Journal of Numerical Cognition

PsychOpen GOLD

Empirical Research

Check for updates

Routine and Adaptive Experts: Individual Characteristics and Their Impact on Multidigit Arithmetic Strategy Flexibility and Mathematics Achievement

Lóa Björk Jóelsdóttir [1,2], Pernille Bødtker Sunde [1,3], Peter Sunde [4], Paul Andrews [1,5]

[1] *Research Centre of Pedagogy and Bildung, VIA University College, Aarhus, Denmark.* **[2]** *Department of Economics and Business Economics, Trygfonden's Centre for Child Research, Aarhus University, Aarhus, Denmark.* **[3]** *Centre for Instructional Psychology and Technology, KU Leuven, Leuven, Belgium.* **[4]** *Department of Ecoscience, Aarhus University, Aarhus, Denmark.* **[5]** *Department of Teaching and Learning, Stockholm University, Stockholm, Sweden.*

Journal of Numerical Cognition, 2024, Vol. 10, Article e14081, https://doi.org/10.5964/jnc.14081

Received: 2024-03-01 • **Accepted:** 2024-11-13 • **Published (VoR):** 2024-12-20

Handling Editor: Geetha Ramani, University of Maryland, College Park, MD, USA

Corresponding Author: Lóa Björk Jóelsdóttir, Research Center of Pedagogy and Bildung, VIA University College, Ceresbyen 24, 8000 Aarhus, Denmark. E-mail: loho@via.dk

Supplementary Materials: Materials [see Index of Supplementary Materials]

Abstract

Motivated by a curriculum privileging number-based strategies but national tests highlighting students' reliance on standard algorithms, this study analyses 2,216 Danish Grade 3, 6 and 8 students' solutions to various multidigit arithmetic tasks, each designed to elicit shortcut strategies, against background variables including sex, ethnicity and familial socio-economic status (SES), and outcomes including strategy flexibility, and national tests for both mathematics and reading. Students offering multiple solutions to a task were defined as flexible, while arithmetic experts (defined by accuracy) were distinguished by their use of shortcut strategies; routine experts never used them, while adaptive experts used them in at least one third of all tasks. With respect to mathematics achievement, experts scored 0.86 *SD*-units higher than non-experts, and within the former, adaptive experts scored 0.49 *SD*-units higher than routine experts. With respect to reading, experts achieved 0.57 *SD*-units higher than non-experts, while adaptive experts achieved 0.19 *SD*-units higher than routine experts. Boys were significantly more adaptive and flexible than girls. The proportion of experts increased from Grade 3 to Grade 8, whereas the proportion of adaptive experts increased from Grade 3 to 6 but then remained constant. Familial SES was significantly higher for experts than for non-experts but not for adaptive experts in relation to routine. Neither quarter of birth nor the existence of older siblings influenced any outcomes, although the proportion of experts was higher for children with Western backgrounds than for children with non-western background. The results suggest a relationship between adaptive expertise, strategy flexibility, and achievement.

Keywords

adaptive expertise, routine expertise, flexibility, adaptivity, arithmetic

Non-Technical Summary

Background

In mathematics education, students often use different strategies to solve arithmetic problems. Some rely on standard algorithms, while others use more flexible approaches, including shortcuts. Developing a strategy repertoire for any given problem underpins not only mathematical reasoning and computational accuracy but also the goals of reform-oriented curricula.

Why was this study done and what did we want to find out?

Our study aimed to explore the differences between two groups of students with high accuracy in multidigit arithmetic. The first group, used flexible approaches when solving arithmetic problems ("adaptive experts", defined by using shortcut strategies in at least one-third of all tasks) while the other group did not ("routine experts", defined by never using shortcut strategies). Specifically, we wanted to investigate whether adaptive experts and routine experts differed with respect to mathematics achievement, but also individual and family characteristics such as socio-economic status (SES).

What did we do?

We analysed data from Danish students in Grades 3, 6 and 8, categorizing them as experts (defined by their accuracy in multidigit arithmetic) vs. non-experts, and within the expert group as adaptive experts vs. routine experts. For each grade and boys and girls, we first described how students divided on non-experts and experts (sub-divided into adaptive experts and routine experts). Second, in a statistical analysis we analysed how the two levels of dichotomies (expert vs. non-expert and adaptive vs. routine expert) related to individual characteristics and mathematics achievement.

What did we find?

With respect to mathematics achievement, experts scored higher than non-experts, and adaptive experts scored higher than routine experts. We also found that flexible problem-solving was more common in adaptive experts than in routine experts. The proportion of students categorized as experts increased from Grade 3 to Grade 8, while the proportion of adaptive experts plateaued after Grade 6. For all grades, three times more boys than girls were categorized as adaptive experts. SES was higher for experts than for non-experts but did not differ between adaptive and routine experts.

What do these findings mean?

The results indicate a link between adaptive expertise, strategy flexibility, and achievement. These findings suggest that encouraging adaptive flexibility in problem-solving could enhance mathematics achievement.

Highlights

- In Danish Grade 3, 6 and 8 students, experts, defined by their accuracy in multidigit arithmetic, scored higher in mathematics achievement than non-experts.
- Within the expert group, adaptive experts, defined by using shortcut strategies in at least one-third of all tasks, scored higher than routine experts, defined by never using shortcut strategies.
- The proportion of adaptive experts increased from Grade 3 to 6 but then remained constant.
- At all grade levels, 3-4 times more boys than girls were categorized as adaptive experts.
- The results suggest a relationship between adaptive expertise, strategy flexibility, and achievement.

Although few would disagree that acquiring subject-related expertise is an important goal of mathematics teaching, it is so rooted in cultural expectations that some have argued that "there can be no fixed conception of what constitutes expertise in mathematics teaching" (Brown & Coles, 2011, p. 862). Moreover, further highlighting the problem, few studies of students' mathematics-related expertise have offered anything other than a token acknowledgement of the cultural context in which they were undertaken, typically assuming that expertise requires no definition (see Jurow et al.'s, 2008 study of US eighth grade and Eronen & Kärnä's, 2018 study of Finnish ninth grade students). Even international tests of mathematics achievement operationalise mathematics in ways that privilege different forms of expertise, whether it is the mathematics of a hypothetical curriculum assessed by TIMSS or the everyday real-world applications evaluated by PISA. In this paper, acknowledging such definitional inconsistency, we aim to contribute to the literature on student expertise in multidigit arithmetic by drawing on the context-independent distinction between adaptive expertise and routine expertise (Hatano & Inagaki, 1984) to examine Danish students' arithmetical solution strategies.

Hatano and Inagaki's (1984) context-independent conceptualisation presented routine experts as having acquired "a body of procedural knowledge... along with the skills necessary for applying that knowledge" (p. 28). Such people, when operating in familiar environments, and despite limited understanding of the principles that underpin their skills, are able to function well with just procedural knowledge. Alternatively, adaptive experts not only have well-developed procedural skills, but "understand the meaning of the skills" (ibid, p. 28) and are able, as a consequence, to manage with confidence "unfamiliar situations and to invent new strategies" (ibid, p. 30). However, highlighting the role of culture in the development of an individual's adaptive expertise, the extent to which members of particular communities become adaptive experts is dependent on whether routine expertise is the privileged form of knowledge and people are discouraged from investigating variations in those skills (ibid).

Adaptive Expertise and Mathematics Education

The development and manifestation of adaptive expertise in the context of mathematics learning has been increasingly researched (Hickendorff, 2018; McMullen et al., 2016; Nemeth et al., 2019; Verschaffel et al., 2009), particularly from the perspective of strategy adaptivity, where evidence suggests that following the introduction of standard algorithms even students with access to multiple strategies typically resort to just one (Sievert et al., 2019). In short, teaching that promotes algorithms tends also to encourage routine experts (Selter, 2009), who, with limited understanding, can solve school exercises efficiently (Baroody, 2003; Hatano & Oura, 2003). Such students acquire a "static, sparsely connected knowledge that can only be applied to typical tasks", often with little transferability to new situations (McMullen et al., 2020, p. 1). By way of contrast, adaptive experts are not only able to integrate both procedural knowledge and conceptual knowledge (Hatano & Oura, 2003) but do so in ways that allow them to go beyond algorithms and manage unfamiliar situations (Hatano & Inagaki, 1984). By drawing on such connections, adaptive experts' understanding of each step of a chosen strategy or procedure, may offer alternative steps within that strategy or procedure, or even invent new ones (Hatano & Inagaki, 1984).

From the particular perspective of arithmetic, the focus of this paper, it is known that adaptive experts are able to identify optimal strategies for a given problem (McMullen et al., 2016), and even young children, when given the opportunity, are able to switch strategies thoughtfully (Lemaire & Callies, 2009; Torbeyns et al., 2018). Developing a strategy repertoire for any given problem, alongside the competence to select appropriately from that repertoire, underpins not only mathematical reasoning and computational accuracy (Sievert et al., 2019; Threlfall, 2009; Torbeyns et al., 2017) but also the goals of reform-oriented curricula. In this respect, the teaching of arithmetic in Denmark, the site of the research reported below, is expected to focus on number understanding and number-based strategies rather than standard algorithms (Ministry of Children and Education, 2019). However, despite such goals, when presented with tasks designed to elicit shortcut strategies (that is, strategies that involve the flexible adaptation of numbers and operations, as with the compensation-related transformation of 199 + 323 to 200 + 323 − 1 or the indirect addition transformation of 703 − 686 to 14 + 3, whereas a non-shortcut strategy or routine strategy could be an algorithmic approach, see for example Linsen et al., 2015), students rarely apply such strategies (Hickendorff, 2018, 2022; Jóelsdóttir & Andrews, 2024b). This lack of shortcut strategy use highlights the extent to which instructional practices may influence which strategies are privileged in mathematics classrooms, particularly as "adaptive expertise involves habits of mind, attitudes, and ways of thinking and organising one's knowledge that are different from routine expertise and that take time to develop" (Verschaffel et al., 2007, pp. 32-33).

In recent years, research into the development of students' adaptive expertise has been augmented by an emphasis on flexibility. Indeed, Nemeth et al. (2019) write of a "consensus among mathematics researchers and educators that the abilities to use various strategies for solving a problem (flexibility) as well as to use efficient strategies (adaptivity) are important mathematical competencies students should gain" (p. 2). However, the distinction between the two constructs has not always been transparent and may be due to Hatano and Inagaki's (1984) earlier observation that adaptive experts possess "flexible and adaptive conceptual knowledge" (p. 30). In this paper, we make a distinction between the concepts of flexibility and adaptivity; with flexibility referring to the existence of a strategy repertoire for solving a particular problem and the ability to switch between those strategies and adaptivity referring to the ability to select the optimal strategy from that repertoire (Hickendorff, 2022; Verschaffel et al., 2009). When viewed in this way, adaptive experts are both flexible and adaptive, as knowing and being able to switch between various strategies are preconditions for adaptivity (Verschaffel et al., 2007).

Adaptive Expertise and Individual Differences

It is generally agreed that students' arithmetic-related strategy choice draws on their prior knowledge of arithmetic (Schulz, 2018; Threlfall, 2009), including both their conceptual and procedural knowledge (Schneider et al., 2011; Star et al., 2015), and the learning opportunities they receive (Sievert et al., 2019). Furthermore, as highlighted in a recent review paper, and of particular relevance to this study, individual and familial variables like, for example, student sex and grade level, alongside socio-economic status (hereafter SES) and other family characteristics have been found to influence strategy flexibility and adaptivity (Verschaffel, 2024).

Sex differences in strategy choice for single-digit arithmetic have been found from the early school years (Carr & Jessup, 1997; Sunde et al., 2020), with girls showing a preference for counting strategies and boys a preference for retrieval strategies. Such differences, in respect of multi-digit arithmetic, resonate with Dutch studies highlighting boys' preferences for mental or short-cut methods and girls' preferences for written methods (Hickendorff, 2018). Elsewhere, at least in the context of eighth and ninth graders' learning of algebra, girls have been found to be more adaptive and flexible than boys (Star et al., 2015). However, research on the influence of sex on strategy choice remains limited.

From the perspective of maturation, instruction, particularly textbooks' timing of the introduction of standard algorithms, may limit not only students' strategy repertoires but their preferred strategies (Hickendorff et al., 2019; Sievert et al., 2019; Torbeyns et al., 2017). Indeed, notwithstanding evidence that students' adaptive strategy choices tend to become more likely with age (Lemaire & Callies, 2009; Torbeyns et al., 2018), in Denmark, the context of this study, both strategy adaptivity and flexibility seem to peak in Grade 6, with students in Grade 6 showing higher levels of both adaptivity and flexibility than those in Grades 3 and 8 (Jóelsdóttir & Andrews, 2024b). Also, children born early in the school year typically achieve higher than those born later and, while some scholars have argued that these differences disappear with age (Milling-Kinard & Reinherz, 1986), the consensus seems to be that they persist (Bell & Daniels, 1990). For example, a recent reanalysis of PISA data found, across mathematics, science and literacy, that at age 15, children born at the start of the school year benefit both cognitively and affectively in comparison with their younger peers (Givord, 2020). Indeed, Smith (2009) has estimated, with respect to Canadian children, that the test scores on core knowledge domains like mathematics and literacy of "older children exceed those of younger children by 0.259 − 0.400 standard deviations (σ) in Grade 4, declining to between 0.104 − 0.242 by Grade 10. Such results complement a Chilean study that found delaying the start of school for the youngest children increases the achievement scores of fourth and eighth grade students by more than 0.3 standard deviations (McEwan & Shapiro, 2008). However, despite such differences, little is known about the impact of birth month on students' multidigit arithmetic strategy choices. In conclusion, while maturity and a child's age relative to his or her cohort are known to impact achievement, little is known about their influence on arithmetic-related strategy choices.

Evidence with respect to the influence of siblings on educational achievement seems ambiguous. For example, a recent Chinese study found, at both grades four and eight, that children raised with no siblings outperformed those with siblings on mathematics, Chinese, science and English (Li et al., 2021). Elsewhere, a study undertaken in Florida, found, with respect to the influence of older siblings, positive influences in low-income households but negative in high income households (Karbownik & Özek, 2023). However, acknowledging our interest in mathematics, Gabay-Egozi et al. (2022) found that women were more likely to study STEM subjects if they had been raised in homes with either small sibling groups, male sibling group dominance, or if they had an older sister with high mathematics achievement. Also, acknowledging the influence of socio-economic status (SES) on mathematics achievement generally (Kalaycioğlu, 2015), it would be reasonable to assume that students from high SES backgrounds would exhibit higher levels of adaptive behaviour than their peers from low SES backgrounds, an assumption that has not, to the best of our knowledge, been investigated. Finally, in addition to highlighting achievement differences located in students' sex and familial SES, international studies like PISA remind us of the achievement gap between students with an immigrant background and their non-immigrant peers, even when controlling for SES (OECD, 2023). Overall, while the above highlights the influence of a range of background characteristics, both individual and familial, on general mathematics achievement, little is known about their influence on the different forms of expertise and students' strategy choices.

The Current Study

Acknowledging such matters, the research reported in this paper examines the influence of a range of individual and familial characteristics on the development of routine expertise and adaptive expertise with respect to Danish Grade 3, 6 and 8 students' approaches to multidigit arithmetic. The study is guided by the question:

> How, in respect of multidigit arithmetic, do adaptive experts and routine experts differ with respect to achievement and both individual and family characteristics?

Specifically, the study examines how adaptive and routine experts differ with respect to age, ethnicity, sex, older sibling, and their parents' socio-economic status (SES). In addition, the study examines the relationships between mathematics achievement and reading, as measured by Danish national tests, and students' arithmetic-related strategy adaptivity and flexibility. We use reading as a proxy for general achievement, to find if the differences between adaptive and routine experts are similar for achievement in other school subjects, compared to mathematics.

Operationalising Expertise — In the following, an expert is defined as a student who is able to calculate accurately across different operations. Experts defined in this way are then distinguished according to whether they are routine experts – students who never use shortcut strategies – or adaptive experts – students who regularly use shortcut strategies. Such an approach should provide new insights into how various individual characteristics impact the two forms of arithmetic-related expertise. It will also extend our understanding of routine and adaptive experts by going beyond an examination of the influence of family background and SES to consider the impact of student flexibility in arithmetic strategy choice and both mathematics and reading achievement.

Method

Participants

Public schools in five demographically different municipalities in Jutland, the central region of Denmark, were invited to participate. This yielded 20 schools, large and small, urban and rural, and a total of 2298 pupils in 37 Grade 3, 39 Grade 6 and 45 Grade 8 classes. These grades were chosen as national tests for mathematics, undertaken in the same grades, and reading, undertaken in Grades 6 and 8, provide the achievement measures used below. In the Danish school system students start in Grade 0 in August of the year in which they turn 6 years, with the consequence that children born in the first quarter (Q1) are almost a year older than their peers born in the fourth. This paper is based on data from the 2,216 pupils (96.4% of the tested sample) for whom family background and other data were available (Table 1). Prior to children's participation, parents/carers (hereafter parents) received two letters. The first, through their child's school's communication platform, outlined the project and confirmed parents' right to withdraw their child for no reason and with no consequences. The second, by means of an email, informed parents about, for example, how personal data would be managed within the framework of the EU's general data protection regulation (GDPR). With the exception of three students whose parent denied permission, all students present on the day of testing participated.

Table 1

Participant Totals, Median Age and Distribution on Grade and Sex

Grade level	Classes (*n*)	Students (*n*)	Median age	Girls (*n*)	Girls (%)
Grade 3	37	722	9	353	49
Grade 6	39	687	12	358	52
Grade 8	45	786	14	380	48
Total	121	2216		1091	50

Operationalising Routine and Adaptive Expertise

All students undertook a Tri-phase Flexibility Assessment (hereafter TriFA) of their multidigit arithmetical competence. Originally designed and validated for evaluating students' equations-related strategy flexibility (Xu et al., 2017), the TriFA has been successfully adapted for evaluating students' multidigit arithmetic strategy choices (Jóelsdóttir & Andrews, 2024a). During Phase 1 students solved each task by whatever strategy they wish; during Phase 2 they solved the same tasks again but with as many different strategies as they can; during Phase 3 they identified the optimal strategy from those they offered. All responses were given in writing. As the aim was to assess strategy repertoire, solution speed was not assessed.

All tasks in this adaptation, which was completed in an hour of class time, were designed to elicit shortcut strategies. Details of each task can be found in Jóelsdóttir and Andrews (2024a). Briefly, acknowledging students' likely experiences of arithmetic, Grade 3 students undertook four addition and four subtraction tasks. Those in Grade 6 undertook three additions, three subtractions and three multiplications, while those in Grade 8 undertook four each of addition, subtraction and multiplication. Where possible, the same tasks were used across the grades, with each designed, as indicated, to elicit different types of shortcut strategies (see Jóelsdóttir & Andrews, 2024a, 2024b). Importantly, division was excluded as the low levels of division-related competence yielded by the Danish national tests indicated that an analysis of division-related strategy selection would be unproductive.

All written responses were coded independently by at least two raters, the first author and research assistants trained to code the various multidigit arithmetic tasks. Students' solutions to Phase 1 tasks were coded first for accuracy (correct, incorrect or no solution) and then for strategy (shortcut or not shortcut). In this context and irrespective of grade, an expert was defined as a student who made no more than one error across all test items and an adaptive expert was an expert who offered a shortcut strategy on the task in question. With respect to Phase 2, the total number of solutions was recorded and each additional solution was coded for strategy type as with Phase 1. Next, the solutions identified as optimal during Phase 3 were coded as either shortcut or not. Any disagreements among coders were discussed and a third colleague invoked when necessary.

As our interest lies in understanding the characteristics and impact of the different forms of expertise, where an expert was any students who submitted no more than one incorrect solution. The figures of Table 2 show the number of tasks experts should solve by means of shortcut strategies to be classed as either routine or adaptive. Here, experts who failed to solve any task by means of shortcut strategies were classified as routine experts. Second, due to the unexpectedly low number of students implementing shortcut strategies, the bar for defining adaptive experts was set lower than originally intended. Consequently, an adaptive expert was defined as one who uses shortcut strategies regularly, that is, on at least three tasks in Grades 3 and 6, and four in Grade 8. This pragmatic definition ensured that shortcut use was unlikely to be accidental. For example, any sixth grade student offering shortcuts on three different tasks would either be offering shortcut solutions to all three tasks addressing the same operation, each of which is designed to elicit a different shortcut strategy, or to tasks from different operations, indicating an ability to use shortcut strategies across the operations (Jóelsdóttir & Andrews, 2024a, 2024b). Experts using shortcuts on fewer tasks cannot with confidence be said to be adaptive and were categorised as mixed experts. Finally, Figure 1 shows the proportions of students at each grade level classified according to the above protocol.

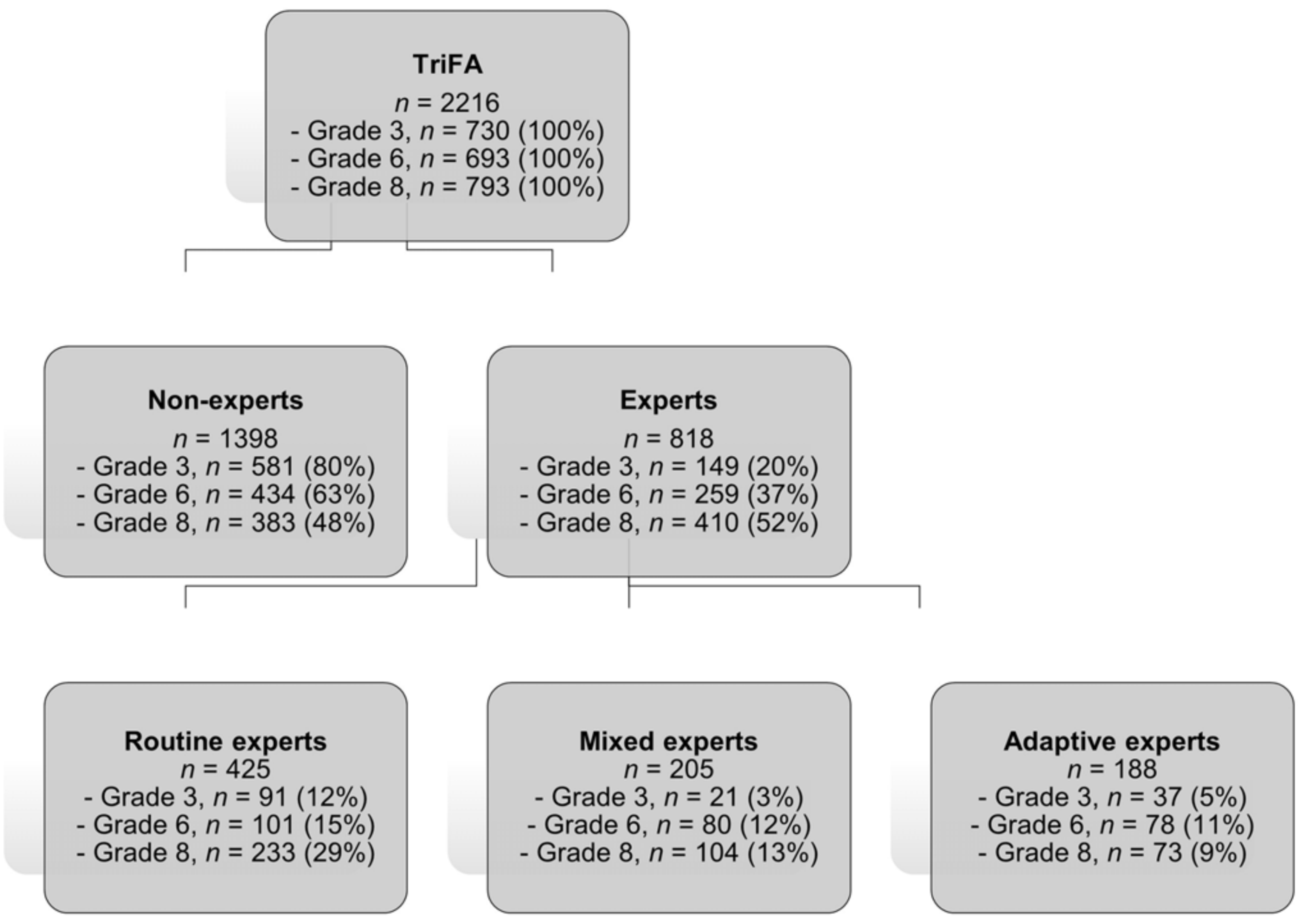
Table 2

Defining Characteristics of Experts, Routine Experts, Mixed Experts and Adaptive Experts

Group			
	Tasks solved accurately		
Grade	Grade 3	Grade 6	Grade 8
(Total number of tasks)	(8)	(9)	(12)
To be an expert (and then)	7-8	8-9	11-12
	Tasks solved adaptively		
Routine experts	0	0	0
Mixed experts	1-2	1-2	1-3
Adaptive Experts	3-8	3-9	4-12

Figure 1

Distribution of Students by Levels of Expertise and Grade



Student Background Characteristics, Family Background and SES

To determine the influence of students' sex (female or male as assigned by birth), ethnicity, family background and SES, register data were obtained from Statistics Denmark. Student ethnicity, following Statistic Denmark's definition, was defined as Western or non-Western, an older sibling student was defined as having at least one older sibling in the student home, and a Q1 student was born in the first three month of the calendar year. Finally, acknowledging its widespread use elsewhere (Shavers, 2007), parents' educational level was used as a proxy for SES. In this instance, a child where at least one parent had a university level qualification was scored 1 otherwise 0.

Student Achievement and Flexibility

Measures of mathematics achievement (Grades 3, 6 and 8) and reading (Grades 6 and 8) were obtained from the Danish National tests, which are mandatory in mathematics for students in Grades 3, 6 and 8 and for reading in Grades 2, 4, 6 and 8 (Ministry of Children and Education, 2019). Finally, based on their responses to the TriFA, three additional scores were calculated for each student. First, flexibility referred to the percentage of TriFA tasks solved by means of more than one strategy. Second, flexibility with shortcut referred to the percentage of TriFA tasks solved flexibly and by at least one shortcut strategy during either Phase 1 or Phase 2. Third, shortcut optimal referred to the percentage of TriFA tasks for which students identified a shortcut strategy as optimal during Phase 3. We did not distinguish between correct or incorrect answers.

Statistical Analysis

The analysis is in two parts. First, the characteristics of the different expert groups in respect of the different background variables were determined. That is, for each student group, pooled across Grade 3, 6 and 8, and classified as (i) either experts or non-experts and (ii) (within the expert group) as either routine, mixed or adaptive experts, we examined, with 95% confidence levels, the proportions that were (a) girls (versus boys), (b) western (versus non-western), (c) in families with older siblings, (d) born in Q1 or (e) in families with at least one parent with a higher education qualification. Next, *t*-scores derived from differences in these proportions relative to their SES enabled us to test (two-tailed hypothesis) whether (i) experts differed from non-experts and (ii) adaptive experts differed from routine experts with respect to these background variables. In addition, drawing on data from the mandatory Danish national tests, we examined the mean and spread of the different student groups' achievement scores in mathematics (Grades 3, 6 and 8) and reading (Grade 6 and 8). All such scores, based on a national sample, are standardised with $M = 0$ and $SD = 1$. Finally, we quantified and tested mean scores for (a) flexibility, (b) flexibility with shortcut, and (c) shortcut optimal.

Second, by means of multiple logistic regressions that account for possible between-class variation as random effects, we quantified the total and partial effects of the above variables as predictors with regard to classifying (i) experts versus non-experts and (within the expert group) as (ii) routine versus adaptive experts. Seven fixed-effect predictor variables were entered in the analyses. Five of these were the same dichotomous variables examined in part one. In addition, two variables were entered as covariates: achievement score in national test in mathematics and flexibility, as determined by the percentage of tasks solved by means of more than one strategy.

National test scores in reading were not included in this initial analysis, as no test scores were available for Grade 3 students. Also, the variables labelled flexibility with shortcut strategies and shortcut optimal were not included as the first is a subgroup of flexibility and the second is partly included in the definition of adaptivity. Prior to the analysis, we created a correlation matrix for all predictor variables to reveal possible collinearity. A priori, we defined a numerical value of Pearson's r of ≥ 0.7 (where 50% of the variation in one predictor variable would be explained by variation in another predictor variable) as a criterion for only including one of any inter-correlated predictors. As all $|r|$ values between eligible predictors were ≤ 0.39 (see Appendix A in the Supplementary Materials), none of the aforementioned predictors were excluded.

For each set of analyses, (i) experts versus non-experts and (ii) adaptive experts versus routine experts, we first established univariate models for each of the aforementioned predictors (yielding total effects of each predictor). In a second round, we established a full model, comprising all seven fixed-effect predictors, yielding partial effects of each predictor alongside the effect of other variables. In a third round, we used a forward stepwise procedure to establish a model consisting of statistically significant terms only (least complex adequate model). As model selection procedure, we used forward stepwise selection with students clustered in classes and with significance level for addition to the model $p < .05$ (default option in Stata). For all models comprising multiple fixed-effect predictors (Round 2 and 3), as post hoc operations we tested for possible two-way interactions between statistically significant main effects.

We first analysed data for Grade 3, 6 and 8 separately. Then, to reveal any variation in fixed effect size between grades, we analysed data for all three grades (3, 6 and 8) in a single model, where we accounted for grade as main effect. We also included interaction terms between grade and the other predictors where such interactions were statistically significant. For an interaction term between grade and a given fixed effect variable (*X*) to be included, the variable *X* should be statistically significant ($p < .05$) as main effect in the first place, as should the interaction term grade-by-variable *X*, as well as the combined statistical significance of the main effect of variable *X* and its interaction term with grade. In so doing, we tested the extent to which effect sizes of significant terms in the full model (β_2) deviated from the initial effect size (β_1), using a *z*-score calculated as $z = (\beta_1 - \beta_2)/(SE_1^2 + SE_2^2)^{0.5}$. Unless otherwise stated, all analyses were carried out in Stata/SE 15.1. Finally, for simplicity, we present only the results for the combined sample; results concerning analyses at the grade level, which are not explicitly focused on our research questions, can be seen in Appendix B (Tables B1 to B6) in the Supplementary Materials.

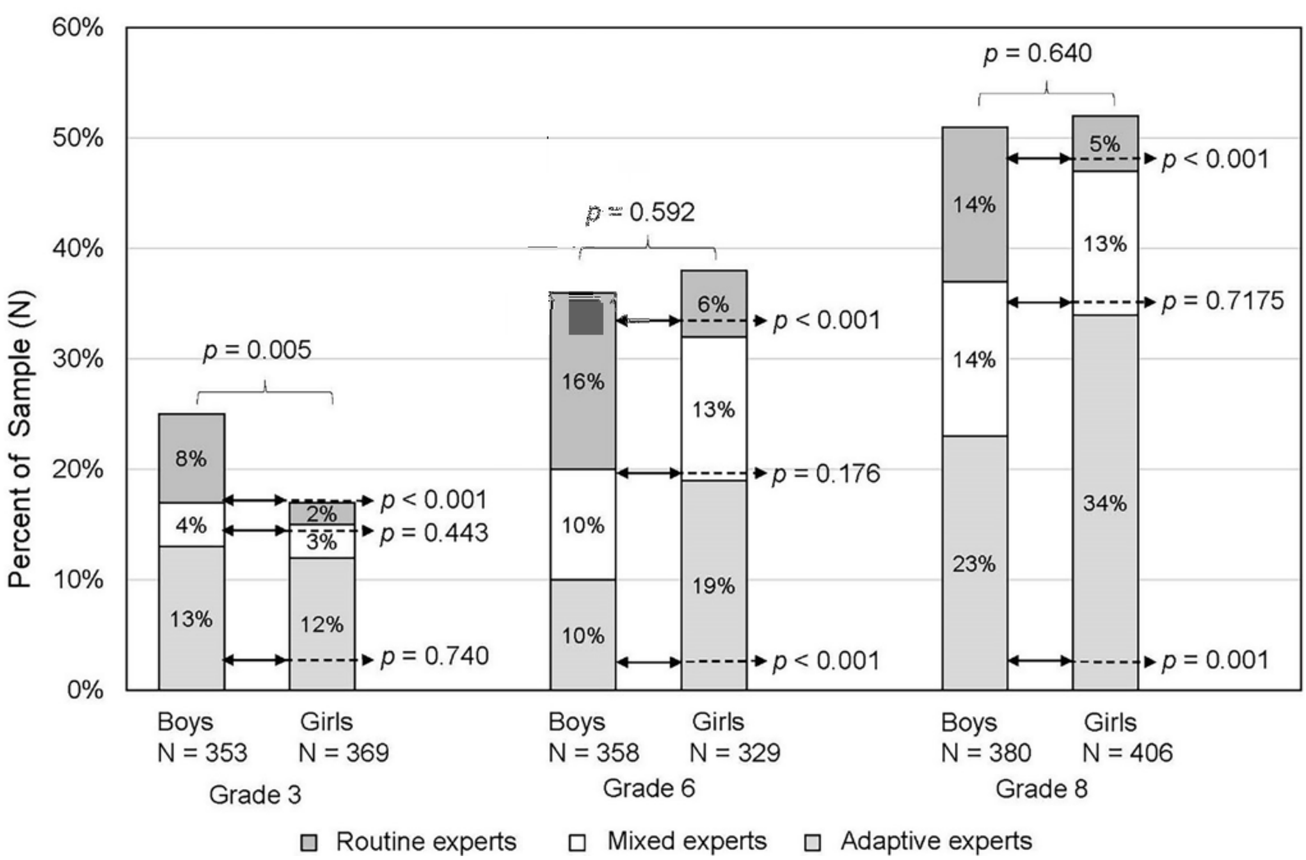
Results

Proportion of Non-Expert and Expert Groups Divided on Grade and Sex

As shown in Figure 1, the proportion of students categorized as experts increased from 20% in Grade 3, to 37% in Grade 6 and 52% in Grade 8 (Pearson's $\chi^2_2 = 34.4$, $p < .0001$). In addition, Figure 2 shows that Grade 3 boys were classified as experts significantly more often than Grade 3 girls (two-tailed *t*-test: $t_{720} = -2{,}80$, $p = .005$).

Figure 2

The Percentage of Students Divided on Sex and Grade Categorized as Experts (Subdivided Into Routine, Mixed and Adaptive Experts)



Note. *p*-values from a two-tailed *t*-test indicate the probability that the proportions of students categorized as (i) experts (ii) routine experts (iii) adaptive experts are similar for boys and girls within a given grade.

However, no such differences were found for Grade 6 (p = .59) or Grade 8 (p = .64). Figure 2 also shows that while the proportions of students classified as routine experts were similar for boys and girls in Grade 3 (13% and 12% respectively), the proportion of routine expert boys was significantly lower than girls in both Grade 6 (10% and 19% respectively) and Grade 8 (23% and 34% respectively). Hence, while a multilevel logistic regression found the proportion of girls categorised as routine experts to increase across the three grades (Grade 3 vs. Grade 6: $\beta = 0.53$, $SE = 0.21$, $p = .012$; Grade 6 vs. Grade 8: $\beta = 0.80$, $SE = 0.18$, $p < .0001$), the proportion of boys categorized as routine experts was constant between Grade 3 and 6 ($\beta = -0.32$, $SE = 0.24$, $p = .17$), but doubled between Grade 6 and 8 ($\beta = 1.04$, $SE = 0.22$, $p < .0001$). By way of contrast, Figure 2 also shows that the proportion of boys identified as adaptive experts was, irrespective of grade level, around three times that of girls: whether in Grade 3 (8% and 2% respectively), Grade 6 (16% and 6% respectively) or Grade 8 (14% and 5% respectively). Despite such differences, the proportions of adaptive experts, irrespective of sex, grew significantly from Grade 3 to Grade 6 – for boys the proportion doubled (from 8% to 16%) ($\beta = 0.75$, $SE = 0.24$, $p = .002$) and for girls it tripled (from 2 to 6%) ($\beta = 1.15$, $SE = 0.45$, $p = .010$) but remained stable from Grade 6 to Grade 8 (boys $p = .29$, girls $p = .72$).

Characteristics of Non-Expert and Expert Groups

Table 3 shows the percentage, by individual characteristics, for non-experts compared with experts, and routine experts compared with adaptive experts. Routine experts differed from adaptive experts in the use of standard algorithms, with means of 91% and 21% respectively of tasks solved in this manner. While the percentage of boys and girls were comparable for both the non-expert and the expert groups, differences within the expert group were evident; more than half of all routine experts were girls (59%) while only one quarter of adaptive experts were girls. Significantly more experts than non-experts had western backgrounds (91% vs. 86%), while the proportion of students with a western background was equal for routine experts and adaptive expert (Table 3). Neither being born in the first quarter of the year nor having an older sibling had any influence on group proportions (Table 3). However, SES, as measured by at least one parent having a university qualification, was significantly higher for experts (68%) than for non-experts (53%) and, albeit to a lesser extent, for adaptive experts (75%) than for routine experts (65%).

Table 3

Percentage of Students, With Standard Error in Parentheses, Classified as Either Non-Expert or Expert, and (Within the Expert Group) as Either Routine, Mixed or Adaptive by Individual Characteristics and Family Background

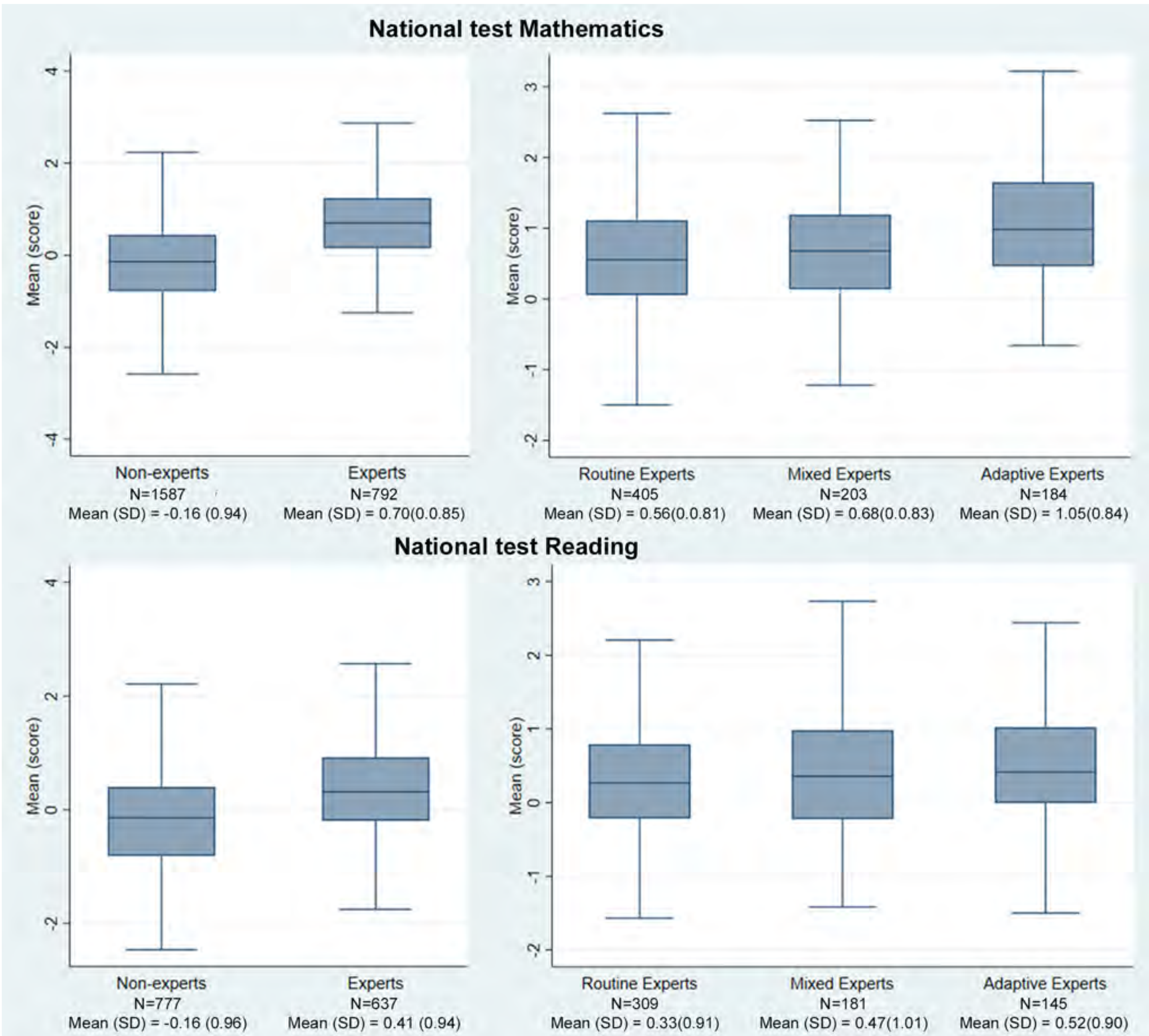
	Experts vs. non-experts					**Routine experts vs. adaptive experts**				
Variable	**Non-expert *N* = 1,398**	**Expert *N* = 818**	***df***	***t***	***p***	**Routine Expert *N* = 425**	**Adaptive Expert *N* = 188**	***df***	***t***	***p***
Girls	51% (1)	49% (2)	2193	-0.70	.485	59% (2)	25% (3)	604	8.24	< .0001
Western	86% (1)	91% (1)	2194	-3.02	.003	90% (1)	93% (2)	604	-1.24	.215
Older siblings	59% (1)	57% (2)	2214	1.28	.202	57% (2)	60% (4)	611	-0.66	.508
Born Q1	23% (1)	27% (2)	2214	-1.93	.054	26% (2)	27% (3)	611	-0.06	.950
High SES	53% (1)	68% (2)	2214	-7.35	< .0001	65% (2)	75% (3)	611	-2.36	.019

Note. The p-values, derived from t-tests, indicate the statistical significance of pairwise differences.

The various charts of Figure 3 show, for the different student groups, the standardised results of national tests for mathematics, based on a mean of zero and a standard deviation of one. With means of, respectively, 0.70 and −0.16, students classified as experts, achieved 0.86 *SD*-units higher than non-experts ($t_{2147} = -21.2$, $p < .0001$). Within the expert group, adaptive experts scored 0.49 points higher than routine experts ($t_{587} = -6.75$, $p < .0001$). Also, further highlighting the potential impact of adaptivity, students in the mixed expert group who, by definition showed adaptivity on at least one task, scored higher than the routine experts but lower than the adaptive experts. With respect to reading, assessed only for Grades 6 and 8, the charts of Figure 3 show that students in the expert group scored 0.57 *SD*-units higher than non-experts (t_{1410} = −11.24, p < .0001). Also, within the expert group, adaptive experts scored 0.19 points higher than routine expert (t_{452} = −2.04, p = .04).

Figure 3

National Test Results for Mathematics and Reading for the Different Student Groups

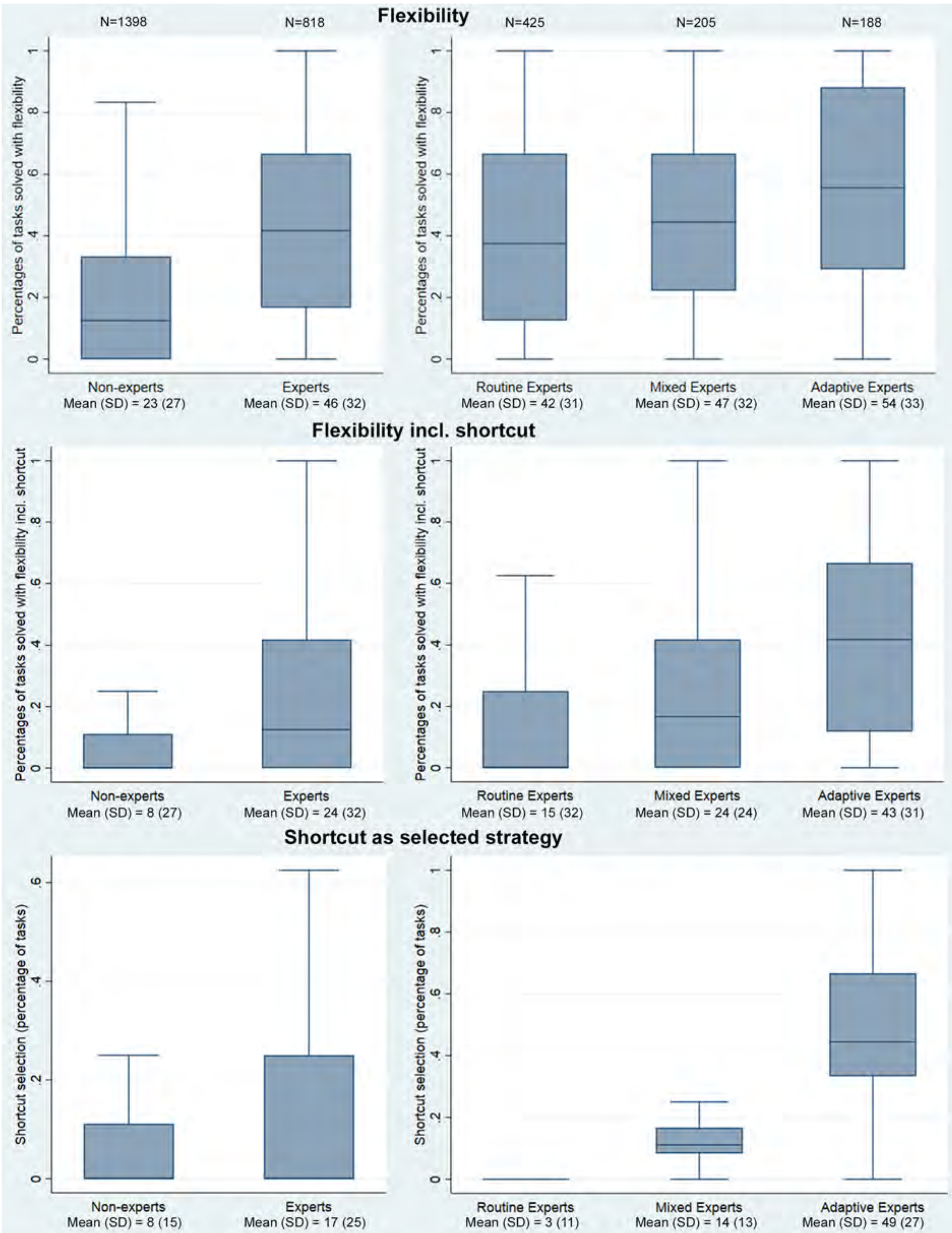


Note. The results are standardised, based on a mean of zero and a standard deviation of one. For legal reasons concerning the remote possibility of identifying individual students, outliers are not shown.

Figure 4 shows the percentages of tasks solved flexibly by the different groups. Experts were significantly more flexible than non-experts (t_{2214} = −18.10, p < .0001), solving on average 46% of all tasks, compared with 23%, by means of more than one strategy. Much of this difference was a consequence of the proportion of students unable to solve any task flexibly − 16% of the expert group were unable to solve any task flexibly compared with 42% for the non-expert group. Within the expert group, adaptive experts solved a significantly higher proportion of tasks flexibly (54%) than did routine experts (42%) (t_{611} = −4.39, p < .0001).

Figure 4

The Percentages of Tasks Solved Flexibly by Each Expert Group, Task Solved Flexibly and by at Least One Shortcut Strategy by Each Expert Groups and Solved by Means of a Shortcut Strategy Identified as Optimal



Note. For legal reasons, outliers are not shown.

Figure 4 also shows, for each student group, the percentages of tasks solved flexibly with at least one shortcut strategy. Experts solved, on average, 24% of all tasks flexibly with at least one shortcut strategy, while the comparable figure for non-experts was 8% (t_{2214} = −16.96, p < .0001). Within the expert group, the figures were 43% and 15% for adaptive and routine experts, respectively (t_{611} = −12.85, p < .0001). Finally, Figure 4 shows the percentages of tasks, for each student group, for which a shortcut strategy was identified as optimal. On average, 17% of tasks fell into such a category for the experts, compared to 8% for the non-experts (t_{2214} = 10.03, p < .0001). Within the expert group, adaptive experts (49%) selected shortcuts as optimal seventeen times as often as routine experts (3%) (t_{611} = 30.20, p < .0001).

Multivariate Analysis

Differences Between Experts and Non-Experts

Table 4 shows the results of three multilevel logistic regression models developed to determine the predictive impact of the different variables under different conditions. The simple model, which adjusted for grade and class nested within school, shows that the probability that a student was classified as an expert was significantly higher for students with western background than for students with a non-western background, higher for students with a parent with higher education (SES) than for students without a parent with higher education. Being an expert also predicted higher achievement in the national test in mathematics and a higher proportion of tasks solved flexibly.

Table 4

Fixed Effects Coefficients of Multilevel Logistic Regression Models (Accounting for Class Nested Within School as Random Factors), Predicting the Conditional Probability That a Student Is Categorized as Expert as Opposed to Non-Expert

	1: Grade + single variables[a]			2: Full model (all variables)			3: Significant effects only		
Predictors	**β (*SE*)**	***z***	***p***	**β (*SE*)**	***z***	***p***	**β (*SE*)**	***z***	***p***
Intercept	-1.590 (0.185)			-2.520 (0.308)			-2.629 (0.221)		
Grade 6 (vs. 3)	0.963 (0.178)	5.42	< .0001	1.356 (0.263)	5.16	< .0001	1.327 (0.261)	5.09	< .0001
Grade 8 (vs. 3)	1.575 (0.179)	8.79	< .0001	2.111 (0.257)	8.20	< .0001	2.095 (0.255)	8.21	< .0001
Type-3 test: Grade (3, 6 or 8);	χ^2_2 = 77.48, p < .0001			χ^2_2 = 67.82, p < .0001			χ^2_2 = 67.96, p < .0001		
Girl (yes)	-0.059 (0.098)	-0.60	.546	-0.073 (0.113)	-0.65	.517	*not included*		
Western (yes)	0.506 (0.180)	2.81	.005	-0.165 (0.214)	-0.77	.441	*not included*		
Older sib (yes)	-0.186 (0.099	-1.87	.062	-0.125 (0.114)	-1.10	.273	*not included*		
Born in Q1 (yes)	0.209 (0.114)	1.83	.067	0.291 (0.132)	2.21	.027	*not included*		
SES (yes)	0.718 (0.109)	6.57	< .0001	0.111 (0.131)	0.84	.399	*not included*		
NT maths score	1.199 (0.072)	16.70	< .0001	1.076 (0.078)	13.71	< .0001	1.068 (0.075)	14.27	< .0001
Flexibility[b]	2.378 (0.180)	13.23	< .0001	[1.418 (0.202)]	[7.01]	[< .0001]	[1.427 (0.228)]	[6.28]	[< .0001]
Test: Grade-by-flexibility	χ^2_2 = 4.68, p = .097			χ^2_2 = 10.20, p = .006			χ^2_2 = 9.59, p = .008		
Flexibility in Grade 3	2.910 (0.348)	8.37	< .0001	2.421 (0.377)	6.42	< .0001	2.398 (0.374)	6.42	< .0001
Flexibility in Grade 6	2.295 (0.304)	7.54	< .0001	1.018 (0.335)	3.04	.002	1.074 (0.332)	3.23	.001
Flexibility in Grade 8	2.078 (0.282)	7.36	< .0001	1.006 (0.317)	3.18	.001	1.022 (0.312)	3.27	.001
Significance of total model:				Wald χ^2_9 = 350.99, p < .0001			Wald χ^2_6 = 349.19, p < .0001		

[a]Effects shown for single variables in models with grade as mandatory fixed effect. [b]In the “Full model” and “Significant effects only” model, ”[..]” indicates main effects of Flexibility if included in the model without interaction terms between Flexibility and Grade.

In the full model, the effects of the national mathematics test score and flexibility remained highly significant, while the significant effects of SES and ethnic background disappeared. Thus, western background and SES were excluded from the significant effects only model. The effect of being born in Q1 was borderline positive significant if entered in isolation and marginally significant in the full model, but not included in the significant effects only model. Furthermore, a statistically significant interaction term between grade and flexibility, suggested that the positive effect of flexibility was stronger in Grade 3 compared to Grade 6 and 8.

From Table 4 it is also clear that the effects of the national mathematics test score were of similar magnitude in the simple model and the full model (β = 1.20 vs. 1.08; z = 1.16, p = .24). The effect of flexibility was not significantly reduced from the simple to the full model in Grade 3 (β = 2.91 vs. 2.42; z = 0.95, p = .34), whereas this was the case for students in Grade 6 (β = 2.30 vs. 1.02; z = 2.64, p = .009) and Grade 8 (β = 2.08 vs. 1.01; z = 2.53, p = .012).

Differences Between Adaptive Experts and Routine Experts

The univariate as well as the complex models (adjusted for variation between grade and random variation between classes) all showed that the probability that an expert was adaptive rather than routine was substantially higher for boys than for girls (β_{girl} = -1.766, odds ratio = exp[β_{girl}] = 1: 4.8) and correlated positively with their national mathematics test and flexibility scores (Table 5). The effects of all three predictors appeared to be additive, as their partial effect sizes did not deviate significantly from the univariate effect sizes (Table 5; sex: z = 0.34, p = 0.73; NT math; z = 0.52, p = .60; Flexibility: z = 0.42, p = .75).

Table 5

Fixed Effect Coefficients of Multilevel Logistic Regression Models (Accounting for Class Nested Within School as Random Factors), Predicting the Conditional Likelihood That an Expert Student Is Categorized as Adaptive Expert as Opposed to Routine Expert

	1: Grade + single variables[a]			2: Full model (all variables)			3: Significant effects only		
Predictors	**β (*SE*)**	***z***	***p***	**β (*SE*)**	***z***	***p***	**β (*SE*)**	***z***	***p***
Intercept	-1.092 (0.422)			-2.174 (0.713)			-1.393 (0.252)		
Grade 6 (vs. 3)	0.792 (0.417)	1.90	.057	1.058 (0.480)	2.20	.028	1.049 (0.471)	2.23	.026
Grade 8 (vs. 3)	0.331 (0.407)	-0.81	.417	0.023 (0.469)	0.05	.961	0.040 (0.457)	0.09	.931
Type-3 test: Grade (3, 6 or 8)	χ^2_2 = 10.02, p = .0067			χ^2_2 = 7.68, p = .021			χ^2_2 = 7.68 p = .022		
Girl (yes)	-1.640 (0.249)	-6.60	< .0001	-1.766 (0.276)	-6.39	< .0001	-1.537 (0.236)	-6.51	< .0001
Western (yes)	0.577 (0.436)	1.32	.186	0.416 (0.526)	0.79	.428	*not included*		
Older sib (yes)	0.062 (0.223)	0.28	.780	0.173 (0.259)	0.67	.506	*not included*		
Born in Q1 (yes)	0.012 (0.245)	0.05	.961	0.306 (0.282)	1.08	.279	*not included*		
SES (yes)	0.401 (0.261)	1.53	.125	-0.023 (0.321)	-0.07	.943	*not included*		
NT math score	0.911 (0.156)	5.84	< .0001	0.788 (0.174)	4.52	< .0001	0.573 (0.125)	4.57	< .0001
Flexibility	1.045 (0.358)	2.92	.004	0.869 (0.422)	2.06	.039	1.071 (0.392)	2.73	.006
Significance of total model:				Wald χ^2_8 = 70.25, p < .0001			Wald χ^2_5 = 69.74, p < .0001		

[a]Effects shown for single variables in models with grade as mandatory fixed effect.

Discussion

In this study, our goals were to determine, in relation to various background variables, the characteristics of student expertise in respect of multidigit arithmetic. In this context, having defined an expert as one who made no more than one error on a test of arithmetical competence, our principal focus was on the distinguishing characteristics of routine experts and adaptive experts, where a routine expert is one who never uses shortcut strategies and an adaptive expert is one who regularly uses shortcut strategies. The results, which we discuss in detail below, make a substantial contribution to the collective knowledge of not only the characteristics of routine and adaptive experts, but also why adaptivity should be a goal of mathematics teaching (Hatano & Oura, 2003). Furthermore, as a basis for the development of instructional practice, the study augments the current limited knowledge about the interplay of individual differences and the different forms of expertise. In the following, we briefly discuss the key differences between non-experts and experts, before discussing the more important differences between routine and adaptive experts.

Non-Experts Versus Experts

Not surprisingly, arithmetic experts achieved significantly higher on national tests of achievement than non-experts, a result that accords with earlier findings that basic arithmetic skills predict general mathematics achievement (e.g. Nunes et al., 2012). Far more interesting is the effect size of the difference, with the experts' mean score in national test, on a normalised distribution of mean zero and standard deviation one, being 0.86 higher than that of the non-experts.

It is also worth noting, that the partial and total effect size of mathematical achievement were of similar magnitude, demonstrating strong proximate effect of this predictor even when entered alongside other predictor variables. It is also worth noting that the national test assesses much more than just arithmetical competence, including evaluations of students' algebraic, geometric, measurement and statistical competence.

The fact that the strong effect of mathematical achievement was not weakened when entered alongside other predictors in the full model is likely to be a consequence of SES and ethnicity, both highly significant univariate effects, disappearing in the multi-predictor models. Hence, being an expert in arithmetic and succeeding on the national mathematics test were two sides of the same coin. That said, the correlation between mathematical achievement and SES accords with large-scale international studies like TIMSS (Kjeldsen et al., 2020).

The positive effect of flexibility on the probability of being an expert was not straightforward, not least because the complex models indicated a stronger effect size for Grade 3 when compared to Grades 6 and 8, which the simple models did not. This could be related to the relationship between flexibility and conceptual and procedural understanding (Schneider et al., 2011), whereby students who develop strategies based on a knowledge of numbers and number relations become arithmetic experts earlier than those who develop an algorithmic expertise.

Routine and Adaptive Experts

The most pronounced differences between significant predictors of routine and adaptive experts concern (i) arithmetic flexibility, where adaptive experts have higher levels of flexibility than routine experts, (ii) mathematics achievement, with adaptive experts outperforming routine experts, and (iii) sex, with boys dominating the group of adaptive experts and girls dominating the group of routine experts.

Flexibility

Adaptive experts not only scored higher on flexibility but also used shortcut strategies more frequently during the first two TriFA phases than routine experts. For example, acknowledging that routine experts did not offer any shortcut strategies during Phase 1 of the TriFA assessment, when given a second opportunity during Phase 2 to offer a shortcut strategy, only 15% of all tasks solved by routine experts involved shortcut strategies. By way of contrast, across Phases 1 and 2 adaptive experts solved 68% of all tasks by means of shortcut strategies. Such findings indicate, as with other studies (Hickendorff et al., 2019; Verschaffel et al., 2007), that students who have learned an algorithm tend to stick to that as their preferred strategy. Importantly, of the 68% of the tasks solved by means of shortcut strategies by adaptive experts, 72% were identified as optimal during Phase 3. In comparison, of the 15% of the tasks solved by means of shortcut strategies by routine experts, only 20% were identified as optimal during Phase 3. In other words, routine experts rarely recognize a shortcut strategy as optimal, even when they offer a shortcut strategy during Phase 2. That might be explained by individual skills (subject variable), or cultural context and what is appreciated in the classroom context (Verschaffel et al., 2009).

General Achievement (National Test in Mathematics and Reading)

Previous studies have shown that conceptual and procedural skills, along with their interconnections, are necessary prerequisites for the development of strategy flexibility and adaptivity (Baroody, 2003; Gilmore et al., 2017; McMullen et al., 2020; Schneider et al., 2011). Moreover, conceptual knowledge is an important predictor of mathematics achievement generally (Rittle-Johnson et al., 2001). In light of such matters, it is of little surprise to find that adaptive experts are generally high achieving in mathematics. However, two elements of our results are particularly striking. First, highlighting the likely importance of acquiring adaptive expertise, the mean achievement of adaptive experts was found to be almost a whole standard deviation higher than the mean for all students. While earlier research has shown a causality between measures of intelligence and strategy choice (Luwel et al., 2011), our data do not necessarily support such a conclusion, not least because it is not clear if adaptivity implies achievement or achievement implies adaptivity. Second, and even more striking, is a difference between adaptive and routine experts of almost 0.5 standard deviations in general mathematics achievement. Thus, by way of warranting the acquisition of adaptive expertise as a goal of instruction, while routine experts are mathematically high achievers, adaptive experts are exceptionally high achievers. Furthermore, highlighting the importance of domain specific aspects of achievement like number understanding (Shanley et al., 2017) and strategy choice (Linsen et al., 2015), differences in achievement between routine and adaptive experts are much greater in mathematics than in reading. This indicates that while general intelligence plays a partial role in explaining high achievement in mathematics, the type of expertise plays a more substantial role. Thus, while routine experts may draw on a static and sparsely connected procedural knowledge (Baroody, 2003), without the prerequisites of appropriate conceptual knowledge and number understanding, they will not have the basis for adaptivity (Baroody, 2003; Hatano & Oura, 2003; McMullen et al., 2020).

Sex, Siblings and Quarter of Birth

While no sex difference was found between non−experts and experts, there was a large difference between the two groups of experts, with boys being significantly more adaptive than girls. Previous studies (e.g. Hickendorff, 2018) have also found boys to be more adaptive than girls, but not as strongly as found in our study. For instance, the figures in Table 3 of Hickendorff's (2018) study suggest an approximate odds ratio of 1.5:1, compared with the 4.8:1 indicated above. The underlying reasons for this substantial difference are not clear, although others have suggested a complex combination of biological, psychological and environmental variables (Zhu, 2007). The fact that the effect size of sex did not differ between the univariate and the complex models, suggest that the sex-effect was purely additive and hence not related to any of the other predictors in the analysis, mathematical achievement and flexibility included. Hence, even though male experts were substantially more likely to be adaptive experts than female, this sex-difference in the proportion of experts being adaptive did not predict any overall difference in mathematical achievement score between boys and girls within the expert group. The results indicates that further studies are needed to explain the sex differences. The lack of influence of early births and older siblings may be a consequence of the particular characteristics of Danish public preschools, whereby, before starting school, young children experience, in age-integrated groups, several years of informal non-directive learning opportunities driven by everyday experiences and free-play (Rothuizen & Harbo, 2017). In other words, the Danish preschool, with its explicit emphasis on socialisation rather than instruction, may ameliorate both age-related cognitive differences and the influence of older siblings.

Implications

The results of this study highlight the importance of adaptive expertise, as measured by students' use of shortcut strategies, in underpinning mathematics achievement (Hatano & Oura, 2003). In the particular context of arithmetic, this implies, we posit, a need to encourage the development of students' use of number-based strategies, particularly shortcut, as opposed to the privileging of standard algorithms. Our results further show that although flexibility, defined as knowledge of several strategies, is related to high achievement (Hästö et al., 2019), flexibility in itself is an insufficient predictor. Students who approach tasks adaptively, particularly recognising shortcut strategies as optimal, perform significantly better on tests of general achievement than routine experts. Thus, acknowledging that the existence of adaptivity is dependent on the existence of flexibility, it is imperative that educators encourage both flexibility and adaptivity. Finally, it is important to note that the correlational nature of what is effectively a cross-sectional study precludes any indication of causality, highlighting a need for longitudinal studies of strategy development and deployment.

Conclusion

Having distinguished between expert and non-expert solvers of arithmetical tasks, this study has examined the characteristics of two groups of experts: routine experts who never use shortcut strategies and adaptive experts who use them regularly. Although parental education and ethnicity initially distinguished between experts and non-experts, their effects disappeared when controlling for national mathematics test score and flexibility. Family variables did not vary at all between adaptive and non-adaptive experts. Flexibility, defined as having access to multiple strategies for a given task, appears a strong predictor of both expertise and adaptivity. Routine experts achieve significantly higher on national tests of reading and mathematics than non-experts, and adaptive experts achieve significantly higher than routine experts. However, in comparison with adaptive experts, routine experts rarely recognise a shortcut strategy as optimal, and boys are significantly more likely to be adaptive experts than girls. Moreover, while students in Grade 6 were more likely to be adaptive than students in Grade 3, indicative of a natural developmental progression, they were also more likely to be adaptive than students in Grade 8, indicative of a didactical privileging of standard algorithms.

Funding: We acknowledge financial support from the Independent Research Fund Denmark (9038-00045B and 0127-00022B).

Acknowledgments: The authors have no additional (i.e., non-financial) support to report.

Competing Interests: The authors have declared that no competing interests exist.

Related Versions: This article is a revised version of Chapter 3 of the first author's PhD dissertation. See Jóelsdóttir, L. B. (2023). *Essays on adaptivity and flexibility in multidigit arithmetic* [Doctoral thesis, Aarhus BSS, Aarhus University]. https://pure.au.dk/ws/portalfiles/portal/316116636/Joelsdottir_2023_Essays_on_Adaptivity_and_Flexibility_in_Multidigit_Arithmetic.pdf

Data Availability: The data that support the findings of this study are available upon request.

Supplementary Materials

The Supplementary Materials contain supplementary results, including tables with results by grade level (see Jóelsdóttir et al., 2024S).

Index of Supplementary Materials

Jóelsdóttir, L. B., Sunde, P. B., Sunde, P., & Andrews, P. (2024S). *Supplementary materials to "Routine and adaptive experts: Individual characteristics and their impact on multidigit arithmetic strategy flexibility and mathematics achievement"* [Additional results]. PsychOpen GOLD. https://doi.org/10.23668/psycharchives.15758

References

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 1-33). Lawrence Erlbaum Associates.

Bell, J. F., & Daniels, S. (1990). Are summer-born children disadvantaged? The birthdate effect in education. *Oxford Review of Education, 16*(1), 67–80. https://doi.org/10.1080/0305498900160106

Brown, L., & Coles, A. (2011). Developing expertise: How enactivism re-frames mathematics teacher development. *ZDM, 43*(6-7), 861–873. https://doi.org/10.1007/s11858-011-0343-4

Carr, M., & Jessup, D. L. (1997). Gender differences in first-grade mathematics strategy use: Social and metacognitive influences. *Journal of Educational Psychology, 89*(2), 318–328. https://doi.org/10.1037/0022-0663.89.2.318

Eronen, L., & Kärnä, E. (2018). Students acquiring expertise through student-centered learning in mathematics lessons. *Scandinavian Journal of Educational Research, 62*(5), 682–700. https://doi.org/10.1080/00313831.2017.1306797

Gabay-Egozi, L., Nitsche, N., & Grieger, L. (2022). In their footsteps or shadow? Gender differences in choosing a stem major as a function of sibling configuration and older sibling's gender and math ability. *Sex Roles, 86*(1), 106–126. https://doi.org/10.1007/s11199-021-01255-0

Gilmore, C., Keeble, S., Richardson, S., & Cragg, L. (2017). The interaction of procedural skill, conceptual understanding and working memory in early mathematics achievement. *Journal of Numerical Cognition, 3*(2), 400–416. https://doi.org/10.5964/jnc.v3i2.51

Givord, P. (2020). *How a student's month of birth is linked to performance at school: New evidence from PISA* (OECD Education Working Papers No. 221). OECD Publishing, Paris. https://doi.org/10.1787/19939019

Hästö, P., Palkki, R., Tuomela, D., & Star, J. R. (2019). Relationship between mathematical flexibility and success in national examinations. *European Journal of Science and Mathematics Education, 7*(1), 1–13. https://doi.org/10.30935/scimath/9530

Hatano, G., & Inagaki, K. (1984). Two courses of expertise. *Research & Clinical Center for Child Development, 82-83*(Ann Rpt), 27–36.

Hatano, G., & Oura, Y. (2003). Commentary: Reconceptualizeing school learning using insight from expertise research. *Educational Researcher, 32*(8), 26–29. https://doi.org/10.3102/0013189X032008026

Hickendorff, M. (2018). Dutch sixth graders' use of shortcut strategies in solving multidigit arithmetic problems. *European Journal of Psychology of Education, 33*(4), 577–594. https://doi.org/10.1007/s10212-017-0357-6

Hickendorff, M. (2022). Flexibility and adaptivity in arithmetic strategy use: What children know and what they show. *Journal of Numerical Cognition, 8*(3), 367–381. https://doi.org/10.5964/jnc.7277

Hickendorff, M., Torbeyns, J., & Verschaffel, L. (2019). Multi-digit addition, subtraction, multiplication, and division strategies. In A. Fritz, V. G. Haase, & P. Räsänen (Eds.), *International handbook of mathematical learning difficulties* (pp. 543-560). https://doi.org/10.1007/978-3-319-97148-3_32

Jóelsdóttir, L. B., & Andrews, P. (2024a). Grade six students' multidigit arithmetic strategy adaptivity and flexibility: Evaluating a novel tri-phase assessment tool. *International Journal of Mathematical Education in Science and Technology*. Advance online publication. https://doi.org/10.1080/0020739X.2024.2328341

Jóelsdóttir, L. B., & Andrews, P. (2024b). Danish third, sixth and eighth grade students' strategy adaptivity, strategy flexibility and accuracy when solving multidigit arithmetic tasks. *European Journal of Psychology of Education, 39*(3), 2363–2382. https://doi.org/10.1007/s10212-023-00786-2

Jurow, A. S., Hall, R., & Ma, J. Y. (2008). Expanding the disciplinary expertise of a middle school mathematics classroom: Re-contextualizing student models in conversations with visiting specialists. *Journal of the Learning Sciences, 17*(3), 338–380. https://doi.org/10.1080/10508400802192714

Kalaycioğlu, D. B. (2015). The influence of socioeconomic status, self-efficacy, and anxiety on mathematics achievement in England, Greece, Hong Kong, the Netherlands, Turkey, and the USA. *Educational Sciences: Theory & Practice, 15*(5), 1391–1401. https://doi.org/10.12738/estp.2015.5.2731

Karbownik, K., & Özek, U. (2023). Setting a good example? Examining sibling spillovers in educational achievement using a regression discontinuity design. *Journal of Human Resources, 58*(5), 1567–1607. https://doi.org/10.3368/jhr.58.5.0220-10740R1

Kjeldsen, C. C., Kristensen, R. M., & Christensen, A. A. (2020). *Matematik og natur/teknologi i 4. klasse: Resultater af TIMSS-undersøgelsen 2019*. A. Universitetsforlag.

Lemaire, P., & Callies, S. (2009). Children's strategies in complex arithmetic. *Journal of Experimental Child Psychology, 103*(1), 49–65. https://doi.org/10.1016/j.jecp.2008.09.007

Li, M., Cai, M., Zhong, H., & Liu, H. (2021). Comparisons of academic achievements of one-only children vs. children with siblings in China. *Current Psychology, 40*(11), 5658–5671. https://doi.org/10.1007/s12144-020-01263-5

Linsen, S., Verschaffel, L., Reynvoet, B., & De Smedt, B. (2015). The association between numerical magnitude processing and mental versus algorithmic multi-digit subtraction in children. *Learning and Instruction, 35*, 42–50. https://doi.org/10.1016/j.learninstruc.2014.09.003

Luwel, K., Foustana, A., Papadatos, Y., & Verschaffel, L. (2011). The role of intelligence and feedback in children's strategy competence. *Journal of Experimental Child Psychology, 108*(1), 61–76. https://doi.org/10.1016/j.jecp.2010.06.001

McEwan, P. J., & Shapiro, J. S. (2008). The benefits of delayed primary school enrollment: Discontinuity estimates using exact birth dates. *The Journal of Human Resources, 43*(1), 1–29. https://doi.org/10.3368/jhr.43.1.1

McMullen, J., Brezovszky, B., Rodríguez-Aflecht, G., Pongsakdi, N., Hannula-Sormunen, M. M., & Lehtinen, E. (2016). Adaptive number knowledge: Exploring the foundations of adaptivity with whole-number arithmetic. *Learning and Individual Differences, 47*, 172–181. https://doi.org/10.1016/j.lindif.2016.02.007

McMullen, J., Hannula-Sormunen, M. M., Lehtinen, E., & Siegler, R. S. (2020). Distinguishing adaptive from routine expertise with rational number arithmetic. *Learning and Instruction, 68*, Article 101347. https://doi.org/10.1016/j.learninstruc.2020.101347

Milling Kinard, E., & Reinherz, H. (1986). Birthdate effects on school performance and adjustment: A longitudinal study. *The Journal of Educational Research, 79*(6), 366–372. https://doi.org/10.1080/00220671.1986.10885707

Ministry of Children and Education. (2019). *Matematik. Læseplan*. Retrieved December 15, 2021 from https://emu.dk/sites/default/files/2020-09/GSK_L%C3%A6seplan_Matematik.pdf

Nemeth, L., Werker, K., Arend, J., Vogel, S., & Lipowsky, F. (2019). Interleaved learning in elementary school mathematics: Effects on the flexible and adaptive use of subtraction strategies. *Frontiers in Psychology, 10*, Article 86. https://doi.org/10.3389/fpsyg.2019.00086

Nunes, T., Bryant, P., Barros, R., & Sylva, K. (2012). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology, 82*(1), 136–156. https://doi.org/10.1111/j.2044-8279.2011.02033.x

OECD. (2023). *PISA 2022 results (Volume I): The state of learning and equity in education.* OECD Publishing.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362. https://doi.org/10.1037/0022-0663.93.2.346

Rothuizen, J. J., & Harbo, L. J. (2017). Social pedagogy: An approach without fixed recipes. *International Journal of Social Pedagogy, 6*(1), 6–28. https://doi.org/10.14324/111.444.ijsp.2017.v6.1.002

Schneider, M., Rittle-Johnson, B., & Star, J. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology, 47*(6), 1525–1538. https://doi.org/10.1037/a0024997

Schulz, A. (2018). Relational reasoning about numbers and operations – Foundation for calculation strategy use in multi-digit multiplication and division. *Mathematical Thinking and Learning, 20*(2), 108–141. https://doi.org/10.1080/10986065.2018.1442641

Selter, C. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM, 41*(5), 619–625. https://doi.org/10.1007/s11858-009-0203-7

Shanley, L., Clarke, B., Doabler, C. T., Kurtz-Nelson, E., & Fien, H. (2017). Early number skills gains and mathematics achievement: Intervening to establish successful early mathematics trajectories. *The Journal of Special Education, 51*(3), 177–188. https://doi.org/10.1177/0022466917720455

Shavers, V. L. (2007). Measurement of socioeconomic status in health disparities research. *Journal of the National Medical Association, 99*(9), 1013–1023.

Sievert, H., van den Ham, A.-K., Niedermeyer, I., & Heinze, A. (2019). Effects of mathematics textbooks on the development of primary school children's adaptive expertise in arithmetic. *Learning and Individual Differences, 74*, Article 101716. https://doi.org/10.1016/j.lindif.2019.02.006

Smith, J. (2009). Can regression discontinuity help answer an age-old question in education? The effect of age on elementary and secondary school achievement. *The B.E. Journal of Economic Analysis & Policy, 9*(1), Article 48. https://doi.org/10.2202/1935-1682.2221

Star, J. R., Newton, K., Pollack, C., Kokka, K., Rittle-Johnson, B., & Durkin, K. (2015). Student, teacher, and instructional characteristics related to students' gains in flexibility. *Contemporary Educational Psychology, 41*, 198–208. https://doi.org/10.1016/j.cedpsych.2015.03.001

Sunde, P. B., Sunde, P., & Sayers, J. (2020). Sex differences in mental strategies for single-digit addition in the first years of school. *Educational Psychology, 40*(1), 82–102. https://doi.org/10.1080/01443410.2019.1622652

Threlfall, J. (2009). Strategies and flexibility in mental calculation. *ZDM, 41*(5), 541–555. https://doi.org/10.1007/s11858-009-0195-3

Torbeyns, J., Hickendorff, M., & Verschaffel, L. (2017). The use of number-based versus digit-based strategies on multi-digit subtraction: 9–12-year-olds' strategy use profiles and task performance. *Learning and Individual Differences, 58*, 64–74. https://doi.org/10.1016/j.lindif.2017.07.004

Torbeyns, J., Peters, G., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2018). Subtraction by addition strategy use in children of varying mathematical achievement level: A choice/no-choice study. *Journal of Numerical Cognition, 4*(1), 215–234. https://doi.org/10.5964/jnc.v4i1.77

Verschaffel, L. (2024). Strategy flexibility in mathematics. *ZDM, 56*(1), 115–126. https://doi.org/10.1007/s11858-023-01491-6

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2007). Developing adaptive expertise: A feasible and valuable goal for (elementary) mathematics education? *Ciencias Psicológicas, 1*(1), 27–35. https://doi.org/10.22235/cp.v0i1.571

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*(3), 335–359. https://doi.org/10.1007/BF03174765

Xu, L., Liu, R.-D., Star, J. R., Wang, J., Liu, Y., & Zhen, R. (2017). Measures of potential flexibility and practical flexibility in equation solving. *Frontiers in Psychology, 8*, Article 1368. https://doi.org/10.3389/fpsyg.2017.01368

Zhu, Z. (2007). Gender differences in mathematical problem solving patterns: A review of literature. *International Education Journal, 8*(2), 187–203.

Journal of Numerical Cognition (JNC) is an official journal of the Mathematical Cognition and Learning Society (MCLS).

leibniz-psychology.org

PsychOpen GOLD is a publishing service by Leibniz Institute for Psychology (ZPID), Germany.