South African Journal of Childhood Education
ISSN: (Online) 2223-7682, (Print) 2223-7674

Pre-service teachers' use of the jump strategy on the empty number line for mental computation

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Dates:
Received: 05 Feb. 2024
Accepted: 12 July 2024
Published: 28 Aug. 2024

How to cite this article:
Lovemore, T.S. & Graven, M., 2024, 'Pre-service teachers' use of the jump strategy on the empty number line for mental computation', *South African Journal of Childhood Education* 14(1), a1526. https://doi.org/10.4102/sajce.v14i1.1526

Background: South African teachers' mathematical knowledge for teaching and number sense are markedly low, evidenced in national and international research. There is a call for supporting teachers in developing their knowledge and confidence in teaching mathematics.

Aim: The aim of this article is to share how pre-service teachers (PSTs) used the empty number line (ENL) to demonstrate the jump strategy (JS) for efficient mental computation, related to the national Mental Starters Assessment Project (MSAP).

Setting: This study was conducted at a tertiary institution in the Eastern Cape province of South Africa, with 40 first-year intermediate phase PSTs, during their mathematics didactics course.

Methods: The data comprised 40 micro-teaching videos, analysed thematically, following an analytical model for video-recorded data. Videos were categorised according to the way they faithfully followed or departed from the MSAP teacher resources.

Results: Findings suggest that more than 70% of the PSTs faithfully followed the teacher resources to teach the JS on the ENL, with regards to use of at least three of the four key elements, namely: ENL, jumps, gesturing, and key phrases and terminology.

Conclusion: We suggest PSTs receive explicit instruction on the above-mentioned key elements for teaching mental calculation strategies with the ENL and opportunities to practise teaching these to their peers.

Contribution: Most PSTs successfully used the ENL to demonstrate the JS for efficient mental computation, enabled by the opportunity to engage in developing (and reflecting on their) micro-teaching videos.

Keywords: mental mathematics; empty number line; number sense; MSAP; pre-service teachers; mental starters; mental strategies; jump strategy; micro-teaching videos.

Introduction

This article focusses on how pre-service teachers (PSTs) used the empty number line (ENL) to teach the jump strategy (JS) for efficient mental computation at the primary school level. The broader study explores intermediate phase (Grades 4-6) PSTs' learning as a result of exposure to the Mental Starters Assessment Project (MSAP) (Lovemore, 2024), which uses six mental computation strategies to develop mental mathematics skills (Graven et al. 2020). While the MSAP resources were originally developed for Grade 3 learners, it is also relevant for intermediate phase PSTs, who may be faced with having to address gaps in Grade 4 and 5, for example, learners' mental computation skills. In fact, adaptations to the materials are being considered by the Department of Basic Education (DBE) and design team for all primary grades. Furthermore, South African curriculum aims, include among others, developing learners who are confident and competent in using mathematics (DBE 2011). National and international benchmarking assessments, however, show that South African levels of mathematics are poor. For example, the Trends in International Mathematics and Science Study (TIMSS) 2019 showed that only 37% of the South African Grade 5 learners could do basic mathematics (Reddy et al. 2022). The results from the Eastern Cape province, where this study was conducted, are significantly lower than the national average, with only 26% of Grade 5 learners demonstrating basic mathematical knowledge. A disconnect therefore exists between the curriculum aims and learner achievement in the intermediate phase, which directly relates to teachers' own confidence and competence. Hence, this study focusses specifically on intermediate phase PSTs.

In-service and PSTs' mathematical knowledge and confidence in mathematics is also of concern. Reports show that teachers experience a low morale with mathematics teaching

Note: Special Collection: Mental mathematics and number sense in the early grades.

(Graven et al. 2013; Graven & Heyd-Metzuyanim 2014; Venkat & Graven 2017) attributed partly to teachers having gaps in their own mathematical pedagogical and content knowledge (Hoadley 2012; Jojo 2019; Spaull 2019). Bowie, Venkat and Askew (2019) found that assessments of PSTs' mathematical ability were poor, mirroring that of the in-service teachers. Both the first- and fourth-year PSTs, from three South African Universities, who participated in Bowie et al.'s (2019) study could not achieve a mean score over 50% for mathematics problems requiring a higher cognitive demand. Fonseca, Maseko and Roberts (2018) found similar results in their study that showed very little improvement in mathematics achievement between first- and fourth-year PSTs. It is therefore not surprising that various authors call for support for teachers in developing their mathematical and teaching proficiency (Jojo 2019; Venkat & Graven 2017).

Number sense strategies are of interest to mathematics education researchers (e.g. Şengül 2013). The Mental Starter Assessment Project (MSAP) is a diagnostic assessment and intervention programme which focusses on developing calculation strategies moving learners beyond the concrete unit counting methods (Graven & Venkat 2021). The MSAP is currently being rolled out in Grade 3 at a national level, and provincial and national piloting has shown positive gains in learners' pre- and post-test performance (Askew, Graven & Venkat 2022; Graven & Venkat 2021; Venkat & Graven 2022). The MSAP model includes six mental calculation strategies (present in the curriculum) that are taught in 10-min sessions at the start of a mathematics lesson. These six strategies are: bridging through ten (BTT), jump strategies, doubling and halving, rounding and adjusting, re-ordering, and linking addition and subtraction. Each strategy's teaching unit comprises a pre-test, eight 10-min scripted lesson starters (guiding teachers to model the strategy), two additional worksheets, and a post-test. The emphasis in the post-test is to focus on learning gains rather than raw scores and are therefore low stakes. Graven and Venkat (2021) explain that each strategy's teaching unit comprises tasks for fluency, strategic calculation, and strategic thinking. Fluency refers to basic skills that should be automatic, for example, bonds to 10. Strategic calculation is the use of the fluencies to perform a calculation in a strategic manner (e.g. 99 + 99 = 100 + 100 − 2). Strategic thinking requires learners to recognise and flexibly use the structural relationship underpinning the strategic calculation (Graven & Venkat 2023; Venkat, Askew & Graven 2023).

The second strategy in the sequence, namely JS (which can incorporate BTT), was the focus of the current study. The JS refers to making use of an ENL (and eventually a mental image thereof) to gesture and visually show the addition or subtraction of a number, using the BTT strategy to support the 'jumps' (Graven et al. 2020). Bridging through ten refers to using knowledge of bonds to 10 to quickly, mentally add on (or subtract) to the nearest multiple of 10, followed by the remaining amount of the addend (or subtrahend) (Graven et al. 2020).

A project began in 2022 with the intention to prepare PSTs across South African higher education institutions to use the MSAP materials, known as the Mental Mathematics–Work-Integrated Learning (MM-WiL) project. Nine South African higher education institutions are actively participating in MM-WiL and are using the resources to prepare their PSTs to work with the materials and improve their own mental calculation strategies (Graven & Venkat 2023). (See, e.g., Brien & Mc Auliffe 2024.)

As mentioned earlier, while the MSAP resources were originally designed for Grade 3, this study explores intermediate phase (Grade 4–6) PSTs' exposure to and use of the resources to prepare them for addressing challenges that their learners may experience with mental computations, and to empower them to grow in their own mental mathematics confidence. According to the Grade 4 to 6 curriculum (DBE 2011:14), learners should use a range of mental calculation techniques, including, rounding off and compensating, doubling and halving, using a number line, and using addition and subtraction as inverse operations. Using the MSAP strategies and resources is therefore appropriate for supporting intermediate phase PSTs. Pre-service teachers can be supported in their own understanding and skills of mathematics by improving their number sense through strengthening strategic mental mathematics calculations. This may in turn support their mathematics knowledge and skills across other content areas in mathematics. With increased skill and knowledge, PSTs may develop a more positive disposition towards mathematics. This is necessary for them to then teach mathematics with deep conceptual understanding, effective procedural fluency, and a productive disposition (Kilpatrick, Swafford & Findell 2001). The aim of this article is to answer the research question, how did PSTs use the ENL to support teaching the JS for efficient mental computation?

Literature review

Poor number sense in primary school

Graven and Venkat (2021) attribute the poor mathematics achievement among primary school learners to a lack of number sense. Number sense refers to 'understanding relationships among numbers and operations – being able flexibly to partition and combine numbers in convenient ways to allow appropriate estimations and mental calculations to be made' (Bobis 2008:4). Rather than developing this understanding and flexible use of numbers, South African researchers report a prevalent reliance on unit counting among primary school learners, where concrete methods for calculations dominate over mental strategies (Graven et al. 2013; Graven & Venkat 2021; Schollar 2008). Number sense is important for learners' development of mathematical concepts, including understanding of number, place value, mental calculations, larger operations and spatial reasoning, and it is a predictor of future mathematical achievement (Bobis 2008; Shumway & Moyer-Packenham 2019). Concerning then is the problem of low number sense among PSTs, as found in various local and international studies (e.g. Aktaş & Özdemir 2017; Bowie et al. 2019; Courtney-Clarke & Wessels 2014; Şengül 2013), that can consequently result in negatively affecting learners.

The empty number line to support mental computation

Şahin and Danaci (2020:2282) define mental computation as the 'practice of finding the actual outcome of operations without the help of any assistive tools such as paper and pencil, and calculators, by using only the properties of operations'. They go on to explain that learners need to have a solid conceptual understanding of number and operations before being proficient in mental computation. One of the reasons learners have difficulty with mental computation is that they cannot visually show their thinking (Şahin & Danaci 2020). Ruiz and Balbi (2019) define mental computation according to what it entails, that is:

> [*T*]he analysis of the numbers involved in the calculation, the activation of memorized facts, knowledge of the properties of the decimal number system, and the operations involved in the computation to transform the initial calculation into an equivalent one. (p. 316)

In this article, we use Ruiz and Balbi's definition adding that mental mathematics can be supported with drawn (and then imagined) visual representations (other than the paper and pencil algorithms). Ruiz and Balbi (2019) further recognised that there is a relationship between mental calculation and number line representation. We argue that developing teachers' and learners' understanding of this relationship and proficiency in using the number line, and in particular the ENL, to do mental calculations is important.

The ENL has been used in the Netherlands since the 1980s (Freudenthal 1983; Gravemeijer 1994; Treffers 1991) and it is widely used by teachers for supporting addition and subtraction to 100 and 1000 (Blöte, Klein & Beishuizen 2000; Van den Heuvel-Panhuizen 2008). Yet, teachers often lack knowledge of the value, strengths and limitations of using the ENL (Bobis & Bobis 2005). Number lines extend infinitely in both positive and negative directions. The ENL is a number line that has no numbers or markers and thus can be used to support mental computations of addition and subtraction in any number range. It is a visual, intuitive model for learners to demonstrate their mental process (Klein, Beishuizen & Treffers 1998).

Literature notes benefits of using the ENL. Bobis (2007) suggests that this linear representation has the potential to stimulate rich discussion around calculation strategies. Van den Heuvel-Panhuizen (2008) notes that using the ENL assists learners in verbalising their thinking as they move from informal to formal representations of calculations. The linear representation supports learning as it allows learners to record and explain their thinking and gives teachers the opportunity to 'see' learners' mental strategies and identify where support is needed (Gravemeijer 1994). Additionally, Vermeulen, Béguin and Eggen (2020) explain that using the ENL can visually support learners and assist them in reducing their cognitive load. The ENL is therefore a useful tool to represent visually what learners will be doing mentally. It has 'temporary value' (Vermeulen et al. 2020:233) to learners, as they will reach a point where they no longer need to draw the visual representation of the ENL, because they can form mental images to solve the calculation. The ENL thus becomes an imagined tool for calculating. By using the ENL, teachers can give learners access to the mental process that they can use for efficient mental computation.

Graven and Venkat (2021) explain their reasoning for choosing the ENL as a key visual representation, used repeatedly in the MSAP resources, as an attempt to scaffold learners away from unit counting to more efficient and fluent strategies. Robertson and Graven (2022) further suggest the benefit of using the ENL for learners who have a limited proficiency in the language of learning and teaching, as is often the case in multilingual South Africa. The ENL enables learners to communicate their thinking and strategies with the support of quick drawings and gesturing. Bobis (2007) and Van den Heuvel-Panhuizen (2008), however, caution against using the ENL rigidly as a procedural tool that can result in a negative experience for learners. They emphasise the importance of meaningfully introducing the strategy of jumping on a number line and encourage its use as a flexible strategy that draws on number relationships. Similarly, Vermeulen et al. (2020:232) state that the ENL should be used as an 'auxiliary tool' to support learners with addition and subtraction, rather than a mandatory procedure.

The ENL facilitates strategies such as the JS, where learners can visually indicate jumping, for example by 10s, from the first number (Blöte et al. 2000). The JS for two-digit addition and subtraction, according to Bobis and Bobis (2005:70), refers to one number being treated as a whole, while the second number is 'added or subtracted in manageable chunks of tens and ones'. Bobis (2007) further suggests that learners do not need to represent the jumps on the ENL proportionally.

Theoretical and conceptual framing

In this study, as the Mathematics Teacher Educator (MTE), the first author facilitated PSTs with engaging in meaningful activities to enhance their own mental calculation skills and strategies, as well as teaching approaches. The MTE intended to facilitate PSTs in developing their Mathematical Knowledge for Teaching (MKfT) (Ball, Thames & Phelps 2008), derived from Shulman's (1986) Pedagogical Content Knowledge (PCK) and Subject Matter Knowledge (SMK). Ball et al.'s (2008:390) model of MKfT includes six knowledge domains that describe the 'work' of teaching and focusses on the types of skills and knowledge needed to carry out the demands of teaching mathematics (Ball et al. 2008; Chikiwa, Westaway & Graven 2019; Hill, Rowan & Ball 2005). However for this article, Shulman's category of PCK suffices to support the analysis.

According to Shulman (1986:9), PCK refers to 'ways of representing and formulating the subject that make it comprehensible to others'. This involves understanding conceptions, pre-conceptions and misconceptions that learners might have, as well as strategies to support learners. Pedagogical Content Knowledge also involves teacher's selection of the most appropriate forms of representation. Therefore, for the purpose of this study, PCK would involve PSTs' mental mathematics skills (*performing* the calculations efficiently), accompanied by their knowledge of how to do mental calculations using the drawn visual representation of the JS on the ENL (*teaching* the strategy effectively).

Research methods and design

The setting of the broader study was a first-year intermediate phase B Ed mathematics didactic course, taught by MTE, and the first author, to approximately 190 PSTs across two classes. The MTE explained to the PSTs that, although the MSAP resources were originally designed for Grade 3, they can be useful to address mental mathematics in the Grade 4 – 6 curriculum (see e.g., DBE 2011:14), and because of the poor achievement levels noted in, for example, Grade 5 (Reddy et al. 2022). Mental mathematics is also a useful life skill for all adults. The MTE implemented the MSAP JS by sharing the MSAP resources and modelling teaching it over six lectures during the course in semester 2 of 2023. Of the group of first-year PSTs, 102 consented to participate in the broader study.

The MTE shared electronic and hardcopy MSAP teacher guides[1] (Graven et al. 2020) – with her PSTs. She demonstrated the use of the JS on the ENL and how to follow the scripted lessons from these resources in English, isiXhosa, and Afrikaans (the primary languages of the province), and she showed the English demonstration videos from the teacher guide. Pre-service teachers were given the opportunity to apply the MSAP mental calculation strategies through a micro-teaching task. In pairs they 'taught' each other the JS through enacting and verbalising the calculation process on the ENL. Each pair then selected two calculations from a provided problem bank to solve, which they then demonstrated and recorded with a cell phone camera focussed on their hands drawing with gestures and recording their verbalised explanation. These aspects are analysed in this article. The PST pairs were prompted to teach and record one example to a fictitious English home-language learner, and one example to a learner whose home language is not English. Participating PSTs uploaded their recorded lesson clips to a secure Google Drive link. A total of 41 PSTs submitted micro-teaching videos and consented for them to be entered into the data set. One video (T4), however, was incomplete and could not be analysed. The data reported on from this stage of the study, thus, include 40 complete micro-teaching video-recordings from PSTs who consented to entering their recordings into the data set. The broader study further collected data, not reported on in this article, in the form of pre- and post-tests, questionnaires, and in-depth interviews.

Data analysis of the micro-teaching video-recordings was guided by Powell, Francisco and Maher's (2003) analytical model for studying video data in mathematics education. The steps followed were: (1) attentively watch the recorded videos; (2) describe the data; (3) identify critical events; (4) transcribe; (5) code the data; (6) construct a storyline; and (7) write the narrative. Videos were organised and labelled according to the language in which the calculation was demonstrated (e.g. Video English 1 [E1]; Video isiXhosa 1 [X1] etc.). While watching and re-watching the videos, the first author made detailed notes describing each video. She recognised patterns of key elements, including the type of calculation. Thereafter, through another watching of the videos, critical moments were identified, and screenshot images captured these moments in the videos. The critical moments of the videos were transcribed. The first author, being fully proficient in Afrikaans and somewhat proficient in isiXhosa, could translate the transcription into English. To ensure accuracy, an expert in isiXhosa teaching checked the isiXhosa transcriptions.

Patterns were noticed during the process of video analysis, based on the extent to which the micro-teaching example was taught in a way that faithfully follows the MSAP teacher resources and demonstration videos – by this we mean the fidelity to the document was maintained in the teaching. It was noticed that micro-teaching examples were either similar to or departed from the MSAP scripted teacher resources in four ways: use of the ENL; use of jumps; use of gestures; and/or use of key phrases and terminology. Micro-teaching videos were categorised in MS Excel Spreadsheets according to similarities to or departures from each of these key elements. The first author did the initial categorising, which was then checked by the second author. Where there were discrepancies on the categorisation, we re-looked at the examples and came to a consensus. Themes were then developed and used to plot the storyline and write the narrative analysis of the data.

Ethical considerations

The first author invited PSTs to participate in the study through a written letter accompanied by her oral explanation. The first author, being the MTE, considered the possible power relations between her and the PSTs. To minimise risk, the first author requested a third-party from the student support services within the faculty to firstly orally translate the invitation to participate in the study and, secondly, to collect the consent forms, and store them in a sealed envelope in a secure cabinet until after the PSTs' course marks were finalised. This was a strategy to mitigate coercion to participate or student fears of obtaining lower grades (Ferguson, Yonge & Myrick 2004; Ridley 2009). While all first-year PSTs taking the course were exposed to the MSAP resources and strategies, data were only collected from the PSTs who volunteered to be

1. https://www.education.gov.za/MSAP2022.aspx

part of the study and submitted complete micro-teaching videos of their teaching of an MSAP strategy, that is, 40 PSTs provided 40 micro-teaching videos that are the data set for this article.

Pre-service teachers were informed that participation in the study was fully voluntary and that their anonymity would be upheld, using teacher numbers (e.g. Teacher 1 [T1]) on written artefacts. While the participants are PSTs, they were practising to be 'teachers' through micro-teaching videos, and we therefore referred to them as Teacher 1 among others, hence the abbreviation of T1. Furthermore, the videos only focussed on the PSTs' hands and voices. Their faces were not in the video shot, supporting anonymity.

Ethical approval to conduct this study was obtained from the Nelson Mandela University Research Ethics Committee (Human [No. H23-EDU-PGE-020]).

Findings and discussion

Description of calculations chosen for micro-teaching videos

The PSTs were given a problem bank from which to select calculations. The 40 micro-teaching videos, therefore, cover a range of calculations, all of which used the JS, except for one video by Teacher 35. This PST chose their own calculation (20 + 4), which does not require JS and is subsequently not included in the description of types of JS calculations. Some of the calculations in the problem bank required BTT (e.g. 36 + 45) and others did not (e.g. 45 + 20). Table 1 summarises the PSTs' selection of calculations in relation to the five different types available to them in the problem bank.

The problem types in Table 1 are ordered from 0 to 5 in terms of the increase in the level of difficulty. As shown in Table 1, of the 39 videos requiring JS, 28 of the chosen

TABLE 1: Description of the type of calculations selected by pre-service teachers for their micro-teaching videos.

Level of difficulty	Types of calculations	Teachers who selected calculation	Total
Sum or difference unknown			
0	Own calculation chosen, not needing JS or BTT	T35	1
1	$y \pm 10n$ =? Without needing BTT e.g. 45 + 20	T8, T9, T10, T11, T12, T13, T14, T16, T23, T24, T25, T26, T31, T32, T33, T34, T36	17
2	$y \pm (10n + x)$ =? $0 < x \leq 9$ Without needing BTT e.g. 74 – 21	T15, T17, T18, T19, T20, T21, T22, T27, T28, T29, T30	11
3	$y \pm (10n + x)$ =? $0 < x \leq 9$ BTT needed e.g. 36 + 45	-	0
Missing addend or subtrahend			
4	Missing addend or subtrahend of 10n Without needing BTT e.g. 48 – ___ = 18	T1, T2, T3, T5, T6	5
5	Missing addend or subtrahend of $10n \pm x$ BTT needed e.g. 36 + ___ = 61	T7, T37, T38, T39, T40, T41	6

JS, jump strategy; BTT, bridging through ten; T, teacher.

calculations were straight forward calculations with the sum or difference unknown ($x \pm y$ = ?) that did not require BTT (type 1 and 2 in the Table). Seventeen PSTs used the ENL to solve an addition or subtraction calculation that only added a multiple of 10 ($y \pm 10n$), and thus did not need BTT. Eleven PSTs used the ENL to solve an addition or subtraction calculation where the addend or subtrahend was a multiple of 10 and a unit ($y \pm (10n + x)$ = ?; $0 < x \leq 9$), again where BTT was not needed (type 2). It was interesting to notice that no calculations requiring JS with BTT, where the sum and difference are unknown, that is, type 3, were selected. Eleven strategic calculations of a missing addend or subtrahend were, however, selected, of which five did not require BTT and six examples did bridge the 10 (see the last two types 4 and 5 of the table). In terms of the progression within the MSAP teacher guide the level of difficulty of a problem increases from type 1 to type 5 calculations. Thus, the majority of the PSTs (28) chose problems within the first two simplest types, avoiding the additional challenge of BTT, while approximately a quarter (11) selected the more challenging types 4 and 5 where the missing addend or subtrahend was needed, six of which required BTT.

Ways in which pre-service teachers used the empty number line to model teaching the jump strategy for efficient mental computation

The authors identified four key elements to categorise the micro-teaching videos: (1) use of the ENL, (2) use of jumps, (3) use of gestures, and (4) use of key phrases and terminology. They identified these four key elements from the teacher resources and the PSTs' micro-teaching videos. Through analysing the micro-teaching videos, the authors recognised variation in extent to which the PSTs followed the scripted teacher resources on using the ENL and JS for calculations. They categorised this into two broad themes: (1) Faithful use of the scripted teacher guide (strong implementation fidelity), and (2) Departing from the teacher guide. Departures from the teacher resources were either modified appropriately and free from errors, in these cases there was a degree of fidelity in terms of the learning intentions though the suggested 'script' was not used, or they were inappropriately implemented in ways that could cause misconceptions (i.e. there was a lack of implementation fidelity). Analysis of the micro-teaching videos, through organising the transcribed video data and images into Excel Spreadsheets enabled graphic representation of the extent to which the PSTs' micro-teaching videos departed from or faithfully followed the teacher resources (see Figure 1).

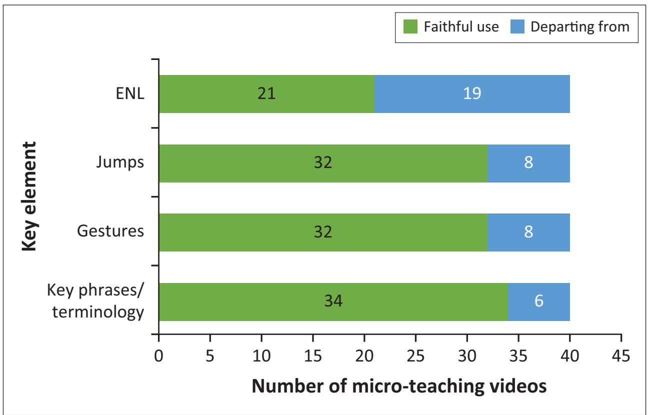
Figure 1 shows that for each of the four key elements, most PSTs faithfully re-enacted the MSAP scripted and demonstrated teaching of the different JS calculation types (in each element the green dominates). Further in the text PST data are discussed in relation to each of these elements. While Figure 1 does not show the number of PSTs who faithfully followed all or none of the four key elements, further analysis showed that:

- 14 PSTs faithfully followed the scripted teacher resources according to all four key elements;
- 15 PSTs according to three of the four key elements;
- 8 PSTs two of the four key elementsv
- 2 PSTs according to only one of the key elements; and
- 1 PST followed none of the key elements.

The next section examines and analyses similarities to and departures from the teacher resources across the four key elements and provides examples from the micro-teaching videos to exemplify these.

Use of the empty number line: Faithful use and departures from the Mental Starters Assessment Project teacher resources

The ENL is the visual representation used for supporting the JS in the MSAP resources and in this study. Being a linear visual representation, it is a tool to support learners in adding or subtracting, helping them develop their conceptual understanding (Bobis 2007; Van den Heuvel-Panhuizen 2008; Vermeulen et al. 2020). The MSAP teacher resources make use of the ENL to support learners with efficient and fluent computation (Graven & Venkat 2021). The teacher resources (the video demonstration and the scripted lesson plan in the teacher guide) emphasise that the ENL is a straight, continuous line with no unit markers. On this line learners can plot the numbers as needed, and then jump forwards or backwards for addition and subtraction respectively. Twenty-one of the 40 PSTs' micro-teaching videos made use of the ENL, as a continuous, initially unmarked line, as shown in the same way as the MSAP teacher resources. Teacher 39 is an example (see Figure 2) showing 75 + 11 by first drawing the continuous ENL with no markers, followed by plotting the starting point, 75, and jumping forwards to solve the sum, only placing the answer on the number line after jumping.

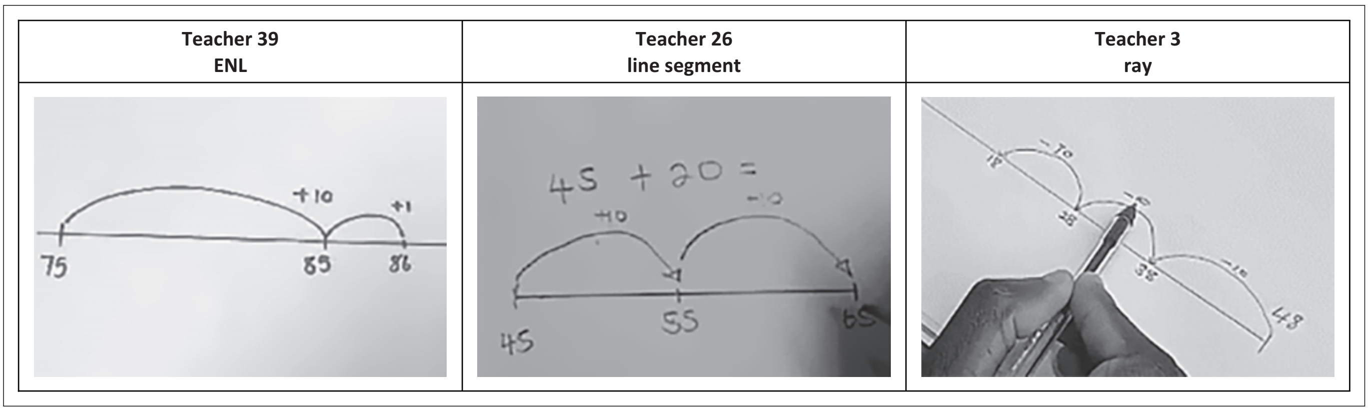


Note: Representation of faithfulness to or departures from the empty number line teacher resources.

FIGURE 1: Representation of the extent to which pre-service teachers followed the teacher resources.

However, 19 of the 40 PSTs' micro-teaching videos departed from the scripted MSAP teacher resource in their use of the visual resources of the ENL. The departure in these videos was that the ENL was represented as a line segment, with the numbers being written as end points on a terminating line, or as a ray, with one endpoint. Figure 2 shows, for example, Teacher 3's use of a ray (with 48 as the endpoint, for calculating 48 – 30). Twelve PSTs used a ray rather than an unmarked, continuous ENL. Nine of the rays were used for subtraction calculations, such as Teacher 3, where the minuend was plotted on the right-hand end of the line, and jumps were made backwards, and three rays were used for addition sums where the first addend was placed on the left end of the line. Figure 2 also shows Teacher 26's use of a line segment to solve 45 + 20, by plotting the predetermined endpoints of 45 and 65. Seven PSTs used a line segment rather than a continuous ENL; four of these were subtraction calculations and three were addition calculations.

While the use of a line segment or ray did not result in an incorrect answer, this is a departure from the MSAP teacher resources, modelled video, and examples by the MTE in class. We argue that this could have pedagogical implications and is linked to the PSTs' PCK of selecting relevant visual representations to support learner understanding. Vermeulen et al. (2020), for example, explain that the ENL offers learners an unstructured, open-ended format to use their variation of a strategy and support mathematical thinking. A line segment or ray may limit the potential of the ENL to guide learners in



ENL, empty number line.

FIGURE 2: Examples of using the empty number line.

visualising a number as a point on a continuous line and engaging in thinking about which direction to jump (as only one direction is possible once the first number is plotted). Teacher 3’s micro-teaching example (in Figure 2), for 48 – 30, plotted 48 at the end of the line. This allowed for jumping backwards to solve the subtraction calculation but would not allow the learners to make that decision for themselves, as 48 was already placed at the end of the line segment.

Teacher 18’s line segment, as shown in Figure 3, is a further example of the difficulty that teachers and learners could encounter should they start with numbers as end points on a line segment. The PST did not draw the jumps with the correct proportions in order to fit the jumps within the endpoints. Starting the thinking process with a continuous ENL, not limited with numbers as endpoint, could allow for flexible use of the JS for computation. The MSAP teacher resources, demonstrating a continuous ENL, can support PSTs in developing their PCK in this aspect, by recognising the importance of an appropriate choice of visual representation to facilitate learner understanding.

Teacher 18

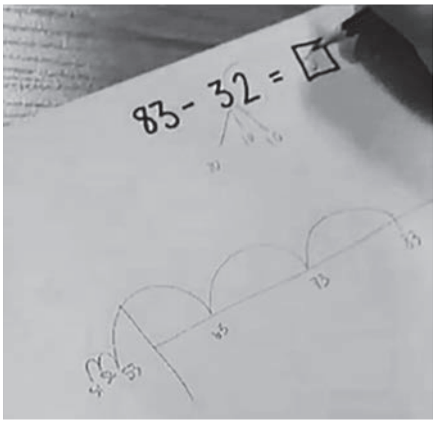


FIGURE 3: Example of line segment.

Use of jumps: Faithful use and departures from the Mental Starters Assessment Project teacher resources

The use of the jumps on the ENL varied across the 40 micro-teaching videos. Pre-service teachers either broke up the jumps into multiples of tens (10 + 10 + 10…) and then the units or did one ‘longer’ jump indicating the tens (10*n*) followed (or preceded) by the units. Of the 17 PSTs who used the ENL to solve the calculation of adding or subtracting a multiple of 10 only, without BTT (type 1 in Table 1), three used one jump for the multiple of 10, and 12 broke the multiple of 10 up into smaller jumps of 10 (e.g. see Teacher 9 in Figure 4). Teacher 31, however, departed from the MSAP JS, for the calculation 16 + 50, by starting with 10 on the ENL, then adding the addend of 50, followed by the 6 ones. This teacher appears to be using the commonly taught and curriculum promoted breaking down and building up strategy (DBE 2011:14) and then demonstrating that strategy on the number line (i.e. demonstrating 10 + 50 + 6 on the ENL). Teacher 12 also departed from the MSAP JS process (of jumping in tens or 10*n*), for the calculation 62 – 30, by breaking the 30 down into two groups of 15 thus not using the ‘friendly number’ (Graven et al. 2020:80) of 10 that would support rapid recall facts of counting backwards in 10 or subtracting a multiple of 10.

From the 11 calculations where the addend or subtrahend was a multiple of 10 and a unit ($y \pm (10n + x) = ?;\ 0 < x \leq 9$) without BTT (type 2 in Table 1), 7 PSTs used one long jump for the multiple of 10, followed (or preceded) by the unit (e.g. see Teacher 20 in Figure 4), and 4 PSTs jumped in multiples of 10, followed by the unit (see Teacher 18). It was interesting to notice that Teacher 18, for the calculation of 83 – 32, also broke up the unit digit into separate jumps (see Figure 4).

Eleven PSTs chose calculations of the more challenging types (type 4 and 5 in Table 1) requiring finding the missing addend or subtrahend. Looking at the missing addend or subtrahend, five of the 10 PSTs broke the multiple of 10 up into jumps of 10 (see, e.g., Teacher 1) and five PSTs jumped the full multiple of 10 (see, e.g., Teacher 6 and Teacher 7). Jumping directly with a multiple of 10 is more advanced than jumping in tens (Klein et al. 1998; Vermeulen et al. 2020), thus a teacher can evaluate the level of a learner’s conceptual and procedural knowledge based on how they jump. One PST (Teacher 35) had not used one of the calculations from the problem bank, but rather attempted to use a single jump on the ENL to demonstrate an addition calculation of 20 + 4. This should be a known bond and does not require the JS.

As mentioned earlier, Shulman’s (1986) PCK is a teacher’s knowledge of learners’ (pre-)conceptions and the most effective way to represent a concept to support deep understanding. In the PSTs’ micro-teaching videos, they made decisions around whether they should break multiples of 10 up into jumps of 10, or whether their fictitious learners would be able to do a ‘longer’ jump of the full multiple of 10. In the teacher resources, the demonstration videos start off with a jump of 10 followed by units. The MSAP demonstration video quickly progresses to grouping the multiple of 10 (10*n*) as one jump, stating in the video ‘some children may need to subtract 10 and subtract 10 to take 20 from 57, but hopefully many of them will now realise I can subtract 20 in one jump.’[2] In the micro-teaching examples, the PSTs are using the ENL to demonstrate the JS, developing their PCK as they make decisions on whether the learners need to have jumps broken up into tens or whether they can do one jump.

According to Vermeulen et al. (2021) the JS used on a linear representation supports learners in visualising addition and subtraction of multi-digit numbers on an ENL. From the sample of 40 PSTs, 32 PSTs made use of proportionate jumps. The two authors agreed upon the criteria for a drawn jump to

2. https://www.youtube.com/watch?v=JQq2zL6pwCM

be considered proportionate: A drawn jump is proportionate where the length or size of each group of 10 would visually appear to be approximately the same, and tens would be clearly larger than ones (which would generally be *at least half* the size of the tens). Initially, the first author categorised the drawn jumps as proportionate or not, which was then checked by the second author.

According to Bobis (2007), learners should not be expected to draw their jumps on the ENL proportionally, but rather use it as a flexible strategy to visualise the calculation, supporting mathematical thinking, and assist in reducing cognitive load. The authors however argue that while children may not be expected to precisely represent their jumps, teachers should avoid confusion and possible misconceptions around number and place value, by drawing jumps disproportionally while teaching learners (Lovemore, 2024). Eight PSTs' micro-teaching videos showed examples where the jumps were not proportionate (see Figure 1). Figure 5 shows three examples of screenshots from three micro-teaching videos of the PSTs.

Teacher 19's visual representation on the ENL, faithful to the teacher resource, shows that the jumps of 10 are visually of approximate equal size. This is in contrast to the much smaller jump of only 2. The proportion between the tens and ones can help learners visualise the values they are working with. Teacher 25, however, chose to do a jump of 30 (rather than breaking it down into three groups of 10). The jump representing 2 is only slightly smaller than the jump of 30, and thus the visual does not clearly show the proportional relationship between a jump of 30 and 2. This may lead to confusion or even misconceptions for learners. Teacher 33 opted to emphasise place value, that 50 is 5 tens and 0 ones. In the micro-teaching video:

> 'And the zero stands for the ones … Now we are going add the ones. As we are adding nothing, the amount is going to be the same'. (Teacher 33, English, 1st year)

This PST has demonstrated PCK that learners may need help with place value, and that it is possible to add only tens without ones values. However, this visual representation of a jump on the ENL representing a jump of 0 can potentially be

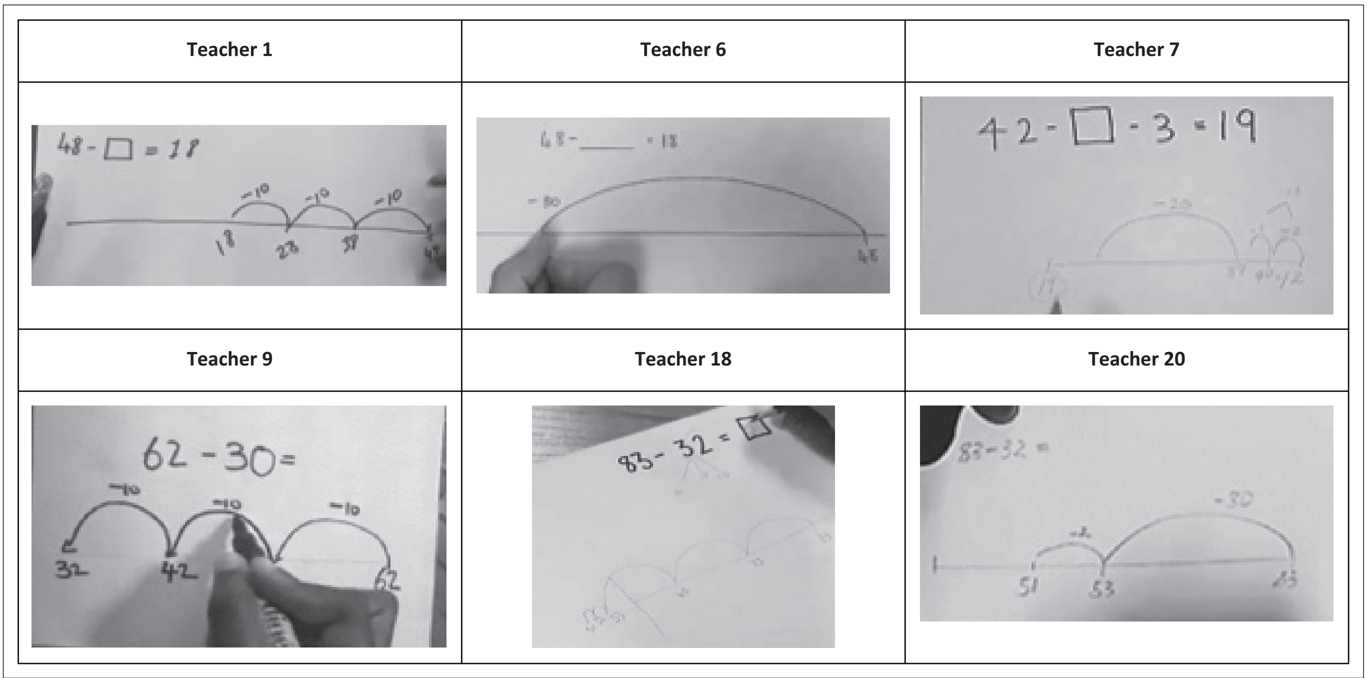


FIGURE 4: Screenshots from micro-teaching videos demonstrating how pre-service teachers used jumps on the empty number line.

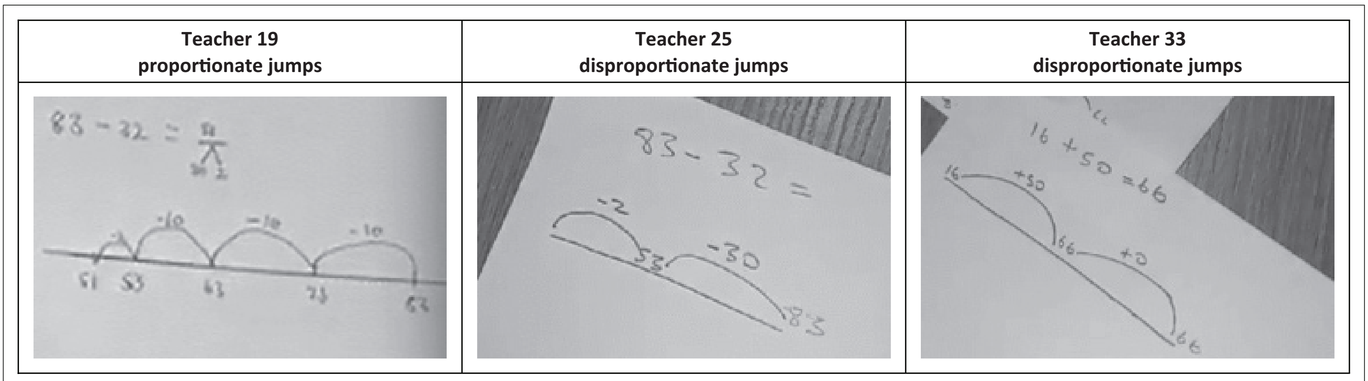


FIGURE 5: Screenshots of the proportionate and disproportionate jumps used on the empty number line.

confusing for learners, and result in misconceptions. The visual linear jump forwards indicates that an amount (of 0) has been added and should move forwards to a new point on the ENL. This new point will however not be the same answer. Teacher 33 predicted his learners may need support in recognising that 50 has zero ones, and he thus decided to explain and represent this. However, he is visually representing a forward jump for the addition of zero, rather than perhaps a vertical jump, landing in the same spot or no jump at all. This representation may hamper learners' conceptual development of using the ENL for JS and in place value. Reflecting on this, we note the importance of MTEs being explicit about the importance of the teacher representing jumps on the ENL proportionately and how this supports learners' visualisation and number sense. In the next section, we discuss the explicit and clear use of gesturing with the jumping on an ENL.

Use of gestures: Faithful use and departures from the Mental Starters Assessment Project teacher resources

Gesturing in mathematics teaching and learning has shown to improve learners' conceptual development and performance of mathematics tasks. Gesturing can entail different visible body movements, such as hand movements, body language or facial expressions, to convey meaning or expression (Kendon 2000; Rasmussen, Stephan & Allen 2004). Gesturing is a visual-motor representation that can support learner understanding of a mathematical process, and reduce cognitive load (Cook et al. 2017; Goldin-Meadows, Cook & Mitchell 2009; Rasmussen et al. 2004), much like the ENL. The MSAP teacher resources, including the video demonstration and the MTE demonstration, used gesturing to emphasise jumps going forwards or backwards on the ENL for addition and subtraction, respectively. The drawing of the jumps on the ENL in itself is a hand movement and could be considered gesturing. For the purpose of this article, however, we chose not to include the drawing of the jumps as gesturing, but rather the emphasis of the jump through two non-drawing hand movements: (1) indicating direction, forwards or backwards on the ENL, and/or (2) pointing to the original calculation, indicating the breaking down of the addend or subtrahend and linking it to the jumps. The movements would reinforce the JS on the ENL, and/or connection to the calculation, visually showing the value getting greater or smaller. If such gestures were clearly visible, the video counted as including gesturing, as it appears in the bar graph in Figure 1.

Thirty-two of the PSTs' micro-teaching videos included *gesturing* that followed the teacher resources and demonstrations faithfully. Of these 32, 16 PSTs gestured only the direction for adding or subtracting, 4 only pointed to a number, and 12 gestured both direction and pointing to the numbers. Eight videos, however, had no gesturing other than drawing the jumps, thus departing from the demonstration of the teacher resources. Teacher 10's micro-teaching video transcript (originally in English) and screenshot images (see Figure 6) are provided next in the text:

> 'You've got a sum of minusing 30 from 62 … The first step is to firstly draw your number line and place 62 on the end of your number line [*points to end of ENL*] since you're going backwards [*gestures backwards on the ENL*]. So you place your 62 at the end of the number line and you move backwards [*emphasises drawing jump arc*] by minusing 10. Since you're not yet by 30, you move back another 10, and you minus 10 once again and you've got 42 [*keeps her right-hand finger on the first ten jump while gesturing and drawing the next jump to 42 - see frame 2*]. So since we see that it's already two tens make up twenty, that's why it takes one more ten to make 30, therefore we move back for the last time [*points with right-hand finger on 30 in the calculation while gesturing jumping the final 10 - see frame 3*].' (Teacher 10, English, 1st year)

Teacher 10, as shown in Figure 6, pointed to the end of the ENL to emphasise that plotting the minuend (62) towards the end of the line will help solve the subtraction problem (Lovemore, 2024). She further gestures the backwards direction to emphasise the subtraction. Teacher 10 also emphasised her drawing of the jumps and then pointed to the subtrahend (30) in the original calculation to show each jump of 10. This example followed the scripted and

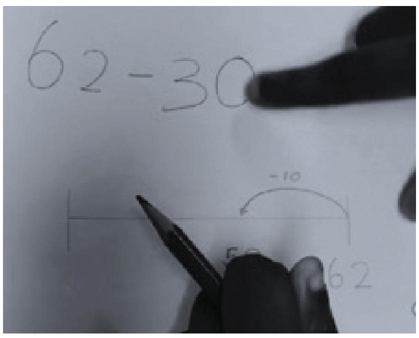
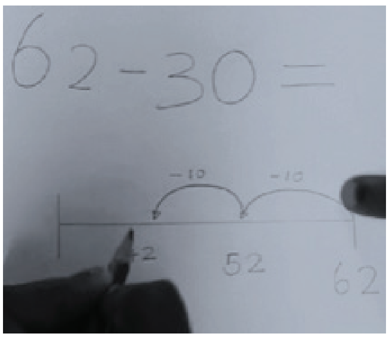
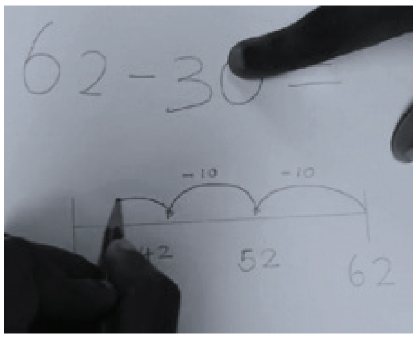
Teacher 10		
Frame 1	Frame 2	Frame 3
		

FIGURE 6: Example of gesturing jumps on the empty number line.

video-supported teacher resource faithfully,[3] also indicating Teacher 10's developing PCK.

Use of key phrases and terminology: Faithful use and departures from the Mental Starters Assessment Project teacher resources

The teacher resources consist of scripted lessons, which were carefully designed to use pedagogically appropriate vocabulary to support conceptual understanding. The teacher demonstration videos use the same language, following the scripted lessons, and so did the MTE (lesson demonstrations in lectures were conducted in English, Afrikaans, and isiXhosa using the three translations of the MSAP materials). An excerpt of a scripted JS lesson from the teacher guide (Graven et al. 2020:29) is discussed next in the text:

> Problem: 36 + 13
>
> Teacher: *Plot* 36 near the *start of the line* (because adding means we will be *jumping forwards*). We have to jump 13 forwards. Let's *break* 13 *down* into 10 and 3. What is 36 + 10?
>
> Learners: 46.
>
> Teacher: (Draw the +10 jump, landing on 46). We still have to jump 3 forward. What is 46 plus 3?
>
> Learners: 49.

See the demonstration video from the teacher guide here for this type of calculation.[4]

To analyse coherence with or departure from the scripted teacher guide and demonstration videos, the authors searched for key phrases and terminology in the micro-teaching videos. Table 2 shows the selected key phrases and terms identified and the number of PSTs that used these throughout the micro-teaching videos.

Based on the identified key terms, most of the 40 PSTs' micro-teaching videos were categorised as highly similar and faithful to the scripted teacher resources. Six PSTs' videos were categorised as departing from the teacher resources. Out of these six, five videos showed PSTs (T12, T13, T24, T30, T38) using the ENL without error, but not using or explaining the JS steps with the key phrases from the scripted lessons in the teacher resources. Two transcripts of PSTs' videos are provided as examples.

TABLE 2: Number of pre-service teachers who used key terminology.

Key phrases and terminology usage	PSTs
Plot	1
Break down or up	6
Jump	20
Forwards or right	2
Backwards or left	9
End of line or right hand side	5
Start or beginning of line or left hand side	5
Friendly number or easier	4

PST, pre-service teacher.

3. See example of MSAP video demonstration of this type of calculation: https://youtu.be/6RkP5bSpINQ

4. https://youtu.be/FPTVolFFd3k

Originally in English:

> '53 plus 17 [*draws jump backwards*]. You add ten, you get 63. 63 plus 7 [*draws jump backwards*], it's 70.' (Teacher 30, English, 1st year)

Translated from isiXhosa to English:

> 'From 36 to 61 … We plus 20. We get 56. We still add 5 to get to 61 [*draws jump*].' (Teacher 38, isiXhosa, 1st year)

These five PSTs used vocabulary such as, 'plus, minus, add, subtract', but did not incorporate the key terminology that makes the ENL and JS pedagogically powerful (e.g. jump, forward or backwards, friendly or easy number etc.). Four of these five PSTs who departed from the key terminology in the teacher resources, taught in isiXhosa, one taught in English. As the key terminology and phrases such as '*tsiba*' (jump) were demonstrated in the lecture sessions in isiXhosa, and they are present in the provided isiXhosa MSAP scripted lessons, the fact that this key term was absent for the teaching of the JS for these five PSTs is somewhat surprising. This may suggest there could be benefit in providing the online video demonstrations in multiple languages rather than only in English. This said, most PSTs who presented in isiXhosa used the term '*tsiba*' and other appropriate translations of key phrases suggested.

One PST departed from the key phrases and terminology in the micro-teaching video in a way that may result in misconception. Teacher 28 attempted to use key terminology to solve 30 + 18:

> 'Then jump 10 units to the right [*gestures*] … then we are done by ten and we are going to jump again, 8 units to the right …' (Teacher 28, English, 1st Year)

She made reference to 'jumps' and 'right' for addition, however, her description of '10 units' and '8 units' may be confusing for learners. The goal is to jump in groups of 10, rather than unit counting – the 10 should be seen as a single unit of 10 rather than as 10 units even while it is indeed made up of 10 units. While Teacher 28 visually represented one jump of 10 and one jump of 8 accurately, her usage of the term 'units' for the 10 jump (rather than 'we jump by a unit of 10 or simply we jump by 10) could be problematic for learners. Using the same description for both the jump of 8 and the 10 masks the key distinction between the tens jumps and the ones jumps.

An excerpt of Teacher 40's micro-teaching video is provided here as an example of faithful use of the teacher guide scripted lessons. The calculation was 75 + ___ = 86:

> 'First, we start by drawing a straight line for the number line. Then we're going to place 75 at *the beginning of the line* … Now we need to see how can we get close to 86? We're going to *jump* to 86 by adding 10 [*draws jump*]. So we're gonna get 85 by adding 10. Then we need to add 1 more to get 86, so we're going to take a *small jump* to get 86.' (Teacher 40, English, 1st Year)

This example includes identified key words, as well as additional phrases such as 'how can we get close to 86?'.

While the PSTs did not use all the identified key terms or phrases, they were able to apply some of the vocabulary emphasised in the teacher resources, demonstration videos, and demonstrations from lectures. Other PSTs who used formal, abstract language only could be supported with more practice and emphasis on the pedagogical value of using key phrases and terminology from the scripted lessons to develop their PCK.

Discussion

After watching demonstrations, in class and online videos, of the MSAP JS on the ENL, PSTs selected a variety of calculations, ranging in complexity, from a problem bank to enact their own demonstration. It was interesting to note that no PSTs chose a calculation requiring BTT with the sum or difference unknown. Twenty-nine PSTs chose easier, straight forward calculations and 10 chose more complex calculations with a missing addend or subtrahend, some of which did bridge the 10.

It is encouraging to note that most PSTs (29/40) faithfully followed the MSAP resources according to at least three of the four key elements (i.e., there was strong implementation fidelity across at least three of the four elements). However, analysis of the 11 who departed from the resources points to some implications for teacher education when using MSAP. From the PSTs' micro-teaching videos, four key elements were identified and analysed to explore how PSTs used the ENL to support teaching the JS for efficient mental computation. The key findings in relation to each of these elements are discussed further in the text.

Just over half the PSTs faithfully followed the MSAP resources and demonstrations to draw an unmarked, continuous ENL, while the other half of PSTs either drew a line segment or a ray. While literature is silent on issues around teachers converting the Empty Number *Line* to an Empty Number *line segment* or *ray*, we have argued that a continuous, initially unmarked, line (the ENL) is a more powerful representation to guide learners in flexibly thinking about directionality with regards to addition and subtraction.

The PSTs used the jumps in a variety of ways, some breaking up the jumps into tens and others jumping in the multiple of 10. We also argue that teachers' representations of jumps should be visually proportionate to support conceptual understanding. While the jumps on the ENL are visually drawn, effective teaching with this representation can support learners in developing mental images for flexible and strategic mental computation.

Gestures were evident in 32 of the 40 micro-teaching videos. Those who gestured for direction *and* pointing to the calculation mostly faithfully followed the MSAP demonstrations. There was also generally good movement between gesturing to the part of the original calculation that was linked to the different steps in drawing and/or writing on the number line. We argue that such gesturing is important for supporting learners' development of number sense and linking what they are 'seeing' to what they are being asked to solve. Given that 20% of PSTs did not use gesturing it appears important that MTEs explicitly bring PSTs' attention to how gesturing can support learner sense-making.

Key phrases and terminology from the MSAP scripted resources were evident in 34 of the videos, while those that departed from the resources were mostly taught in isiXhosa. This points to a need for video demonstrations being in multiple languages rather than only in English. Emphasis on key phrases such as 'jump', 'forwards', 'friendly or easy numbers' is important. In each of these four key elements are points that MTEs should make explicit to the PSTs when using MSAP.

The micro-teaching videos show that the PSTs were able to implement the MSAP resources, thus building their PCK. Some adaptations or departures from the resources were noted, such as Teacher 33 who intended to demonstrate the place value by drawing a jump for 0 ones. Teacher 31 also departed from the MSAP JS, for the calculation 16 + 50, by starting with 10 on the ENL, then adding the addend of 50, followed by the 6 ones. These PSTs may have reverted to the breaking down and building up strategy taught in the curriculum, but they then tried to combine it with the JS. These examples suggest that PSTs' existing knowledge of curriculum expectations from their own schooling may in some cases hinder the implementation of the MSAP strategies, while in other case may support it.

Conclusion

This article sought to answer the research question: How did PSTs use the ENL to support teaching the JS for efficient mental computation? This is part of a broader study relating to the national MSAP. The MSAP teacher resources have been rolled out nationally, and it is intended that teachers will use the tasks as designed, following the lesson scripts, guidelines and even demonstration videos. The PSTs in this study were exposed to all the teacher resources and were guided by their MTE. They practised using the ENL to teach the JS through micro-teaching videos, of which 40 were analysed. Findings suggest that most PSTs in this group could implement the ENL JS in a way that was highly similar to the teacher resources, with regards to the use of the ENL, jumps, gesturing, and key phrases and/or terminology. This points to the potential advantages of using micro-teaching as an opportunity for PSTs to engage in developing and reflecting on their own teaching. In some instances, PSTs departed from the scripted resources and video demonstrations, which either did not change the meaning of the procedure, but could hamper the pedagogical richness of the representation, or the departures could cause confusion and misconceptions. These instances pointed to the importance of MTEs bringing explicit attention to: the way in which the ENL is a continuous line and not a line segment; that specific gestures are useful to support learner understanding (particularly of the connection between the original problem and the 'work' on the ENL), that approximate proportionality of the jumps on the ENL is important and that consistent use of appropriate and specific terminology and phrases will aid clear communication. We conclude by suggesting that PSTs receive explicit instruction on how to use and teach mental calculation strategies and opportunities to practise their teaching with attention to each of these aspects.

Acknowledgements

This article is a product of a number of individuals' contributions who offered their time and insight into the various aspects of the study. The authors thank the Faculty and PSTs where the study was conducted, and are also grateful for the support and training received from the South African Numeracy Chairs at Rhodes and Witwatersrand University.

Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

Authors' contributions

T.S.L was the principal investigator for this study. She conducted the data collection, initial analysis and draft write-up, as well as editing and reviewing. M.G. assisted with the conceptualisation of the study, discussed and assisted with further analysis, wrote sections under the findings and discussion, and conducted multiple rounds of editing and reviewing.

Funding information

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Data availability

Data sharing is not applicable to this article, as no new data were created or analysed in this study.

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