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A didactic engineering for the study of the Padovan's combinatory model

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ABSTRACT

Considering the content of history of mathematics textbooks, it's evident that their emphasis is primarily on the illustrative aspects of recurring numerical sequences, with a particular focus on the Fibonacci sequence. Unfortunately, this limited approach results in the neglect of other sequences akin to the Fibonacci numbers, thus rendering the subject challenging for teaching purposes. This study aims to address this gap by offering a concise exploration of the combinatorial aspects of the Padovan numbers, specifically through the concept of a board as initially examined by mathematicians. In line with the research methodology of didactic engineering and the teaching theory of the theory of didactic situations, two problem situations have been developed, centered on the Padovan combinatorial model, thereby contributing to the enrichment of mathematical education within initial teacher training programs. Within this framework, various strategies are introduced that rely on visualization and counting, with the objective of illustrating specific mathematical identities suitable for potential classroom applications.

Keywords: didactic engineering, the Padovan's combinatorial model, theory of didactic situations

INTRODUCTION

The discussion of mathematical curiosities and illustrative content within the realm of the Fibonacci sequence is heavily dominated by authors of history of mathematics textbooks (Burton, 2007). Regrettably, many of these authors often neglect to consider the profound mathematical contributions and pioneering work of the renowned Leonardo Pisano (1170-1217), the creator of the Fibonacci sequence.

This oversight becomes apparent when examining the work of Gullberg (1997), which delves into the famous problem of "immortal rabbits," without delving into the relevant mathematical contributions or the latest evolutionary studies related to these numerical patterns. It's crucial to recognize the historical context that propels the epistemological and evolutionary advancement of specific mathematical entities and, more broadly, a myriad of recurring numerical sequences and their corresponding mathematical theories.

Generally, there is a discernible interest in the diverse approaches and generalizations of recurring numerical sequences, with the Fibonacci sequence taking the spotlight due to its contemporary significance. This sequence, which belongs to the second order, exhibits numerous relationships with the golden number $\varnothing \approx 1.61$. One such relationship stems from one of the roots of the characteristic the Fibonacci polynomial, which equals the value of the golden number. Further associations with the Fibonacci sequence are explored in Dunlap's (1997) work, spanning from the description of plant growth to the development of computer algorithms for data retrieval.

In contrast, the Padovan sequence, bearing the name of Italian architect Richard Padovan (born in 1935), constitutes a third-order recursive numerical sequence. One of the solutions to its characteristic polynomial is the plastic number, with a value of $\Psi \approx 1.32$. This underpins a clear connection between the Padovan sequence and the plastic number, also referred to as the radiant number (Padovan, 1994; Vieira, 2020).

Marohnic et al. (2013) point out that the Padovan's exploration was built upon the groundwork of the architect Hans van der Laan (1904-1991), who stumbled upon the discovery of a novel irrational number–the plastic number. However, it's worth mentioning that previous research credits the study of this number to Gérard Cordonnier (1907-1977), and thus, this sequence is also known as the Hans van der Laan or Cordonnier sequence (Alves & Catarino, 2022).

Studies examining the properties of the golden number, also known as the divine proportion, have led to the definition of morphic numbers. These numbers only possess two solutions, one being the plastic number and the other the golden number. This highlights the connection between the two sequences: Fibonacci and Padovan (Aarts et al., 2001; Ferreira, 2015).

Given this intricate interplay between these numerical sequences, the impetus emerged to explore other sequences, fostering their respective epistemological and mathematical advancements. The unprecedented evolution of the Padovan and the Perrin sequences is evident, offering a unique opportunity for research participants. This substantiates the drive to create a pedagogical proposal for delving into the Padovan and the Perrin sequences, with a focus on generalization, complexity, and the combinatorial model of these numerical entities.

Considering these considerations, there is an imperative to create instructional scenarios that align with students' needs. Consequently, mathematics educators face the challenge of making their lessons more engaging, igniting students' interest in constructing mathematical concepts through didactic situations designed to evoke curiosity. In response to this necessity, among the theories of mathematics didactics, which originate from France, we have sought one that not only scrutinizes obstacles within the teaching process but also explores didactic and cognitive aspects. This quest led us to the theory of didactic situations, as conceptualized by Brousseau (1986). The selection of the research methodology follows the same French perspective, aligning with didactic engineering, a framework developed by Artigue (1988) and further elaborated upon by Artigue and Glorian (1991). This methodology stems from Brousseau's (1986) insights into didactic practices.

Didactic engineering serves as a structured approach to the knowledge underpinning a given educational activity. It commences with the formulation of hypotheses, followed by their analysis. This approach facilitates pedagogical innovation, marked by the design of research procedures for use in the classroom. By introducing various didactic situations to students, providing them opportunities for constructing knowledge, this methodology proves instrumental in overcoming obstacles within the realm of mathematics.

In this context, it is of utmost concern for educators and researchers in initial teacher training programs who introduce the study of the Padovan sequence in the history of mathematics course. Their intention is to critically assess and adapt their teaching practices. This illustrates the enriching contribution of French didactics to the field of mathematics didactics, particularly in connection with the study of mathematical objects and their relationships as experienced within the classroom.

Drawing on the works of Benjamin and Quinn (2003) and Spivey (2019) concerning the combinatorial model of the Padovan sequence, the research is guided by the following question: "How can we delve into the epistemological and evolutionary trajectory of the Padovan sequence and enhance its combinatorial model?" This central query underpins the overarching goal of this research, which is to investigate the combinatorial model of the Padovan sequence, scrutinizing its epistemological and evolutionary development within the framework of the theory of didactic situations.

Consequently, a problem situation, informed by Brousseau's (1986) didactic situation theory, is developed around the combinatorial model of the Fibonacci sequence, aiming to contribute to initial teacher training programs and mathematics educators alike.

PRELIMINARY ANALYSES

In the initial phase of this study, an extensive bibliographic exploration was conducted, focusing on Fibonacci's combinatorial model and the concept of a board. This groundwork was essential to pave the way for the investigation into the Padovan's combinatorial model. The search encompassed works within the domains of pure mathematics and the history of mathematics, enabling the examination of the sequence's origins, with particular attention to key elements such as the concept of a board, mathematical properties, and related concepts. This process served to compile essential epistemological elements that could be translated into instructional content for students. Additionally, a comprehensive review of literature pertaining to the didactic engineering research methodology and the teaching theory of didactic situation theory was carried out.

Among the pertinent studies focused on mathematical concepts related to numbers and Padovan, notable works include those by Benjamin and Quinn (2003), Craveiro (2004), and Koshy (2001, 2019, 2014). These authors conducted extensive research on combinatorial approaches and interpretations related to sequences such as Fibonacci, Lucas, Jacobsthal, and Pell, alongside parallel studies exploring the extension of these sequences. Notably, we emphasize the foundational idea of tiling and chessboards, as inspired by the work of Spreafico (2014), which is primarily concerned with combinatorial interpretations in the realm of recurring numerical sequences.

In the context of mathematics education, noteworthy studies and research efforts that contribute to this work include da Silva (2019) and Oliveira (2015). These studies examine the application of the didactic engineering research methodology to the study of analytical geometry, operations with rational numbers, and the introduction of initial activities for the study of probability and calculus. Additionally, the book authored by Blum et al. (2016) delves into the European tradition of mathematics didactics.

Within this framework, the proposed initiative revolves around the development of mathematical properties and definitions, with a specific focus on the Padovan and the Perrin sequences. The approach involves employing didactic teaching scenarios for students in initial mathematics training. This educational endeavor entails an in-depth exploration of the mathematical-epistemic domain of these numbers, emphasizing their respective combinatorial models and addressing generalization and complexification. Subsequently, didactic teaching situations are designed and executed in the classroom, offering a means to scrutinize and compare the anticipated outcomes with the empirical results. As such, this research incorporates the didactic engineering research methodology and the teaching theory of didactic situation theory, fostering an environment, where research participants can explore novel pedagogical approaches, in tandem with an exploration of the mathematical object itself.

DIDACTIC ENGINEERING & THEORY OF DIDACTIC SITUATIONS

Certainly, the process of teaching and learning mathematics is far from trivial for many students, demanding alternative approaches to stimulate their thinking and foster interest. In this context, during the mid-1980s, a novel research methodology emerged in France with the dual purpose of refining prevailing educational practices and deepening our comprehension of mathematical learning events.

This gave rise to didactic engineering, a research methodology characterized by its similarity to the work of an engineer. As noted by Artigue (1988), the practice aligns with that of a technical-scientific engineer, leveraging knowledge within their domain and necessitating the use of more intricate elements than those distilled by science alone. Importantly, didactic engineering facilitates an examination of existing classroom practices, offers resources for teacher training, and delves into the analysis of both teacher activity and the didactic transposition of subject matter (Chevallard, 1991).

In the context of exploring the role of the teacher in mathematics didactics, the focus extends to the teaching of the Padovan's combinatorial model, with a particular emphasis on the initial training of mathematics educators. Consequently, this research leans on didactic engineering, which is delineated as either micro-engineering or macro-engineering. The former presents a more focused perspective on classroom practices, while the latter assumes a broader, more holistic view. For this work, micro-engineering is employed with the objective of effectively teaching the mathematical object of interest. Artigue (1995) underscores that the application of this research methodology is somewhat intricate due to the challenges involved in practically developing classroom-derived data.

Didactic engineering empowers the analysis of prevailing classroom phenomena and affords resources for teacher training, thereby facilitating an assessment of the teacher's role and their capacity to transpose didactic content derived from scientific knowledge. Within the realm of French mathematics didactics, this research further enriches the scientific conditions and classroom experiments. As such, we have approached the combinatorial study of the Padovan sequence by employing didactic engineering methodology in tandem with the theory of didactic situations in the context of initial training courses for mathematics educators. This approach places significant emphasis on both the training and learning experiences of mathematics teachers and the evolutionary trajectory of the Padovan sequence.

The research is structured into four distinct phases: preliminary analysis, design and a priori analysis, experimentation, and a posteriori analysis and validation. It harmonizes theory with practice. The preliminary analysis stage identifies issues related to teaching and learning, conducting an extensive review of the literature, encompassing relevant works and texts related to the mathematical object under examination. This phase culminates in the enumeration of epistemological, cognitive, and didactic elements (Artigue, 1995).

The a priori conception and analysis phase entails the selection of variables (micro didactic or macro didactic, as discussed further), culminating in the development of teaching scenarios grounded in the epistemic-mathematical domain. The aim is to fulfill the research objectives. Almouloud (2007) aptly states that,

> "a priori analysis is extremely important because the success of the problem situation depends on its quality; in addition, it allows the teacher to control the performance of the students' activities, and to identify and understand the observed facts. In this way, the conjectures that emerge can be considered, and some can be the subject of scientific debate" (p. 176).

In experimentation, the teaching situations developed in the previous phase are applied and the data collected must be recorded (Alves, 2016).

Lopes et al. (2018) state that,

> "Initially, it consists of the period of application and experimentation with the previously planned activities, collecting data on the investigation. Secondly, it refers to the analysis of the results obtained in the research. This phase is based on the analysis of all the data obtained in the experimentation during the teaching sessions, as well as productions inside or outside the classroom" (p. 164).

Furthermore, it is imperative to establish a didactic contract that delineates the roles and responsibilities of both the teachers and the research participants. It is worth noting that there are instances, where the didactic contract is breached due to student disinterest in the learning process.

Lastly, the ultimate phase involves a posteriori analysis and validation, which scrutinizes the data collected in the preceding phase, comparing it with the a priori analysis, thereby validating the previously formulated hypotheses. This validation can be conducted internally, focusing solely on the participating students, or externally by comparing participants who employed the research methodology with those who did not (Laborde, 1997).

During the application phase, as Almouloud (2007) suggests, certain adjustments and corrections may be necessary. Subsequently, the results obtained are thoroughly evaluated, allowing didactic knowledge to contribute to content transmission.

This is succeeded by a validation of the elements during the experimentation phase, where the outcomes discussed are analyzed to determine if engineering has evolved.

To support the various phases of didactic engineering, a teaching methodology is adopted that provides students with a platform for learning and knowledge exchange. This methodology is rooted in the theory of didactic situations, which centers around didactic teaching scenarios. It encourages students to engage in these scenarios, fostering investigation throughout the teaching and learning process in mathematics (Brousseau, 1986). A teaching situation, according to Brousseau (2006), is "the model of interaction between a subject and a specific environment that generates particular knowledge" (Brousseau, 2006, p. 19). Thus, we must acknowledge the presence of the milieu, representing the environment in which the didactic situation is applied. The theory of didactic situations comprises four key situations: action, formulation, validation, and institutionalization.

In the action situation, as described by Alves (2016), participants encounter the proposed problem situation for the first time. This scenario comprises a set of questions characterized by direct and concise statements. Participants endeavor to solve these problems by drawing upon their existing knowledge. The formulation situation involves participants converting their ideas into a more technical and formal language, with the aim of formulating theorems and properties (Vieira et al., 2019). The validation situation is centered on verifying the solutions presented in the action phase, often through discussions and peer assessments. Finally, the institutionalization situation entails the teacher's analysis of the resolutions put forward, elucidating the core objective of the problem situation (Alves, 2019).

Subsequently, the first phase of didactic engineering commences, involving the establishment of a theoretical framework surrounding the object of study and the delineation of the respective epistemic-mathematical domain.

EPISTEMIC-MATHEMATICAL FIELD

The study of recurring numerical sequences is often centered around the Fibonacci sequence, emphasizing its historical significance while disregarding other crucial mathematical contributions (Burton, 2007). Consequently, numerous other recurring numerical sequences remain relatively unknown and are often excluded from initial teacher training courses, such as the Padovan sequence.

Given this educational landscape, exploring the combinatorial interpretation of the Padovan sequence holds great potential for enhancing mathematics education. It enables a more meaningful approach to teaching, facilitated through the application of didactic engineering. According to Zborowski and Pigatto (2018), didactic engineering emerged from discussions within the field of mathematics didactics, driven by the need for improvements in mathematics education in French schools.

Mathematics education should be about fostering lifelong learning, not merely achieving prescribed objectives, or executing algorithms devoid of deeper understanding. There is a clear imperative to adopt methodologies that encourage critical reflection on teaching practices, consequently enhancing the learning process. It is within this context that didactic engineering offers valuable guidance for exploring the mathematical concept of the Padovan sequence. This methodology is complemented by the teaching theory of didactic situation theory, further enriching research guided by didactic engineering.

Didactic situation theory plays a pivotal role in the analysis of classroom dynamics, empowering teachers to embrace an investigative approach to their teaching. It places the student at the center of the learning process, emphasizing the need to consider the initial training of mathematics teachers. This involves introducing research methodologies and teaching theories that can be effectively applied in their future teaching practices.

The Padovan sequence is of third order, with recurrence given by $Pn=P_{n-2}+P_{n-3}$, $n\geq3$, with $P_0=P_1=P_2=1$, where P_n is the n^{th} term of the Padovan sequence. Considering this, an in-depth exploration of these numerical sequences is undertaken, with a particular focus on a combinatorial approach rooted in the work of Vieira et al. (2022).

A board consists of an arrangement of squares, referred to as cells, each of which is assigned a unique enumeration corresponding to its position. Such a board is referred to as an n-board (Spreafico, 2014). Building upon this foundational definition, we can delve into the Fibonacci combinatorial model, which investigates the count of ways (f_n) of covering a 1×*n* board with 1×1 squares and 1×2 dominoes is equal to $f_n=F_{n+1}$ (Spivey, 2019).

As a result of the discussions elucidated by Benjamin and Quinn (2003) and Koshy (2001), who dissect the combinatorial behavior of the Fibonacci sequence through tiling, a specific rule is formulated for the theorem associated with the Padovan tiling.

Applying a similar concept, while introducing the notion of an extended domino, defined as a 1×3 rectangle, a black square measuring 1×1, and the traditional 1×2 domino, any Padovan tiling can be constructed through the arrangement of these defined shapes.

Definition 1. By considering the inclusion of an extended domino, defined as a rectangle with dimensions of 1×3, a black square with dimensions of 1×1, and the classic 1×2 domino, any Padovan tiling can be constituted through the strategic arrangement of these prescribed shapes. The purpose of the black square is to complement the vacant tiles, subject to the rule that it is inserted solely at the commencement of the tiling and appears only once in each tile. These specific rules are established for the Padovan tiling theorem.

Thus, an *n*-board is considered, incorporating the following tile shapes: a 1×1 black square, blue 1×2 dominoes, and gray 1×3 extended dominoes, all of which are assigned a weight of one. In **Figure 1**, on the left-hand side, various illustrative examples are provided to demonstrate the filling of the *n*-board corresponding to the Padovan sequence. On the right, you will find the terms corresponding to the Padovan numbers. The term represents the count of tile shapes within the *n*-board, following the rules, which ultimately defines the ratio: $P_n=P_n$, $n\geq0$. The following is fixed $P_0=1=P_0$, when there is no board of size 1×*n*, and $P_1=1=P_1$, when there

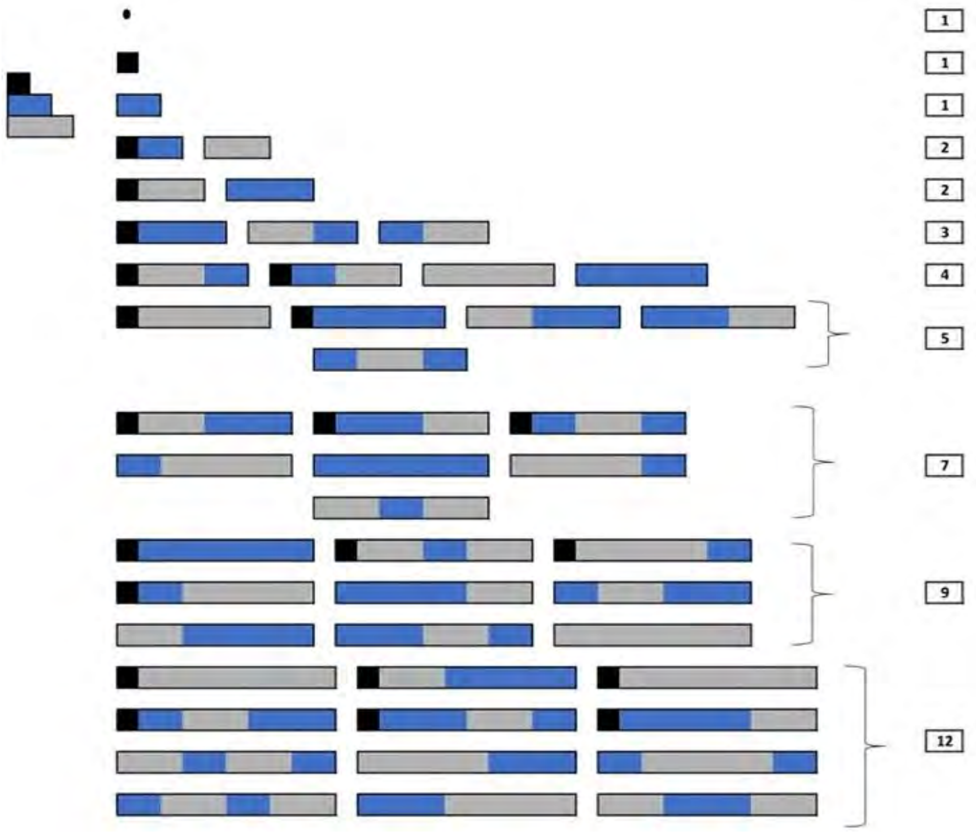

Figure 1. The Padovan tiling (adapted from Vieira et al., 2022)

is only one black square. This is the exception to the rule, since in this case, the black square will be the only figure, being at the beginning and therefore at the end (simultaneously).

Theorem 1. For $n \geq 0$, the possible tiling of a $1 \times n$ board with black square, blue domino and gray extended domino tiles is given by (Vieira et al., 2022): $P_n = P_n$, where P_n is the number of ways to fill the $1 \times n$ board and P_n is the n^{th} term of the Padovan sequence.

The demonstration of the theorem can be seen in the work of Vieira et al. (2022).

Some of the Padovan's identities are associated with the discussions arising from the interpretation of Fibonacci in Benjamin and Quinn (2003).

Identity 1. The number of pieces on a board of size n that use at least one domino is P_n or P_{n-1}, depending on the value of n.

Identity 2. The number of pieces on a board of size n that use at least one extended domino is P_{n-1}.

Moreover, a careful curation of identities will be made for an activity proposal that integrates the theory of didactic situations. This activity aims to delve into the combinatorial interpretation of the Padovan sequence by scrutinizing the logical and intuitive reasoning employed by students in their preliminary mathematics courses.

DESIGN & A PRIORI ANALYSIS OF TEACHING SITUATIONS

This section encompasses the development of two problem situations, which will be scrutinized through the lens of micro-didactic variables. This examination aligns with the principles of didactic engineering, as described by Almouloud (2007).

Nonetheless, the groundwork laid out in the previous section encompasses an investigation of the epistemic-mathematical domain. This exploration offers an opportunity to analyze the Padovan's combinatorial interpretation from the perspectives of epistemology, cognition, and didactics within the context of initial mathematics teacher training. This analysis sets the stage for an in-depth examination of the didactic situations, constructed as problem situations, and grounded in the theory of didactic situations. These scenarios enable an evaluation of the methodological approaches and pedagogical conceptions addressed.

In the formulation of problem situation 1 and situation 2, we will delve into the didactic variables related to the content previously outlined in the epistemic-mathematical field:

- notion of a board (Spreafico, 2014),
- the Fibonacci combinatorial model (Benjamin & Quinn, 2003),
- rules for configuring the pieces in the Padovan's combinatorial interpretation (see definition 1),
- the Padovan's combinatorial model (see theorem 1), and
- Identity 1 and identity 2.

Problem Situation 1.

Based on the studies indicated by the Padovan's combinatorial approach, seen in definition 1 and theorem 1, what is the number of pieces on a board of size n that use at least one domino?

Action phase

In this initial phase, students should revisit their prior knowledge, specifically focusing on the foundational concepts surrounding the notion of a board, as elucidated by Spreafico (2014), and the Padovan's combinatorial model. It is advised that students reacquaint themselves with the work of Benjamin and Quinn (2003), which offers a comprehensive combinatorial approach to Fibonacci, along with an examination of the initial identities presented in this work. This approach is expected to mitigate potential challenges and facilitate a smooth transition to the formulation phase. Ultimately, students should aim to grasp the essence of definition 1 and theorem 1, thereby acquiring a comprehensive understanding of the Padovan's combinatorial framework, enabling them to progress to the formulation phase.

Formulation phase

The formulation phase necessitates careful observation and interpretation of the Padovan tiles. Specifically, students should focus on identifying instances, where at least one domino is incorporated. It is essential for students to discern that when the value of n is divisible by two, the count of tiles is determined to be P_n. For n divisible by three or for n-1 divisible by three, it is necessary to subtract one unit from the value of P_n. Thus, according to the value of n, the number of tiles will be P_n or P_{n-1}.

Validation phase

Validation should be divided into cases:

Case 1. If n is divisible by two, then there will always be at least one domino, so the number of pieces will be P_n.

Case 2. If n is divisible by three, then there will be a case, where you do not use dominoes, so the number of pieces will be P_{n-1}.

Case 3. If n-1 is divisible by three, then there will be a case, where you do not use the dominoes, so the number of pieces will be P_{n-1}.

Institutionalization phase

The institutionalization phase plays a crucial role in validating the insights garnered during the problem-solving process. The teacher's role is to facilitate a discussion that encompasses both accurate conjectures and any misconceptions that may have arisen. At this juncture, the teacher reassumes a central position in the activity and underscores the primary goal of the proposal, which is to derive identity 1.

Problem Situation 2

Building upon the discussions in problem situation 1, the second problem situation challenges students to determine the number of pieces on a board of size n that incorporate at least one extended domino.

Action phase

Much like in problem 1, students are expected to draw upon their prior knowledge concerning the notion of a board defined by Spreafico (2014), the combinatorial models of Padovan and Fibonacci, as well as the various identities elucidated by Benjamin and Quinn (2003). Given the similarity of this activity to the preceding problem situation, any potential obstacles should be minimal. With definition 1 and theorem 1 as their foundation, students should embark on the task of constructing the initial Padovan tiles.

Formulation phase

Following the construction of the Padovan board for the initial values, students should scrutinize the cases in which at least one extended domino is present. This phase differs from the previous activity in that it emphasizes that the value of n does not affect this aspect of the problem. Thus, the number of the Padovan tiles with at least one extended domino will be P_{n-1}, since the value is always one unit greater than that contained in the Padovan tiling.

Validation phase

Validation should be divided into cases. For identity validation, you have the following cases:

Case 1. If n is even, then there will be a case in which the domino is not used, so the number of pieces will be P_{n-1}.

Case 2. If n is odd, then there will be a case, where you do not use dominoes, so the number of pieces will be P_{n-1}.

Institutionalization phase

During this phase, the teacher assumes a central role and engages the students in a discussion of their mathematical analyses. Furthermore, the primary objective of the activity is clarified, which is to derive identity 2. For a better understanding of the identities seen, look at **Table 1**, where one represents the black square of size 1×1, two represents the blue domino of size 1×2 and three represents the gray extended domino of size 1×3. Based on theorem 1, it was possible to perform the Padovan's combinatorial interpretation for the first terms.

Table 1. Interpretation of the Padovan board (prepared by the authors)

$P_2=1$	$P_3=2$	$P_4=2$	$P_5=3$	$P_6=4$	$P_7=5$	$P_8=7$	$P_9=9$	$P_{10}=12$
2	12	22	23	222	223	2222	12222	22222
	3	13	32	33	232	1223	3222	13222
			122	123	322	1232	2322	12322
				132	1222	1322	2232	12232
					133	233	2223	12223
						323	1332	3322
						332	1323	3232
							1233	3223
							333	2332
								2323
								2233
								1333

The research has provided a platform for an in-depth exploration of the combinatorial aspects of the Padovan's numbers and has successfully rekindled an examination of mathematical identities. This endeavor was accomplished by employing the didactic situation theory's teaching framework and the didactic engineering research methodology. Through this marriage of pedagogical theory and research methodology, two problem scenarios involving the Padovan's combinatorial model have been meticulously developed. These scenarios now offer pedagogical strategies that hold promise for application in initial teacher training programs, catering to the study of mathematics' historical context.

It's worth noting that the remaining phases of didactic engineering remain unaltered. The study has considered the potential behaviors exhibited by students and contemplated any epistemological hurdles that might surface during the classroom implementation.

CONCLUSIONS

Mathematical knowledge plays a pivotal role in comprehending various subjects within the academic realm. Consequently, it becomes imperative to tap into students' intuitive faculties during their initial educational journey, enabling them to grasp the subject matter and its practical applications (Masolanorma & Allevato, 2019).

Hence, it becomes necessary to shift away from the mere execution of algorithms as problem-solving tools since this approach often hinders students from truly grasping the underlying significance of the material they are learning. It's important to note that an algorithm is defined as "a set of meticulously outlined, step-by-step instructions to be followed by the student until the desired outcome is achieved" (de Araújo, 2020, p. 22).

Rocha and Aguiar (2012) further emphasize the need for an instructional paradigm shift, incorporating pedagogical knowledge into the training processes. This amalgamation of theoretical knowledge and effective teaching methods contributes significantly to enhancing students' comprehension and learning experience.

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