/10.1016/0304-4076(86)90021-7
Nolen, S. B. (2020). A situative turn in the conversation on motivation theories. *Contemporary Educational Psychology, 61*, 101866. https://doi.org/10.1016/j.cedpsych.2020.101866
Nolen, S. B., Horn, I. S., & Ward, C. J. (2015). Situating motivation. *Educational Psychologist, 50*(3), 234–247. https://doi.org/10.1080/00461520.2015.1075399
Peng, P., Namkung, J., Barnes, M., & Sun, C. (2016). A meta-analysis of mathematics and working memory: Moderating effects of working memory domain, type of mathematics skill, and sample characteristics. *Journal of Educational Psychology, 108*(4), 455–473. https://doi.org/10.1037/edu0000079
Phelps, E., & Damon, W. (1989). Problem solving with equals: Peer collaboration as a context for learning mathematics and spatial concepts. *Journal of Educational Psychology, 81*(4), 639–646. https://doi.org/10.1037/0022-0663.81.4.639
Pohl, A. J. (2020). Strategies and interventions for promoting cognitive engagement. In A. L. Reschly, A. J. Pohl, & S. L. Christenson (Eds.), *Student engagement: Effective academic, behavioral, cognitive, and affective interventions at school* (pp. 253-280). Springer Cham. https://doi.org/10.1007/978-3-030-37285-9_14
Pollastri, A. R., Epstein, L. D., Heath, G. H., & Ablon, J. S. (2013). The collaborative problem solving approach: Outcomes across settings. *Harvard Review of Psychiatry, 21*(4), 188–199. https://pubmed.ncbi.nlm.nih.gov/24651507/
Polya, G. (1945). *How to solve it.* Princeton University Press.
Ponitz, C. C., McClelland, M. M., Matthews, J. S., & Morrison, F. J. (2009). A structured observation of behavioral self-regulation and its contribution to kindergarten outcomes. *Developmental Psychology, 45*(3), 605–619. https://doi.org/10.1037/a0015365
Radvansky, G. A., & Copeland, D. E. (2006). Memory retrieval and interference: Working memory issues. *Journal of Memory and Language, 55*(1), 33–46. https://doi.org/10.1016/j.jml.2006.02.001
Rau, M. A., & Matthews, P. G. (2017). How to make 'more' better? Principles for effective use of multiple representations to enhance students' learning about fractions. *ZDM Mathematics Education, 49*, 531–544. https://doi.org/10.1007/s11858-017-0846-8
Ray-Riek, M. (2013). *Powerful problem solving: Activities for sense-making with the mathematical practices*. Portsmouth, NH: Heineman.
Renningr, K. A., Corven, J., De Dios, M.C., Hogan, M. R., Kyaw, M.H., Michels, A.G., Nakayama, M., Werneck, H., & Yared, F. (manuscript in preparation). *Collaborative problem solving and executive functions in middle-schoolers' work in the Virtual Math Teams environment*.
Renninger, K. A., & Hidi, S. E. (2016). *The power of interest for motivation and engagement.* Routledge.
Rogat, T., Hmelo-Silver, C., Cheng, B., Traynor, A., Adeoye, T., Gomoll, A., & Downing, B. (2022). A multidimensional framework of collaborative groups' disciplinary engagement. *Frontline Learning Research, 10*(2), 1–21. https://eric.ed.gov/?id=EJ1369028
Romero-López, M., Pichardo, M. C., Bembibre-Serrano, J., & García-Berbén, T. (2020). Promoting social competence in preschool with an executive functions program conducted by teachers. *Sustainability, 12*(11), 4408. https://doi.org/10.3390/su12114408
Salmela-Aro, K., Upadyaya, K., Cumsille, P., Lavonen, J., Avalos, B., & Eccles, J. (2021). Momentary task-values and expectations predict engagement in science among Finnish and Chilean secondary school students. *International Journal of Psychology, 56*(3), 415–424. https://doi.org/10.1002/ijop.12719
Salminen-Saari J. F. A., Garcia Moreno-Esteva, E., Haataja, E., Toivanen, M., Hannula, M. S., & Laine, A. (2021). Phases of collaborative mathematical problem solving and joint attention: A case study utilizing mobile gaze tracking. *ZDM Math Education, 53*(4), 771–784. https://doi.org/10.1007/s11858-021-01280-z).
Sankaranarayanan, R., Kwon, K., & Cho, Y. (2021). Exploring the differences between individuals and groups during the problem-solving process: The collective working-memory effect and the role of

collaborative interactions. *Journal of Interactive Learning Research, 32*(1), 43–66. https://psycnet.apa.org/record/2021-80316-002
Schoenfeld, A. H. (1992). On paradigms and methods: What do you do when the ones you know don't do what you want them to? Issues in the analysis of data in the form of videotapes. *The Journal of the Learning Sciences, 2*(2), 179-214. https://doi.org/10.1207/s15327809jls0202_3
Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education, 196*(2), 1–38. https://doi.org/10.1177/002205741619600202
Siegler, R. S. (1998). *Emerging minds*. Oxford University Press.
Skaguerlund, K., Bolt, T., Nomi, J. S., Skagenholt, M., Västfjäll, D., Träff, U., & Uddin, L. Q. (2019). Disentangling mathematics from executive functions by investigating unique functional connectivity patterns predictive of mathematics ability. *Journal of Cognitive Neuroscience, 31*(4), 560–573. https://doi.org/10.1162/jocn_a_01367
Skinner, E. A., & Pitzer, J. R. (2012). Developmental dynamics of student engagement, coping, and everyday resilience. In S. L. Christenson, A. L. Reschly, & C. Wylie (Eds.), *Handbook of research on student engagement* (pp. 21-44). Springer. https://doi.org/10.1007/978-1-4614-2018-7_2
Stahl, G. (2013). *Translating Euclid: Designing a human-centered mathematics*. Springer Cham. https://doi.org/10.1007/978-3-031-02200-5
Straub, S., & Rummel, N. (2021). Promoting regulation of equal participation in online collaboration by combining a group awareness tool and adaptive prompts. But does it even matter? *International Journal of Computer-Supported Collaborative Learning, 16*(3), 67–104. https://doi.org/10.1007/s11412-021-09340-y
Su, Y., Li, Y., Hu, H., & Rosé, C. P. (2018). Exploring college English language learners' self and social regulation of learning during wiki-supported collaborative reading activities. *International Journal of Computer-Supported Collaborative Learning, 13*(1), 35–60. https://doi.org/10.1007/s11412-018-9269-y
Sun, J., Anderson, R. C., Lin, T. J., Morris, J. A., Miller, B. W., Ma, S., Nguyen-Jaheil, K. T., & Scott, T. (2022). Children's engagement during collaborative learning and direct instruction through the lens of participant structure. *Contemporary Educational Psychology, 69*, 102061. https://doi.org/10.1016/j.cedpsych.2022.102061
Swanson, H. L., & Beebe-Frankenberger, M. (2004). The relationship between working memory and mathematical problem solving in children at risk and not at risk for serious math difficulties. *Journal of Educational Psychology, 96*(3), 471–491. https://doi.org/10.1037/0022-0663.96.3.471
Symonds, J. E., Kaplan, A., Upadyaya, K., Salmela-Aro, K., Torsney, B., Skinner, E. & Eccles, J. S. (2021). Momentary engagement as a complex dynamic system. PsyArXiv. https://doi.org/10.31234/osf.io/fuy7p
Symonds, J. E., Schreiber, J. B., & Torsney, B. M. (2019). Silver linings and storm clouds: Divergent profiles of student momentary engagement emerge in response to the same task. Journal of Educational Psychology, 113(6), 1192–1207. https://doi.org/10.1037/edu0000605
van den Bossche, P., Gijselaers, W., Segers, M., Woltjer, G., & Kirschner, P. (2011). Team learning: Building shared mental models. *Instructional Science, 39*(3), 283–301. https://doi.org/10.1007/s11251-010-9128-3
van Leeuwen, A., & Janssen, J. (2019). A systematic review of teacher guidance during collaborative learning in primary and secondary education. *Educational Research Review, 27*, 71-89. https://doi.org/10.1016/j.edurev.2019.02.001
Vandenberg, J., Zakaria, Z., Tsan, J., Iwanski, A., Lynch, C., Boyer, K. E., & Wiebe, E. (2021). Prompting collaborative and exploratory discourse: An epistemic network analysis study. *International Journal of Computer-Supported Collaborative Learning, 16*(3), 339–366. https://doi.org/10.1007/s11412-021-09349-3
Veraksa, A., Bukhalenkova, D., & Almazova, O. (2020). Executive functions and quality of classroom interactions in kindergarten among 5-6-year-old children. *Frontiers in Psychology, 11*, 603776–603776. https://doi.org/10.3389/fpsyg.2020.603776

Verbruggen, F., & Logan, G. D. (2008). Automatic and controlled response inhibition: Associative learning in the go/no-go and stop-signal paradigms. *Journal of Experimental Psychology, 137*(4), 649–672. https://doi.org/10.1037/a0013170

Viterbori, P., Traverso, L., & Usai, M. C. (2017). The role of executive function in arithmetic problem-solving processes: A study of third graders. *Journal of Cognition and Development, 18*(5), 595–616. https://doi.org/10.1080/15248372.2017.1392307

Webb, N. M., Franke, M. L., Ing, M., Turrou, A. C., Johnson, N. C., & Zimmerman, J. (2019). Teacher practices that promote productive dialogue and learning in mathematics classrooms. *International Journal of Educational Research, 97*, 176–186. https://doi.org/10.1016/j.ijer.2017.07.009

Yeniad, N., Malda, M., Mesman, J., van IJzendoorn, M. H., & Pieper, S. (2013). Shifting ability predicts math and reading performance in children: A meta-analytical study. *Learning and Individual Differences, 23*, 1-9. https://doi.org/10.1016/j.lindif.2012.10.004

Younger, J., O'Laughlin, K., Anguera, J., Bunge, S., Ferrer, E., Hoeft, F. Mccandliss, B., Mishra, J., Rosenberg-Lee, M., Gazzaley, A., & Uncapher, M. (2023). Better together: Novel methods for measuring and modeling development of executive function diversity while accounting for unity. *Frontiers in Human Neuroscience, 17*, https://doi.org/10.3389/fnhum.2023.1195013

Zambrano, J., Kirschner, F., Sweller, J., & Kirschner, P. A. (2019). Effects of group experience and information distribution on collaborative learning. *Instructional Science, 47*(5), 531–550. https://doi.org/10.1007/s11251-019-09495-0

Zhang, S., Chen, J., Wen, Y., Chen, H., Gao, Q., & Wang, Q. (2021). Capturing regulatory patterns in online collaborative learning: A network analytic approach. *International Journal of Computer-Supported Collaborative Learning, 16(*1), 37–66. https://doi.org/10.1007/s11412-021-09339-5