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How the Number Line Can Be Used to Promote Students' Understanding of the Normal Distribution

Danri H. Delport
Central University of Technology, Free State, ddelport@cut.ac.za

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How the Number Line Can Be Used to Promote Students' Understanding of the Normal Distribution

Abstract

A strong foundation in early number concepts is crucial for students' future success in statistics. Despite its importance in statistics, many first-year students struggle to comprehend the normal distribution due to a lack of basic number sense. Students get confused about the order and magnitude of negative *z*-scores on a standard normal curve or when problems about normally distributed random variables are presented in word questions which involve phrases that indicate inequalities. As a result, students shade wrong areas on the bell-shaped curve when they have to calculate probabilities for normally distributed variables. Visual representations such as the number line can support students' development of quantitative literacy or number sense by helping them create a mental representation of the order and magnitude of numbers as well as inequalities. Based on a comprehensive investigation of evidence demonstrating this weakness, this experienced-based perspective proposes a framework that demonstrates how the number line can be used as a powerful teaching tool to promote students' conceptual understanding about the normal distribution. The framework illustrates with authentic examples how the number line relates to the horizontal axis of the normal and standard normal curve and how it can be used to address erroneous quantitative reasoning when students are required to calculate probabilities for normally distributed variables. To determine if the number line teaching intervention affects student performance, the researcher adopted a non-equivalent pretest-posttest design with two intact classes. The results of the analysis show that students who received the intervention performed significantly better in the posttest.

Keywords

quantitative literacy, number sense, number line, the normal distribution, probabilities, bell-shaped curve

Cover Page Footnote

Danri H. Delport is a Senior Lecturer in the Department of Mathematical and Physical Sciences at Central University of Technology, Free State.

Introduction

Introductory statistical concepts are some of the most challenging to convey in quantitative literacy courses (Ancker & Begg 2017). When the transition is made from data analysis to inferential statistics in introductory statistics courses at tertiary institutions, many first-year students encounter problems when it comes to reasoning about the normal distribution (Batanero et al. 2004). Probabilities of events occurring are often presented in word questions. Students particularly experience difficulty translating these words into the correct probability notation which often involves the use of letters and inequality symbols. As a result, wrong areas are shaded on the bell-shaped curve when students have to calculate the probability of an event occurring. For many students, word problems appear to be more problematic than other forms of mathematical competence because they require students to first decipher the text describing the problem situation and then derive the number sentence representing the situation (Fuchs et al. 2015, 204). Furthermore, one cannot decipher text without adequate reading skills (Gomez et al. 2020, 1329). Research shows that only 36% of learners in South Africa are achieving the reading and numeracy outcomes expected of a grade 3 learner while the vast majority are performing two to three years below expectation. Even children in the first years of primary school are performing far below the expected grade levels (Spaull and Kotze 2015). Furthermore, a 2015 study conducted by Trends in International Mathematics and Science (TIMMS) revealed that 65% of grade 5 learners in South Africa cannot add or subtract whole numbers (Shay 2020). According to Suellen Shay, from the University of Cape Town, only 54% of pupils who wrote their matric exam in 2019 passed it (Shay 2020). The minimum score for a pass rate in South Africa is only 30%, meaning that only 54% of candidates achieved a mark of at least 30% in their mathematics exam. This finding is of great concern as performance in mathematics matters for university entrance. The decline of quantitative literacy skills among students that enter university is not only a concern in South Africa, but worldwide.

The ability to perform these tasks has to do with a set of early numerical competencies which is often referred to as "number sense or early numeracy competencies" (Lembke and Foegen 2009; Lloyd et al. 2009; Powell and Fuchs 2012; Woods et al. 2018). The learning of number sense is underpinned by the element of quantifying numbers, meaning that "a student becomes increasingly able to count, recognise, read and interpret numbers expressed in different ways" (Australian Curriculum, Assessment and Reporting Authority n.d.). Currently, little research is available that focuses on students' quantitative thinking when reasoning about the normal distribution. According to Batanero et al. (2004, 274), "the normal distribution is a very complex idea that requires the integration and relation of many different statistical concepts and ideas." Furthermore, conceptual understanding and fluency are considered core foundations of any introductory statistics course at the university level. Once students become fluent in the fundamentals of numeracy through varied and repeated practice with increasingly complex problems over time, they will gain conceptual understanding as well as the capacity to recall and apply knowledge quickly and accurately (McClure 2014). Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which is it useful. In the context of the normal distribution, students need to be able to read and shade an inequality statement and understand what it means in terms of the numbers around it on the normal and standard normal curve. It is therefore imperative that teachers design learning activities to facilitate number sense and conceptual understanding about the normal distribution.

The number line is one of the most important models that can be used as a powerful visual tool to ensure a relatively seamless transition from school maths to undergraduate university introductory statistics courses. Despite the fact that number line estimation appears to be a basic and specific skill that is seldom used in everyday life, research have found correlations between task performance and a wide variety of other, more complicated and advanced mathematical ability measures (Siegler 2016). These measures include counting (Östergren and Träff 2013) and arithmetic tasks (Torbeyns et al. 2015), as well as standardised school achievement tests (Ashcraft and Moore 2012). Currently, there is a lack of research showing how the number line can be used as a powerful tool to promote students' understanding about the normal distribution. The horizontal axis of the number line provides an excellent example, particularly for combining inequalities and negative numbers, because the normal distribution involves the graphical presentation of finding probabilities under the normal curve by locating *z*-scores on the bell-shaped curve and shade areas on the graph. Visual representations such as the number line "support students' development of number sense by helping them create a mental representation of the order and magnitude of numbers" (Woods et al. 2018, 229). Furthermore, this transition from "concrete to visual to abstract representations" may supports students' conceptual understanding about shading areas under the normal and standard normal curve (Woods et al. 2018, 229).

This research is important because misunderstandings about the normal distribution among students are high and ongoing, which in turn, affect the ability of students to understand further concepts with regard to inferential statistics. In the first section, I review some evidence showing that number sense is a weakness among students that needs attention. Thereafter, I present the framework that I use to explicitly teach students how to grasp probabilities for normally distributed data by first understanding order, magnitude, and inequalities on the number line. Finally, I use the proposed framework to see whether it has a positive effect on students' test performance on the normal distribution. The activities in this article will help to develop students' understanding of the normal distribution and how it relates to the number line. This in turn may provide students with a deeper understanding when they are required to calculate probabilities for normally distributed random variables.

Evidence of the Problem

Although numerous educators agree that first-year students struggle with the normal distribution, few studies are available that tap into students' understanding of this concept. Earlier studies examined a few isolated aspects in the understanding of the normal distribution, which include the concept of symmetry (Piaget and Inhelder 1951), as well as students' misconceptions about *z*-scores (Huck et al. 1986)—students believe that all standard scores will always range between -3 and +3. From my own experience, a few students fail to take into account the properties of the normal distribution when they are required to calculate probabilities for random normal variables. However, the majority of students have misconceptions about the order and magnitude of negative *z*-scores on the horizontal axis of the standard normal curve. Students also find it difficult to comprehend letters and inequality symbols in probability notation and use these incorrectly when calculating probabilities of events that are presented in word questions. These misconceptions have to do with a lack of basic number sense, as discussed later.

Order, Magnitude and Decimal Z-scores

Positive and negative numbers both have order and magnitude. Although students have no problem in locating positive integers on a number line, negative *z*-scores often cause confusion among students with regard to magnitude and order (Fig. 1).

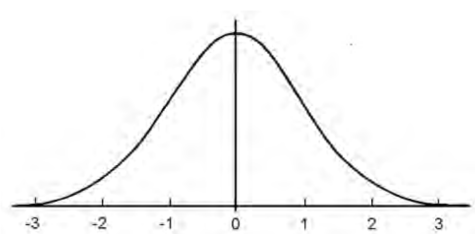


Figure 1. The standard normal curve

For example, because 3 is more than 2 on the number line ($3 > 2$), many students think that -3 is also greater than -2. However, $-3 < -2$. This concept is confusing for many students since 3 appears to have a magnitude of more than 2 (Bofferding 2014). Furthermore, *z*-scores are often in the form of decimal numbers such as -1.22 or -2.58. Many students often face problems in locating such *z*-scores on the horizontal axis of the standard normal curve and incorrectly plot -1.22 to the left of -2.58. In fact, research shows that students often struggle with the placing of decimals on a number line (Akhtar 2018). Schneider et al. (2018, 8) argue that "learning about numerical magnitudes and their interrelations is a central component of mathematical learning and competence tests during pre-school years, but gets progressively less central in instruction and competence tests for older children". Students' misconceptions about order and magnitude, in tertiary educational settings, clearly reflects this statement.

Letters, Signs, and Inequalities

Students are often required to find probabilities that an observation from normally distributed data will fall within a specific interval, or more or less than a specific value. These intervals are indicated with letters x_1 and x_2 when dealing with normally distributed variables, and z_1 and z_2 when raw x scores are transformed into standardised z-scores. To find the probability that a normal random variable is within any specific interval (or more, or less), the area under the normal curve over that interval should be computed (Williams et al. 2012, 261). These inequalities are often expressed in words and involve phrases like "more than," "less than," "smaller than," "greater than," or "between" certain data values. These words can be expressed by making use of inequality symbols such as >, <, or < between <. Students often experience confusion with the use of such inequality symbols (Rowntree 2009).

In my own statistics class, I have seen on numerous occasions how students incorrectly write "between" in probability notation, for example, the probability for a random variable to be between say 30 and 40 should be written as $P(30 < x < 40)$. However, I have seen cases where students write it as $P(30 > x > 40)$, or $P(30 > x < 40)$, or even omitting one of the equality signs $P(30 < x\ 40)$. A thorough understanding about the use of such symbols is ingrained in basic number sense. Students need to be able to read and shade an inequality statement and understand what it means in terms of the numbers around it on the normal and standard normal curve. As such it is crucial for students to be familiar with inequality expressions and direction when it comes to shading areas under these curves.

Probability within a Given Interval. If a set of data has a normal distribution (X), the z-table is used to find the probability that an event will occur within a defined set of parameters. Students should understand that the probability of observations that takes on a value between x_1 and x_2 is the area under the normal curve between these two points. If the endpoints of the interval lie on the opposite sides of the mean, areas are added (meaning a negative z-score and positive z-score on the standard normal curve), while if the two endpoints lie on the same side of the mean, the smaller area is subtracted from the bigger area (meaning both negative z-scores or both positive *z*-scores). It is often at this specific step that students do not know whether they should add or subtract probabilities. For example:

$P(1.25 < z < 2.58)$ is calculated by students as 0.3944 + 0.49506 instead of 0.49506 – 0.3944,

$P(-2.58 < z < -1.25)$ is calculated as 0.3944 + 0.49506 instead of 0.49506 – 0.3944, and

$P(-2.58 < z < +1.25)$ is calculated as 0.49506 – 0.3944 instead of 0.49506 + 0.3944.

Students make these mistakes because they struggle with the concept of calculating "areas" (probability) between two points, whether these points are both positive, both negative, or negative and positive. According to Makonye and Fakude (2016, 1), learners seem obsessed with positive numbers and addition operation frames. In fact, as the authors state "they could not easily accommodate negative numbers or the subtraction operation involving negative numbers." The area between such two points can be compared with finding distances between two points on a number line. However, students fail to make that connection and as such, are unable to understand this concept. In a few cases, students simply just look up z-scores from a table without completing the rest of the calculation. For example:

$P(1.25 < z < 2.58)$ = 0.3944 and 0.49506.

Probabilities Greater or Less Than a Given Value. Students are often required to find the probability that a normal random variable is more or less than a certain value, for example $P(x > 40)$ or $P(z < 2)$. Many students are familiar with comparing 2 numbers, for example 3 > 2, but when one of these numbers is replaced with a letter such as *x* or *z*, students get confused. In fact, when letters are present in algebraic entities, this is seemingly difficult for students. Studies also indicate that sometimes these letters, such as the variables *x* and *z*, are simply ignored or replaced with numerical values (Kuchemann 1981; Samo 2009). A thorough understanding about the use of such symbols is ingrained in basic number sense. This has implications for answering questions about normally distributed variables.

Another common error among students is the use of inequality signs in probability notation and whether to add probabilities with 0.5 or subtract probabilities from 0.5. For example, when students are required to find $P(z > -1.25)$ they are able to look up 1.25 correctly from the standard normal distribution table (0.3944). However, they do not know whether they should subtract 0.3944 from 0.5

or add 0.3944 with 0.5. Further, the > sign in front of the negative number -1.25 is difficult for students to comprehend and they do not know what it means in terms of shading the correct area under the standard normal curve in order to determine whether they should add or subtract 0.3944 with or from 0.5. This again demonstrates students' weakness with regard to number sense. In fact, research points out that students have difficulties in understanding negative numbers and the operation associated to it (Peled and Carraher 2008). Because -1.25 is a negative number, the inequality sign (more than >) is confusing for some students. Students often reason that (less than <) is associated with the negative side of the number line, and (more than >) is associated with the positive side of the number line. When students see greater than signs before negative numbers in particular, they become totally confused. The order and magnitude of especially negative numbers also plays a role in this regard.

When students do not know how the inequality symbols relate to the shading of an area of interest on the bell-shaped curve, they will calculate probabilities incorrectly and make wrong conclusions. When students are required to sketch $P(z > -1.25)$ on a standard normal curve, the area to the right (>) of -1.25 should be shaded.

The examples of misunderstanding that students have about the normal distribution clearly show how critical it is for students to have a strong foundation in early numeracy concepts to be able to calculate areas under the standard normal curve. The next section focuses on the proposed framework for teaching the normal distribution. This includes the properties of the normal distribution, how the number line relates to the normal and standard normal curve, and how it can be used to promote understanding about the normal distribution.

Framework for Teaching the Normal Distribution

What Is the Normal Distribution?

According to Walpole (1982, 184), "the normal distribution is often referred to as the Gaussian distribution" in honour of the famous German mathematician Karl Gauss (1777–1855). The normal probability distribution is the most important probability distribution for describing continuous random variables, for example, the heights and weights of people, blood pressure, cholesterol levels, IQs, metabolic rates, scientific measurements, test scores and other similar values (Williams et al. 2012, 257). The graph of the normal distribution is called the "normal curve," which is bell-shaped, and "describes many sets of data that occur in nature, industry, and research" (Walpole 1982, 184). The normal distribution shows up widely in statistics as a result of the Central Limit Theorem, which argues that "even if individual values come from a very non-normal population, the sample means tend to have a normal distribution" (Croucher 2013, 467). However, "it should be noted that there is virtually no naturally occurring system that follows an exact normal distribution (Croucher 2013, 450). As such, the normal distribution can often provide a sufficiently accurate approximation to enable accurate conclusions to be drawn. For this reason, a set of observations may be said to have an *approximate* normal distribution." The normal distribution can therefore be seen as a mathematical model that tries to resemble reality.

One of the requirements of students is to describe the normal curve in general terms, and use its properties to answer questions about data sets that are assumed to be normally distributed. When I introduce the normal distribution to my students, I provide them with a clear guideline of what is expected from them with regard to the normal distribution. Students should be able to:

- identify the properties of the normal distribution and normal curve,
- identify the characteristics of the standard normal curve,
- understand examples of normally distributed data,
- read z-scores from tables and find areas under the normal curve, and
- understand that the probability of observations that take on a value between x_1 and x_2 is the area under the curve between the two points.

The normal distribution is a probability function that describes how the values of a variable are distributed. When I introduce my students to the normal distribution, I make a graphical representation of the normal distribution (Fig. 2), and explain its properties to my students as such (Wegner, 2010, 141):

- The curve of the normal probability distribution is bell-shaped.
- The curve is symmetrical about a central value, the population mean μ.
- The tails of the bell-shaped curve are asymptotic.
- The distribution is described by two parameters, the population mean (μ) and the population standard deviation (σ).
- The probability of a value x between two points x_1 and x_2 occurring, is the corresponding area under the curve, between the two end points in question.
- The area under the curve of the normal distribution represents probability and the total area under the curve sums to one. However, because of symmetry, the area above the mean is 0.5 and the area below the mean is 0.5 (50%)
- The empirical rule is used to calculate the probability of randomly obtaining a score from a normal distribution (McLeod 2019). When data values for a normal distribution are converted to standard scores (z-scores) the empirical rule describes the percentage of the data that fall within specific numbers of standard deviations (σ) from the mean (μ) on a standard normal curve. The empirical rule states that:
    - 68% of the data falls within 1 standard deviation of the mean,
    - 95% of the data falls within 2 standard deviations of the mean, and
    - 99.7% of the data falls within 3 standard deviations of the mean.

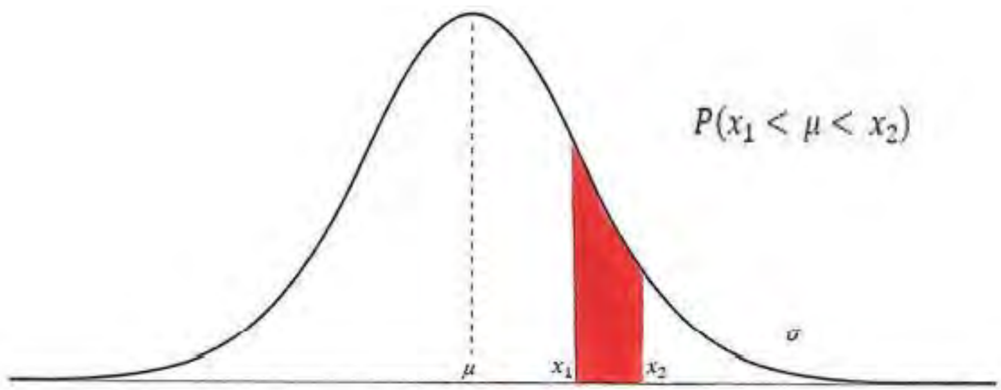


Figure 2. A graphical representation of the normal distribution.

Standardising *X* Scores. One important learning goal of the normal distribution relates to being able to transform raw scores from the normal distribution into standardised scores.

$$z = \frac{x - \mu}{\sigma}$$

Where z is the standard score:

- x represents the score of the measurement of interest,
- μ is the population mean, and
- σ is the population standard deviation.

In this process of converting a raw score to a z-score, the mean of a standardised variable becomes zero and its standard deviation becomes one. The standard normal curve is the corresponding curve for this special distribution. Many students understand this important step and are quite capable of converting raw scores to standardised z-scores.

Finding Probabilities under the Standard Normal Distribution Curve. One of the tricky parts about the normal distribution is the idea of the probability being the area under the bell-shaped curve. As the normal distribution is a probability distribution, the probability of the area that falls under this curve between two points on a probability distribution plot indicates the probability that a certain value will fall within this specific interval (Fig. 2). By looking up z-scores in a Standard Normal Distribution Table, areas can then be calculated. To calculate these probabilities, it is imperative for students to use the values in such tables *in conjunction* with the properties of the normal distribution (see above). Many students look up values from the z-table but fail to take into account the properties of the normal distribution. Although several forms of statistical tables exist for this curve, I made use of the standard statistical table for the standard normal distribution in Wegner (2010, 472), which is also used for our introductory statistics course at the Central University of Technology (CUT).

How the Number Line Can Be Used to Promote Understanding about the Normal Distribution

Students find it relatively easy to look up the values from the *z*-table. However, when it comes to drawing the normal curve and shading the area (probability) of interest they often experience difficulty. Before I teach my students to find probabilities for normal random variables (*x*) or standardised scores (*z*), I draw a number line with negative and positive numbers on the white board (Fig. 3).

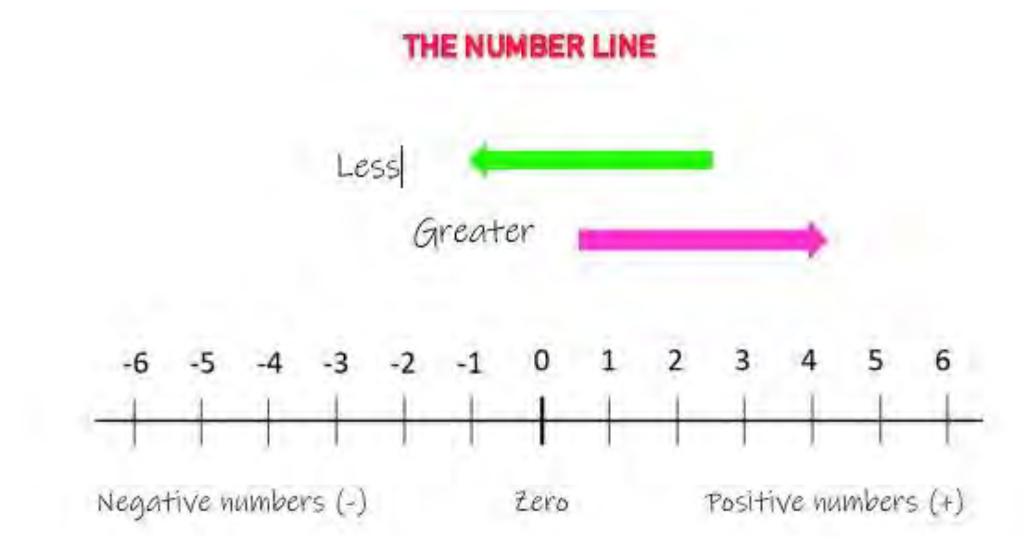


Figure 3. A graphical representation of the number line.

Order, Magnitude, and Signs on a Number Line. What I really like about the number line is that it is a simple model that can be used to explain the concept of symmetry about the normal distribution, as well as to represent relationships between numbers. I first use examples to make sure that students understand order and magnitude on the number line before we progress to the normal distribution itself. For example,

- $-10 < -9$ and not $-10 > -9$
- $-5 > -7$ and not $-5 < -7$
- $4 < 5 < 6$ and not $4 > 5 > 6$
- $-3 < -2 < -1$ and not $-3 > -2 > -1$

I also emphasize that the correct sign must accompany the correct word, when probability problems are presented in words. For example:

- <, < means between, within
- < is for strict inequalities such as: less than, lower, smaller
    - non-strict inequalities such as: not more than, at most
- > is for strict inequalities such as: more than, higher, greater
    - non-strict inequalities such as: not less than, at least

The Relationship between the Number Line and the Standard Normal Distribution. Only after students are familiar with certain concepts on the number line, I explain to them how the number line relates to the horizontal axis of the standard normal curve (Fig. 4).

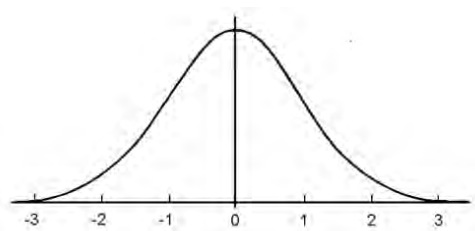
The Number Line	The Standard Normal Distribution
 • Numbers are indicated with equally spaced intervals. • The line is centred at the origin, or zero. • Positive numbers increase from 0 to the right direction, while negative numbers increase from 0 to the left direction. • The number line is infinite in length as well as in mutually opposite directions. • All distances are measured from the origin, 0.	 • *Z*-scores are indicated with equally spaced intervals on the horizontal scale of the standard normal distribution. • The mean is 0 and centred in the middle. • If the data value lies above the mean, the *z*-score will be positive, and if the *z*-score falls below the mean, the *z*-score will be negative. • The tails of the bell-shaped curve of the normal distribution are asymptotic. • *Areas* (probabilities) are measured from 0 under the standard normal distribution (see Appendix). • The probability for each half of the bell-shaped curve is 0.5.

Figure 4. How the number line relates to the horizontal axis of the standard normal distribution.

How the Number Line Can Be Used in the Computation of Probabilities. I usually use examples from my textbook to demonstrate to students how questions are asked about the normal distribution. The following is such an example:

> Example 1: A sales representative for a cosmetics company has a regular customer to whom she must make deliveries every Monday morning. To ease the boredom of the drive

each week, she records the time that it takes her to travel the distance. She finds that these times follow an approximate normal distribution with a mean of 20 minutes and a standard deviation of 4 minutes. Find the probability of mornings in which the time taken for the drive is between 16 and 18 minutes.

Explanation: After reading the problem, students should write the question in probability notation and indicate the mean and standard deviation:

$P(16 < x < 18)$, $(\mu = 20, \sigma = 4)$.

Hereafter, we sketch a normal bell-shaped curve and indicate the mean in the middle of the curve (μ = 20). We plot x_1 = 16 and x_2 = 18 also on the curve and shade the area on the graph between 16 and 18 (Fig. 5).

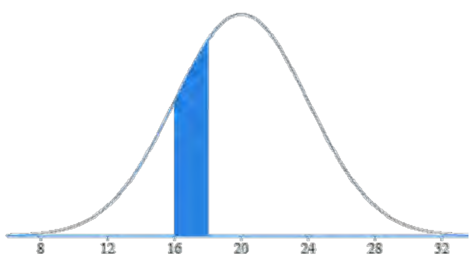


Figure 5. $P(16 < x < 18)$

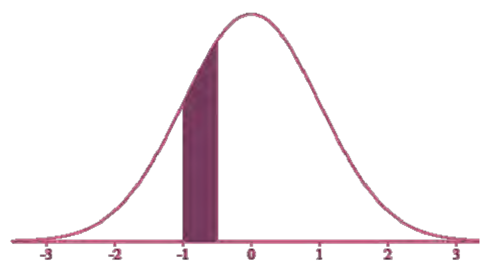


Figure 6. $P(-1 < z < -0.5)$

We then transform both x-limits into the corresponding z-limits by making use of the standard score formula. Because 2 limits are given, we do this calculation twice, first for 16 and then 18:

$$z_1 = \frac{x_1 - \mu}{\sigma} \qquad z_1 = \frac{16 - 20}{4} = -1$$

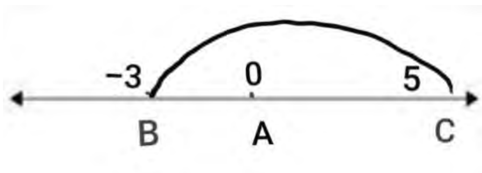
$$z_2 = \frac{x_2 - \mu}{\sigma} \qquad z_2 = \frac{18 - 20}{4} = -0.5$$

We rewrite the question in probability notation for the standardised scores, so the problem in probability notation becomes $P(-1 < z < -0.5)$. Hereafter we plot these z-scores (-1 and -0.5) on the horizontal axis of the standard normal curve and shade the region on the graph between -1 and -0.5 (Fig. 6). We read up both z-scores from the z-table in the Appendix and write them down.

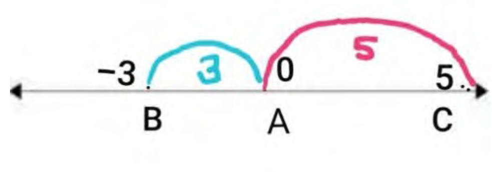
$z = 1$ is 0.3413 (ignore the negative in front of 1)
$z = 0.5$ is 0.1915 (ignore the negative in front of 0.5)

It is at this step that students struggle to comprehend whether they should add these two probabilities or subtract them from each other. I explain this concept by illustrating to my students how to calculate the distance between two points on a number line when one number is negative and one is positive, when both numbers are negative, or when both numbers are positive.

Example 2: What is the distance between -3 and +5? (B and C)

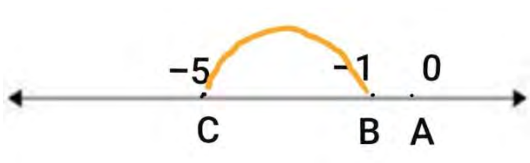


The distance between two points A and B is 3 units, and the distance between A and C is 5 units. Therefore, the distance between point B and C is the distance from A to B *plus* the distance from A to C, 3 + 5 = 8 units.



The same concept can be applied when working out the area between a negative and positive *z*-score under the standard normal curve. Recall that *distances* are presented on a number line, whereas *areas* are calculated under the standard normal curve. Furthermore, *z*-scores have to be looked up first in the Appendix table in order to find probabilities. The example above illustrates the use of whole numbers on a number line, while areas are in decimal form. However, the same concept applies. The reason why whole numbers are used in the illustration is to promote conceptual understanding.

Example 3: What is the distance between -5 and -1? (B and C)



The distance between A and B is 1 unit. The distance between A and C is 5 units. Therefore, the distance between B and C is the *difference* between the distance from A to C and the distance from A to B, 5 – 1 = 4 units.

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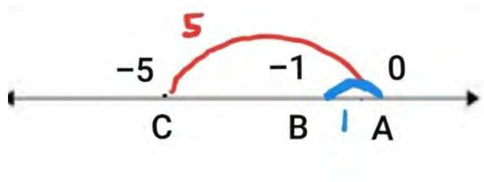
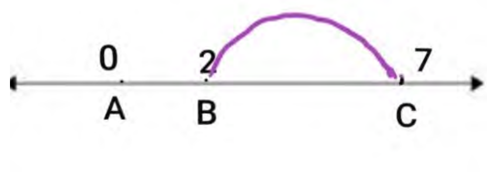
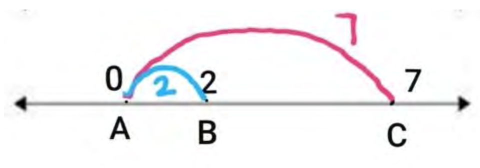


Example 4: What is the distance between 2 and 7? (B and C)



The distance between A and B is 2 units. The distance between A and C is 7 units. The distance between B and C can be found by *subtracting* the distance between A and B from the distance between A and C, 7 – 2 = 5.



Once students are familiar with calculating distances on a number line, they can continue to calculate *probabilities* within a given interval. It should be made clear to students that distances and areas are not the same, and that distances are only used for illustrative purposes.

We continue then with Example 1 and subtract the smaller probability from the bigger one [compare with example 3].

$$P(-1 < z < -0.5) = P(-1 < z < 0) - P(-0.5 < z < 0)$$
$$= 0.3413 - 0.1915$$
$$= 0.1498$$

Finally, we interpret our answer: On approximately 15% of mornings, the time taken to travel the distance is between 16 and 18 minutes.

Explaining Probabilities Greater or Less Than a Given Value. The Appendix table can also be used to determine what probability of a normal population is greater than a certain value or less than a certain value. The raw score should be converted first into a *z*-score. To determine the probability of a normal distribution *greater* than a value of x, calculate the z-score corresponding to x and find the area to the *right* of this score (sketch the region on the bell-shaped curve). If z is negative, add the area in the Appendix table with 0.5. If z is positive, subtract the area in the table from 0.5 (see properties of the normal distribution).

> Example 5: Use the data in Example 1 to calculate the probability of mornings in which the time taken for the drive exceeds 18 minutes.

Explanation: The word "exceeds" clearly indicates (more >). As such, the question in probability notation is $P(x > 18)$, ($\mu = 20$, $\sigma = 4$).

When we draw a normal bell-shaped curve, the mean ($\mu = 20$) is in the middle of the curve, while 18 will be on the left side of 20 on the curve. We shade the region on the graph more than (>) 18 (Fig. 7).

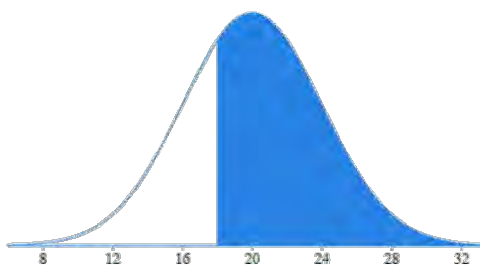


Figure 7. $P(x > 18)$

Figure 8. $P(z > -0.5)$

We transform $x = 18$ into the corresponding z-limit by making use of the standard score formula:

$$z = \frac{x - \mu}{\sigma} \qquad z = \frac{18 - 20}{4} = -0.5.$$

The problem in probability notation becomes $P(z > -0.5)$.

We then plot the z-score -0.5 on the horizontal axis of the standard normal curve and shade the region on the graph that is (>) -0.5 (Fig. 8). A z-score of 0.5 has a value of 0.1915 from the Appendix table. Before we go any further, I explain this concept to my students with the following example on a number line.

> Example 6: Ask students to Plot $x > -2$ on the number line.

The greater than sign (>) indicates that the values must be more than -2. On a number line, all numbers greater than -2 should be included. Plot the number -2 and shade everything to the right of -2 on the number line. Since there is no upper bound (>), the arrow starts at -2 and points to the right of -2.

$x > -2$ (read as x is more than -2)

I then ask my students how can we work out the distance between -2 and 4 on this number line? We know the distance between 0 and -2 is 2 units, and the distance between 0 and 4 is 4 units. So the total units is 2 + 4 = 6. The same concept applies when working out the area under the standard normal curve. The distance between 0 and -2 on this number line can be compared with the area between 0 and -0.5 under the standard normal curve (Fig. 8). On the number line the distance is 2 units. However, on a standard normal curve the z value of 0.5 needs to be looked up in the Appendix table to calculate the required area. The distance on this number line between 0 and 4 can be compared with the area between 0 and the right tail of the standard normal curve. On this number line the distance is 4. However, on the standard normal curve this area is 0.5.

After students understand this concept they are ready to proceed to Example 5:

$P(z > -0.5) = 0.5 + P(-0.5 < z < 0)$ [the area of the right half of the curve is 0.5]
$= 0.5 + 0.1915$
$= 0.6915$

Interpretation: On approximately 69% of mornings, the time taken to travel exceeds 18 minutes.

Example 5 clearly shows how probabilities are added even when a z-score is negative and on the negative side of the horizontal axis of the standard normal curve. To determine the probability of a normal distribution *less* than a value of x, calculate the z-score corresponding to x and find the area to the *left* of this score. If z is negative subtract the area in the z-table from 0.5. If z is positive add the area in the Appendix table with 0.5.

> Example 7: Use the data in Example 4 to calculate the probability of mornings in which the time taken for the drive is less than 19 minutes.

Explanation: The question in Example 7 contains the word "less." As such, the question in probability notation is $P(x < 19)$, ($\mu = 20$, $\sigma = 4$).

When we sketch a normal bell-shaped curve and indicate the mean in the middle of the curve ($\mu = 20$), the number 19 is on the left side of 20 on the curve and we shade the region on the graph less than (<) 19 (Fig. 9).

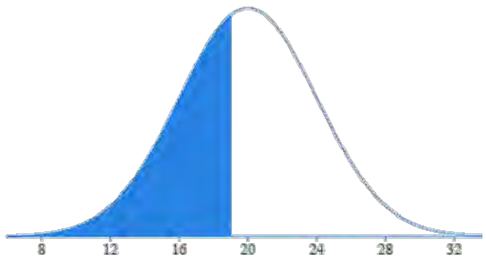


Figure 9. $P(x < 19)$

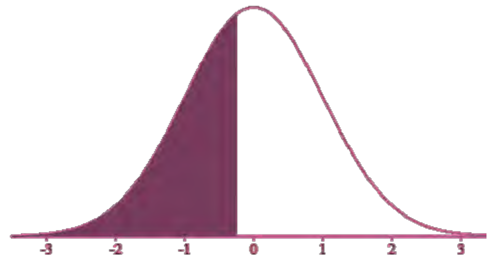


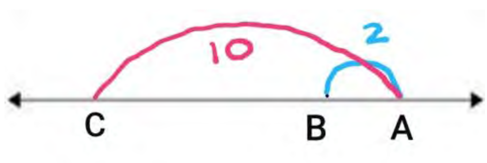
Figure 10. $P(z < -0.25)$

We then transform the x-limit of 19 into the corresponding z-limit by making use of the standard score formula:

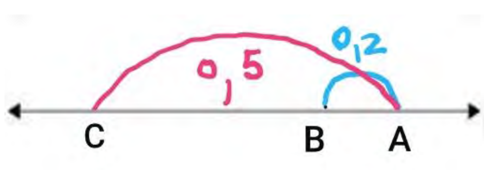
$$z = \frac{x - \mu}{\sigma} \qquad\qquad z = \frac{19 - 20}{4} = -0.25.$$

The standardised score is -0.25 and the problem in probability notation is $P(z < -0.25)$. We plot the z-score -0.25 on the horizontal axis of the standard normal curve and shade the region on the graph that is less than (<) -0.25 (Fig. 10). We look up the z-score of 0.25 (ignore the minus sign) from the Appendix table, which is 0.0987. I compare this probability with the following number line.

Suppose the distance between two points, A and C is 10 units. The distance between A and B is 2 units. What is the distance between Points B and C? Students find this example very easy and can calculate the distance between B and C as 10 – 2 = 8 units.



After students understand this concept by using integers, I demonstrate another example with decimals. For example, if the distance between A and C is 0.5 units, and the distance between A and B is 0.2 units, what is the distance between B and C? The answer is 0.5 – 0.2 = 0.3.



DOI: https://doi.org/10.5038/1936-4660.15.2.1403

After students understand this concept they are ready to proceed to Example 7:

$P(z < -0.25) = 0.5 - P(-0.25 < z < 0)$
[the area of the left half of the curve is 0.5]
$= 0.5 - 0.0987$
$= 0.4013$

Interpretation: On approximately 40% of mornings, the time taken to travel is less than 19 minutes.

To determine whether the number line teaching intervention has any impact on students' academic performance, I decided to analyse students' academic performance with regard to the normal distribution.

Research Methodology

Research Design

A quasi-experimental quantitative research technique was used to determine if the post-test performance of students exposed to the number line teaching intervention differed from students who only received traditional instruction (Leedy and Ormrod 2001, 236). To establish plausible causation, a non-equivalent pre-test/post-test design with two intact classes was adopted (McMillan and Schumacher 2006, 273). The pre-test was used to compare both groups of students at baseline before the number line teaching intervention.

Population

Two groups of students at the CUT participated in this study. The first group of students were enrolled for a Business Statistics module at the main campus of the CUT, which is situated in Bloemfontein (BFN). The second group of students were enrolled for the same module at the CUT, but were situated on the satellite campus of the CUT in Welkom. The two towns are 150 km apart. As it was impossible to make use of a random sample selection strategy, a non-probability sampling method was employed. Both groups of students were African and were in the age range between 20 and 23. The experimental group consisted of 95 students from Bloemfontein who were taught the suggested number line teaching technique. Students from Welkom (n=59) served as the control group and were given traditional instruction.

Data Collection

To assess the impact of the proposed number line teaching intervention on students' academic performance in the statistics module, quantitative data was collected using two self-developed instruments designed to produce highly reliable and accurate scores. The researcher relied on numerical data (test scores) to determine if the experimental group's average post-test score in a business statistics module differed from the control group's average post-test score. The researcher acquired quantitative data from students' pre-test and post-test scores during the second semester of 2021. The pre-test was a practice test on the normal distribution chapter, which was given to students in both groups in October before the number line teaching intervention. The items on the test focused only on the normal distribution and were derived from the textbook that students were required to use. The pre-test questions are presented here:

> A survey indicates that for each trip to the supermarket, a shopper spends an average of 45 minutes with a standard deviation of 12 minutes in the store. The length of time spent in the store is normally distributed and is presented by the variable *x*. A shopper enters the store. Find the probability that the shopper will be in the store:
>
> Between 24 and 54 minutes?
> More than 39 minutes?
> Less than 54 minutes?
> Less than 39 minutes?
> More than 54 minutes?

Students were also told that their results for the practice test will not contribute toward their course mark in the statistics module. After the pre-test, the Bloemfontein students (*n*=95) received explanations of the normal distribution with the number line examples, while the Welkom students (*n*=59) only received traditional instruction from the textbook. After the intervention, both groups of students wrote the main test (post-test questions) about the normal distribution:

> A radar unit is used to measure speeds of cars on a motorway. The speeds are normally distributed with a mean of 90 km/hr and a standard deviation of 10 km/hr. What is the probability that a car picked at random is traveling:
>
> More than 100 km/hr
> Less than 120 km/hr
> Between 80 and 100 km/hr
> Less than 70 km/hr
> Between 100 and 120 km/hr

Both the pre-test and post-test comprised of only five questions about the normal distribution. The question of interest was to see whether students who received additional explanations with the number line improved more in the post-test than students who received traditional instruction only. If students in the experimental group did significantly better in the post-test than the control group, it can be argued that the number line teaching intervention may have contributed to students' understanding about the normal distribution.

Validity and Reliability

Both groups of students studied online, received identical learning material at the start of the module, utilised the same textbook, and completed the same tests online on the same day and time. The researcher was the lecturer for the Bloemfontein group of students, while another lecturer, which is also situated at the CUT, was appointed to facilitate the subject online for the Welkom group of students in this semester subject. The researcher developed all the study material as well as instructional videos for each chapter in this statistics module. The researcher posted her videos on her own WhatsApp group which students from the Bloemfontein campus have joined, while the Welkom lecturer used exactly the same videos to share with his students on his own WhatsApp group. Both lecturers communicated with their students on WhatsApp, without any face-to-face classes.

Although Welkom had a different lecturer, which could be a source of variation, the internal validity of this study increased as both groups of students received identical learning material as well as instructional videos. In instances where the Welkom group of students had certain questions, the Welkom lecturer first consulted with the Bloemfontein lecturer for clarity.

The researcher developed a pre-test and a post-test based on the normal distribution chapter. In this study, content validity was established by including only questions from the required textbook that are relevant to the normal distribution topic area. Both assessments' items adequately reflected the content domain on which students were assessed. The content validity of the items in both tests was improved by having another statistics lecturer assess them for clarity and completeness (Bell 2005, 118). The Cronbach's alpha reliability coefficients for the pre-test (0.718) and post-test (0.812) were both adequate, and the SPSS statistical software program (version 27) was used to analyse the data.

The extent to which the research findings are generalisable is a source of concern for the external validity of this study (Saunders et al. 2003, 102). Because this study was done at a single University of Technology (the CUT) and no randomisation was utilised, the findings may not be applicable to all students at other universities (Saunders et al. 2003, 102). Students were not informed that their test results would be utilised for data analysis in order to maximize the internal validity of this study. The researcher believes that if students realised they were taking part in this study, the Hawthorne effect would have lowered the study's validity. As McMillan and Schumacher (2006, 143) purport "some educational research is quite unobtrusive and has no risk for the subjects."

Data Analysis

The purpose of the assessment was to investigate the effects of a teaching technique on students' academic performance in a statistics-related course at CUT. The number line teaching intervention was specified as the independent variable, while the pre- and post-test scores were the dependent variables. To investigate whether there was any difference between the pre-test and post-test performance *within* each group of students, the researcher conducted the paired samples *t*-test (in SPSS) for each of the two groups of students. This test compares the means for two measurements (pre-test and post-test) taken from the same individual. To investigate whether there was any difference between the pre-test and post-test performance *between* the two groups of students, the researcher conducted the independent samples *t*-test.

Results and Discussion

The results in Table 1 show that both groups of students performed very much the same on the pre-test. The control group of students (Welkom, n = 59) had an average pre-test score of 39.32, while the experimental group of students (Bloemfontein, n = 95) had an average pre-test score of 38.53. The results from the post-test show that the experimental group of students (BFN) had an average post-test score of 75.37, while the control group of students (Welkom) had an average post-test score of 61.36.

Table 1
Pre-test and Post-test Descriptive Statistics

	Group	*N*	Mean	Std. Deviation	Std. Error Mean
Pre-test	Welkom	59	39.32	33.21	4.32
	BFN	95	38.53	33.42	3.43
Post-test	Welkom	59	61.36	28.01	3.65
	BFN	95	75.37	30.38	3.12

The paired sample *t*-test was used to test whether a significant difference exists between the pre-test and post-test *within* each group of students.

From Table 2 it can be seen that, on average, the post-test scores were 22.03 points higher than the pre-test scores for the Welkom group of students (control). For the Bloemfontein group of students (experimental), the post-test scores were 36.84 points higher than the pre-test scores. The results of the paired samples *t*-test show that there was a significant difference between the pre-test and post-test scores for the Welkom group of students ($t(58) = 4.73, p < 0.0001$), as well as for the Bloemfontein group of students ($t(94) = 9.49, p < 0.0001$).

Table 2
Paired Samples Test for Each Group of Students

		Paired Differences					
		Mean	Std. Deviation	Std. Error Mean	*t*	*df*	Sig. (2-tailed)
BFN	Post-test – Pre-test	36.84	37.85	3.88	9.488	94	0.000
Welkom	Post-test – Pre-test	22.03	35.76	4.66	4.733	58	0.000

The independent sample *t*-test was used to determine whether the pre-test scores, as well as post-test scores, differed significantly *between* the two groups of students (see Table 3).

Table 3
Independent Sample *t*-test for the Pre-test and Post-test

	Levene's Test for Equality of Variances		*t*-Test for Equality of Means				
	F	Sig.	*t*	*df*	Sig. (2-tailed)	Mean Difference	Std. Error Difference
Pre-test	0.002	0.961	0.144	152	0.886	0.80	5.53
Post-test	1.51	0.221	-2.87	152	0.005	-14.01	4.89

Since the sample for each group was larger than 25, the normality assumption has not been violated. Furthermore, Levene's test confirms the assumption of equal variances (pre-test, $p = 0.961$ and post-test, $p = 0.221$). The independent samples *t*-test confirm no significant difference with regard to the pre-test between the control group of students (Welkom) and the experimental group of students (Bloemfontein), $[t(152) = 0.144, p = 0.886]$. However, the Bloemfontein group of students who received the number line teaching intervention performed significantly better than the Welkom group of students who received no intervention $[t(152) = -2.87, p < 0.05]$.

From the results in Table 1 it can be seen that both groups of students did not perform well in the pre-test. Reasons for this may be because students knew that the practice test would not count towards their course mark, and as such did not study as hard. Both groups of students performed significantly better in the post-test, compared to the pre-test. It is obvious that students studied more for this test as they were aware that the post-test would count towards their course mark for this statistic module at the CUT. Although both groups of students performed significantly better, students who received the number line teaching intervention outperformed students who received only traditional instruction, without any intervention. It can therefore be argued that the proposed number line teaching intervention may have had a positive effect on students' assessment in a statistics module at the CUT.

Conclusion

Calculating probabilities for continuous random variables is an important learning outcome with regard to the normal distribution. The inclusion of different examples as illustrated in this article can provide teachers with examples of how to move from a visual and concrete example, such as a number line, to more abstract representations, such as calculating probabilities for normally distributed variables. Before it is required of students to calculate these probabilities, students should be familiar with the order and magnitude of numbers on a number line, as well as how inequality symbols are used when problems about the normal distribution are presented in word questions. When students are familiar with how distance work between two points on a number line, they will know when to add or subtract areas under the standard normal curve when calculating probabilities. The illustration of inequalities on a number line may help students when they have to shade certain regions on the normal or standard normal curve, which is crucial to determine whether probabilities should be added to 0.5 or subtracted from 0.5.

The normal distribution is important for inferential statistics as described by the Central Limit Theorem previously, which states that as the sample size increases, the distribution of possible means approaches normal. This in turn, allows students to compute the probability of obtaining sample means in any range of interest, given that the sample is drawn at random from a population with a specified mean and standard deviation. Ignoring student's misunderstandings about the normal distribution may have implications when they move on with more advanced statistical concepts. Furthermore, the normal distribution is "widely used in statistical inference," which is important because it "provides a description of the likely results obtained through sampling" (Williams et al. 2012, 257). The number line is not limited to the normal distribution and has wide applications in statistics, for example, indicating the lower and upper limit of confidence intervals, displaying outcomes for whole numbers when one deals with discrete variables, displaying the minimum, first and third quartiles, median and maximum of box plots, displaying the boundaries of bars in a histogram, as well as indicating the acceptance and rejection areas in hypothesis tests.

Although some educators may use different normal distribution tables or computer software, some instructors may benefit from this article. As Hung-Hsi Wu (2009) states so eloquently, "we want students to be exposed, as early as possible, to the idea that beyond the nuts and bolts of mathematics, there are unifying undercurrents that connect disparate pieces." This saying goes for statistics students as well. Having a strong foundation in statistics is like a tall building with a solid foundation underneath. If students do not have a strong foundation in quantitative literacy skills, they are unable to flourish in a world that requires, more than ever, a substantial understanding of statistics.

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