Pedagogical Research
2024, 9(3), em0208
e-ISSN: 2468-4929
https://www.pedagogicalresearch.com

MODESTUM

Review Article **OPEN ACCESS**

State of the art on the Leonardo sequence: An evolutionary study of the epistemic-mathematical field

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Citation: Mangueira, M. C. d. S., Alves, F. R. V., Catarino, P. M. M. C., & Spreafico, E. V. P. (2024). State of the art on the Leonardo sequence: An evolutionary study of the epistemic-mathematical field. *Pedagogical Research, 9*(3), em0208. https://doi.org/10.29333/pr/14476

ARTICLE INFO

Received: 08 Aug. 2023

Accepted: 10 Apr. 2024

ABSTRACT

This work is a segment of an ongoing doctoral research in Brazil. The Leonardo numbers and the Leonardo sequence have gained attention from mathematicians and the academic community. Despite being a relatively new sequence within mathematical literature, its discussion has intensified over the past five years, giving rise to other branches, with contributions and associations to other topics in mathematics. Thus, the aim of this study was to construct and present the state of the art of the Leonardo sequence, considering its historical aspects and highlighting works on its evolutionary process in the epistemic-mathematical field, regarding its generalization, complexification, hyper complexification, and combinatorial model during the last five years (2019-2023). The methodology used was a bibliographic study, where the state of the art was carried out through the mapping of publications on the subject. Twenty-four research works related to the key descriptors “Leonardo sequence”, “Leonardo numbers”, “complexification”, “generalization”, “hybrids”, and “combinatorial model” were found, cataloged, and discussed. From the analysis of these studies, it is noted that its development in pure mathematics has advanced to other branches and discoveries, and that, albeit timidly, research on the subject has emerged directed towards the field of education, especially in the initial teacher training and, particularly, in Brazil.

Keywords: Leonardo numbers, Leonardo sequence, mathematical evolution

INTRODUCTION

The study of numerical sequences is quite common in the realm of pure mathematics, with the Fibonacci sequence being the most famous and extensively explored. On the other hand, there are other mathematical relationships such as the Leonardo sequence, which despite originating from the Fibonacci sequence and both possessing mathematical properties that draw them closer, the Leonardo sequence is seldom discussed in mathematical literature (Mangueira et al., 2021a, 2021b, 2021c).

Recent studies, dated from 2019 onwards, demonstrate that the Leonardo sequence has been mathematically explored, as well as there is a significant interest among mathematicians regarding the study of the Leonardo numbers. In this perspective, this work constitutes a segment of an ongoing doctoral research in Brazil, which focuses on the discussion and development of the subject.

Given the limited research on the topic, alongside the mathematical interest in exploring the possibilities of the Leonardo sequence, both in the realm of pure mathematics and in mathematics education, we delved into existing investigations. Our aim was to comprehend the current landscape and construct a theoretical framework to broaden its discussion within the academic community, through the construction of a state of the art on the subject. With that said, the objective of this work was to construct and present the state of the art of the Leonardo sequence, considering its historical aspects and highlighting works concerning its evolutionary process in the epistemic-mathematical field, regarding its generalization, complexification, hypercomplexification, and combinatorial model over the last five years.

Research on the state of the art concerning a specific topic has commonly been defined as bibliographic in nature, where certain academic productions in a given field of knowledge are mapped and discussed, aiming to understand which aspects and dimensions have been highlighted in different times and places, as well as how and under what conditions these investigations have been produced (Ferreira, 2002; Palanch & Freitas, 2015).

The Leonardo sequence is a second-order non-homogeneous linear recursive sequence, with initial values equal to one, and its recurrence relation bears great similarity to the Fibonacci sequence. Historically, it is considered a relatively new sequence,

Figure 1. Leonardo Pisano (Fibonacci) (Silva, 2017, p. 3)

and there is no exact confirmation about the author of its creation in mathematical literature. With that said, we start from the following research question: *What are the main characteristics of the research that has been developed around the Leonardo numbers and its sequence?*

Based on the survey conducted, there has been a significant interest in recent years in understanding and exploring these numbers, with investigations discussing their generalization, complexification, hypercomplexification, and combinatorial model within the realm of pure mathematics (Alves & Vieira, 2020; Catarino & Borges, 2019; Diskaya et al., 2023; Kuruz et al., 2021).

Furthermore, it was also possible to identify the Leonardo sequence in educational settings, where some productions within the realm of pure mathematics were adapted for didactic transposition, particularly in the field of History of Mathematics in teacher education programs (Alves et al., 2021; Mangueira, 2022; Mangueira et al., 2021d). These studies bring an identity based on theories and teaching methodologies stemming from the French didactics of mathematics, such as didactic engineering (DE) and the theory of didactic situations (TDS), being developed in pre-service mathematics teacher education programs.

Vieira et al. (2021a, p. 125) state that "although recent in Brazil, studies regarding the state of the art have become indispensable concerning the magnitude of the academic production of the object of study". So, we aim to present in this study the most relevant productions on the Leonardo sequence, bringing forth the aspects most addressed by the authors, perspectives on their investigations, and contributions in the academic sphere. Finally, we provide an analysis of this sequence, highlighting its historical, mathematical, and evolutionary aspects, discussed in the following sections.

THE LEONARDO SEQUENCE

The Leonardo sequence is a linear and recurring sequence, with a relation like the Fibonacci sequence, however, there is little information about its historical origin (Mangueira, 2022). Alves et al. (2020) explain that these numbers were studied by Leonardo Pisano (1180-1250) (**Figure 1**), known as Leonardo Fibonacci, due to the similarity between these sequences. Both have similar recurrence relations, differing only in their initial values and the addition of a "one" value in the case of the recurrence of the Leonardo sequence.

Leonardo Pisano, an Italian mathematician born in Pisa, Tuscany, accumulated experiences in the areas of algebra and arithmetic due to his travels (Mangueira, 2022), becoming well-known for the problem of immortal rabbits, described, as follows: "How many pairs of rabbits will be produced in a year, starting with just one pair, if in each month each pair generates a new pair that becomes productive from the second month?" (Boyer & Merzbach, 2012, p. 174). By constructing an answer to this question, the Fibonacci sequence is developed, a sequence that bears a great resemblance to that of Leonardo.

Matematically, Catarino and Borges (2019) present the Leonardo sequence as a second-order non-homogeneous sequence, with the recurrence relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \geq 2, \quad (1)$$

and present its first two terms as equal to one. Thus, the first ten terms of the Leonardo sequence are given by: 1, 1, 3, 5, 9, 15, 25, 41, 67, and 109. Catarino and Borges (2019) also present another relation, defined, as follows:

$$Le_n = 2Le_{n-1} - Le_{n-3}, n \geq 3, \quad (2)$$

where it is possible to obtain the Leonardo numbers, while preserving their initial values, by transforming it into a third-order homogeneous sequence. Furthermore, a relationship between the Leonardo numbers and the Fibonacci numbers can be found, which can be mathematically described as: twice a Fibonacci number subtracted by one equals a Leonardo number (Mangueira, 2022; Spreafico & Catarino, 2023; Vieira et al., 2023).

The characteristic equation of this sequence is the cubic equation, as follows:

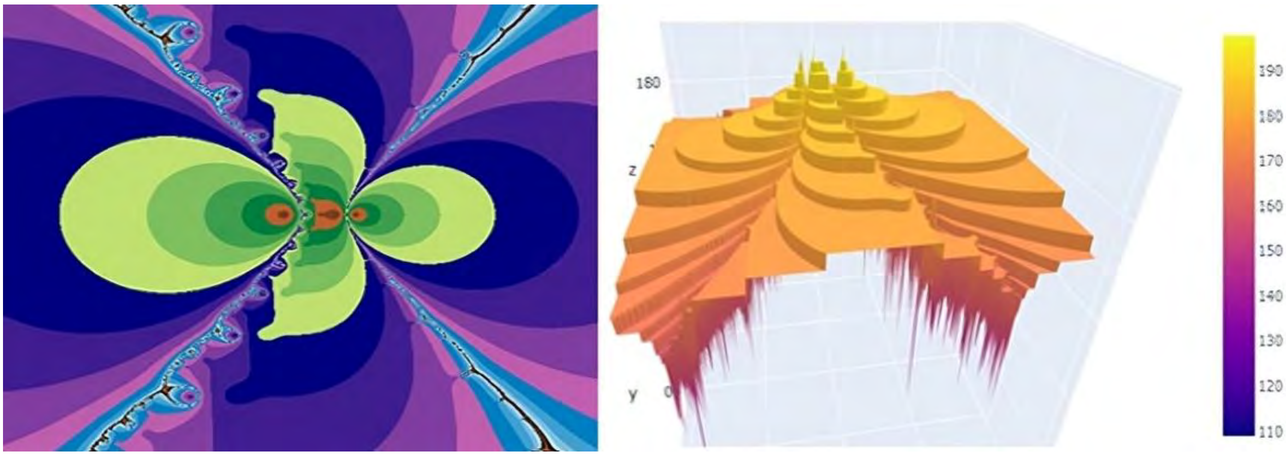

Figure 2. Leonardo's fractal in 2D & 3D (Alves & Vieira, 2020, p. 6)

$$x^3 - 2x^2 + 1 = 0. \quad (3)$$

This equation has three real roots, which are, as follows:

$$x_1 = \frac{1+\sqrt{5}}{2}, \quad (4)$$

$$x_2 = \frac{1-\sqrt{5}}{2}, \quad (5)$$

$$x_3 = 1. \quad (6)$$

One of the roots has an approximate value of 1.61, known as the golden ratio (Shannon, 2019). Alves and Vieira (2020), through the characteristic equation, illustrate a representation of this sequence based on the Newton fractal, providing its visualization in 2D and 3D forms (**Figure 2**), using the Google Colab tool.

Based on the definition presented earlier, some authors have begun to develop other mathematical aspects of these numbers, which we discuss in the following sections, respecting the chronological line of the works within the epistemic and historical-evolutive field of this sequence.

METHODOLOGY: THE STATE OF ART

The state of the art is a research method that maps the academic production on a specific subject, consisting of a bibliographic review that allows for broadening the view, discussion, and perspectives on the topic in question. Let's consider the definition proposed by Romanowski and Ens (2006, p. 41):

> A state of the art can constitute surveys of what is known about a particular area, development of prototypes for research analysis, evaluation of the state of knowledge production in the focused area [...]. It can also establish relationships between previous productions, identifying recurring themes and pointing out new perspectives, consolidating a knowledge area and providing guidelines for pedagogical practices to define parameters for professional development [...]. It can further investigate the multiplicity and plurality of approaches and perspectives, providing clues to clarify and solve historical problems [...]. Similarly, it makes it possible to recognize the importance of research, the significant contributions to the construction of theory and pedagogical practice [...].

Ferreira (2002) reports that a significant amount of this type of research has been produced in Brazil, defined as bibliographic in nature, and that even with its limitations, the state of the art is capable of mapping and discussing academic productions on various types of knowledge, including mathematical knowledge, contributing to its evolution.

In the case of this article, the state of the art was developed based on a bibliographic survey of investigations that discuss the Leonardo sequence, considering works produced between 2019-2023. Within these studies, we sought data such as its historical origin, the evolution of the epistemic-mathematical field, and its application in mathematics education. The steps developed for this research are presented in the scheme of **Figure 3**.

The search for material was conducted in Coordenação de Aperfeiçoamento de Pessoa Superior [Superior Person Improvement Coordination] database and on Google Scholar platform, using keywords associated with topic like "Leonardo sequence", "Leonardo numbers", "complexification", "generalization", "hybrids", and "combinatorial model". Also, social networking sites linked to researchers' profiles, such as ResearchGate and Academia.edu, were consulted.

Figure 3. Research steps (Source: Authors' own elaboration)

We cataloged the found works, highlighting the authors, title, year of publication, nature (whether it is related only to pure mathematics or focused on teaching/education), the journal, and the online research site, in chronological order. The organization of the research is presented in **Table 1**.

Table 1. Research on the Leonardo sequence in last five years

Title	Author/year	Nature	Journal	Website for research
On Leonardo numbers	Catarino and Borges (2019)	Pure mathematics	Acta Mathematica Universitatis Comenianae [Mathematical Journal of the Comenian University]	http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1005
A note on generalized Leonardo numbers	Shannon (2019)	Pure mathematics	Notes on Number Theory and Discrete Mathematics	https://doi.org/10.7546/nntdm.2019.25.3.97-101
The Newton fractal's Leonardo sequence study with the Google Colab	Alves and Vieira (2020)	Pure mathematics	International Electronic Journal of Mathematics Education	https://doi.org/10.29333/iejme/6440
Didactic engineering to teach Leonardo sequence: A study on a complexification process in a mathematics teaching degree course	Alves et al. (2021)	Teaching/ education	International Electronic Journal of Mathematics Education	https://doi.org/10.29333/iejme/11196
Generalized Leonardo numbers	Soykan (2021)	Pure mathematics	Journal of Progressive Research in Mathematics	https://doi.org/10.20944/preprints202110.0101.v1
On Leonardo Pisano hybrinomials	Kuruz et al. (2021)	Pure mathematics	Mathematics	https://doi.org/10.3390/math9222923
Os biquaternions elípticos de Leonardo [Leonardo's elliptical biquaternions]	Mangueira et al. (2021a)	Pure mathematics	Revista Eletrônica Paulista de Matemática [São Paulo Electronic Mathematics Magazine]	https://doi.org/10.21167/cqdvol21202123169664mcsmfrvapmmcc130139
Os números híbridos de Leonardo [Leonardo's hybrid numbers]	Mangueira et al. (2021b)	Pure mathematics	Ciência e Natura [Science and Nature]	https://doi.org/10.5902/2179460X63773
A generalização dos duais e sedenios de Leonardo [The generalization of Leonardo's duals and sedenios]	Mangueira et al. (2021c)	Pure mathematics	Revista Eletrônica Paulista de Matemática [São Paulo Electronic Mathematics Magazine]	https://doi.org/10.21167/cqdvol20ic202123169664mcsmrpmvfrvapmmcc1327
Uma experiência da engenharia didática no processo de hibridização aa sequência de Leonardo [An experience of didactic engineering in the hybridization process of Leonardo's sequence]	Mangueira et al. (2021d)	Teaching/ education	Revista Binacional Brasil-Argentina: Diálogo Entre as Ciências [Binacional Magazine Brazil-Argentina: Dialogue Between Sciences]	https://doi.org/10.22481/rbba.v10i02.9560
Hybrid quaternions of Leonardo	Mangueira et al. (2022a)	Pure mathematics	Trends in Computational and Applied Mathematics	https://doi.org/10.5540/tcam.2022.023.01.00051
Os números híbridos de *k*-Leonardo [*K*-Leonardo's hybrid numbers]	Mangueira et al. (2022b)	Pure mathematics	Brazilian Electronic Journal of Mathematics	https://doi.org/10.14393/BEJOM-v3-n5-2022-61534
Leonardo's bivariate and complex polynomials	Mangueira et al. (2022c)	Pure mathematics	Notes on Number Theory and Discrete Mathematics	https://doi.org/10.7546/nntdm.2022.28.1.115-123

Table 1 (Continued). Research on the Leonardo sequence in last five years

Title	Author/year	Nature	Journal	Website for research
Engenharia didática: Um processo de hibridização e hipercomplexificação de sequências lineares recursivas [Didactic engineering: A process of hybridization and hypercomplexification of recursive linear sequences]	Mangueira (2022)	Teaching/ education	[Master's thesis, Federal Institute of Education, Science and Technology of Ceará]	http://biblioteca.ifce.edu.br/mobile/detalhe.asp?idioma=ptbr&acesso=web&codigo=103094&tipo=1&detalhe=0&busca=0
On hybrid numbers with Gaussian Leonardo coefficients	Kara and Yilmaz (2023)	Pure mathematics	Mathematics	https://doi.org/10.3390/math11061551
Pauli–Leonardo quaternions	Isbilir et al. (2023)	Pure mathematics	Notes on Number Theory and Discrete Mathematics	https://doi.org/10.7546/nntdm.2023.29.1.1-16
A new family of number sequences: Leonardo-Alwyn numbers	Gokbas (2023)	Pure mathematics	Armenian Journal of Mathematics	https://doi.org/10.52737/18291163-2023.15.6-1-13
On the hyperbolic Leonardo and hyperbolic Francois quaternions	Diskaya et al. (2023)	Pure mathematics	Journal of New Theory	https://doi.org/10.53570/jnt.1199465
Hyper-Leonardo hybrinomials	Mersin and Bahsi (2023)	Pure mathematics	Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B-Teorik Bilimler [Eskisehir Technical University Journal of Science and Technology B-Theoretical Sciences]	https://doi.org/10.20290/estubtdb.1226488
Ordered Leonardo quadruple numbers	Nurkan and Guven (2023)	Pure mathematics	Symmetry	https://doi.org/10.3390/sym15010149
On Leonardo sedenions	Ozimamoglu (2023)	Pure mathematics	Afrika Matematika	https://doi.org/10.1007/s13370-023-01065-5
Some new families of generalized *k*-Leonardo and Gaussian Leonardo numbers	Prasad et al. (2023)	Pure mathematics	Communications in Combinatorics and Optimization	https://doi.org/10.22049/cco.2023.28236.1485
On Leonardo numbers and Fibonacci fundamental system	Spreafico and Catarino (2023)	Pure mathematics	Proceedings of the Mathematical Methods for Engineering Applications	https://doi.org/10.1007/978-3-031-49218-1_6
A note on Leonardo's combinatorial approach	Vieira et al. (2023)	Pure mathematics	Journal of Instructional Mathematics	https://doi.org/10.37640/jim.v4i2.1862

We found 24 works on the subject, including 22 scientific articles published in national and international journals, one book chapter, and one master's dissertation. We prioritized only the research already published in open-access journals, discarding works in the pre-print model or with paid access.

We noticed that discussions on the topic become more pronounced from the year 2019 onwards. The listed research encompasses the mentioned key descriptors and mostly focuses on the use of the Leonardo sequence and its mathematical advancement within other branches in the field of pure mathematics (21 research works). Only three research works concerned themselves with bringing the discussion to the realm of mathematics education at the level of teacher training. Although we found works with empirical data, among these, we did not find investigation of the sequence in the field of applied mathematics.

In the next sections, we present the state of the art of the Leonardo sequence, based on the discussion of the listed works related to this sequence, addressing its historical, mathematical, and evolutionary aspects, as well as the authors' perspectives.

A STATE OF ART ON THE LEONARDO SEQUENCE: PANORAMA OF CURRENT RESEARCH

As mentioned earlier, the Leonardo sequence is a numerical sequence that has recently attracted the attention of mathematicians. In this state of the art, we examine its mathematical properties, its relationships with other sequences, and its potential use in mathematics education, based on the works cataloged and listed in the methodology section of this paper.

Although the Leonardo numbers and the Leonardo sequence are relatively recent in mathematical literature compared to the Fibonacci numbers and sequence, they have already inspired several investigations and developments in mathematics. We note that these investigations have become more frequent since the year 2019 and have been expanding, contributing to the construction of a theoretical corpus that has contributed to the development of the sequence and new discoveries about its pattern and properties.

Some of the main mathematical ramifications associated with its study include the analysis of its arithmetic properties, its relation to other sequences, exploration of combinatorial properties, possibilities of its application in computational algorithms, and fractal patterns. A synthesis scheme on the evolutionary process of the Leonardo sequence, highlighting the most relevant works, is presented in **Figure 4**.

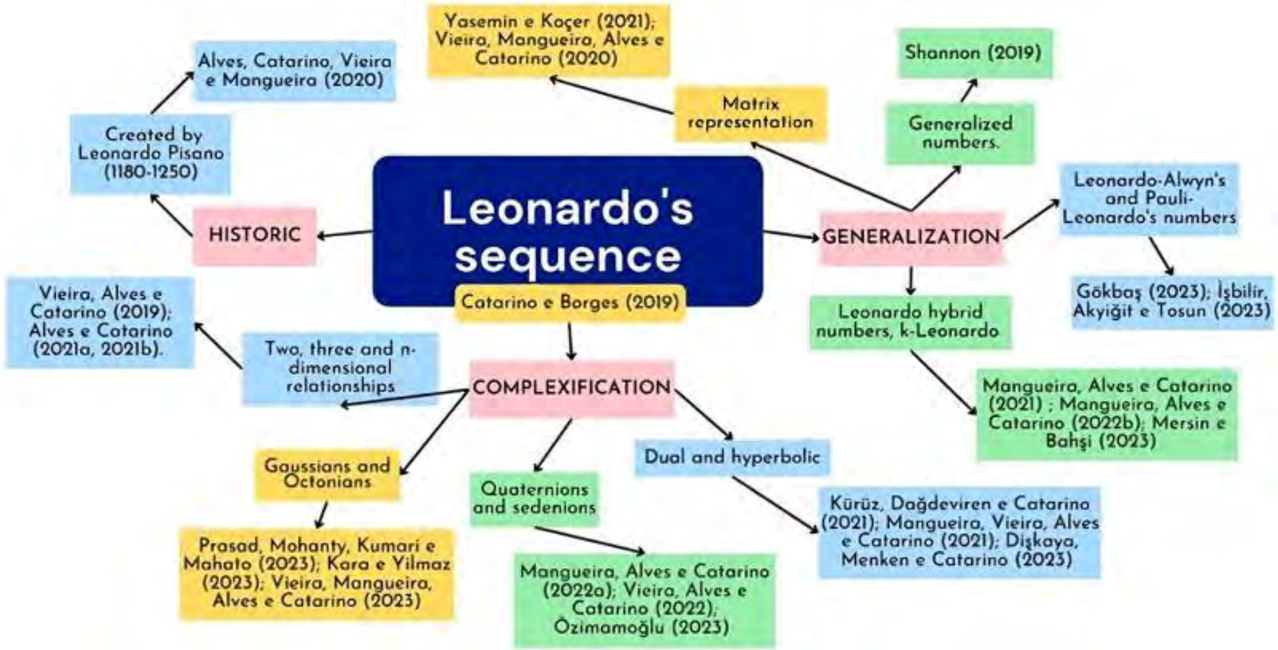

Figure 4. Evolutive process of the Leonardo sequence (Source: Authors' own elaboration)

Although there has been significant progress in a short period regarding the epistemic-mathematical panorama of the Leonardo sequence in the field of pure mathematics, it is noticeable that in terms of its teaching perspective and possibilities for classroom approach, the discussion has still been developing timidly.

In the field of mathematics education, we can highlight works developed especially in Brazil, such as Alves et al. (2021), Mangueira (2022), and Mangueira et al. (2021d). The main characteristic of these research is the development of didactic proposals that use the Leonardo numbers and the Leonardo sequence to introduce mathematical concepts, such as recursion, numerical sequences, arithmetic properties, and mathematical patterns.

In the field of pure mathematics, there is a wide-ranging discussion that began with the work of Shannon (2019). This author wrote a note on the generalized Leonardo numbers in which he examined a generalization of the Horadam numbers (Horadam & Shannon, 1988) and replicated such generalization for the Leonardo numbers.

Regarding the complexification of its terms, Vieira et al. (2019) present the bidimensional relations of Leonardo from its one-dimensional model. In the authors' research, this process occurred around the insertion of the imaginary unit i and the dimensional expansion of its algebraic representations. In other works, by Vieira et al. (2021b, 2021c), the study was continued in which they presented the tridimensional and n-dimensional relations of Leonardo and inserted more imaginary units, there was another dimensional expansion, and some identities involving these numbers were developed and discussed.

Vieira et al. (2020) also brought a discussion about the matrix form of Leonardo, which consists of adding a vector containing its respective initial values. Thus, the authors defined another matrix to be multiplied by the vector, presenting as a product six matrices of the Leonardo numbers and six matrices for the field of non-positive integers, besides addressing some properties related to the matrices presented.

Continuing the mathematical evolution of the sequence, Vieira et al. (2021d) presented a model for the complexification of the Leonardo numbers through hyperbolic numbers, which are a bidimensional extension of real numbers, defined similarly to complex numbers. The generalization of hyperbolic Leonardo numbers was also carried out for terms with non-positive integer indices, and some properties around these numbers were developed.

Other properties were presented by Yasemin and Kocer (2021). The authors used the Binet formula to find identities inherent to this sequence, as well as its matrix form. Additionally, they also present identities linked to the Fibonacci and Lucas sequences.

In the same year, Kuruz et al. (2021) presented a generalization of complex, dual, and hyperbolic numbers, the so-called hybrid numbers, along with the polynomial Leonardo sequence, resulting in the Leonardo hibrinomial numbers. In this work, the authors proved new definitions inherent to these numbers and the polynomial Leonardo numbers, further reaffirming that the Leonardo sequence would have indeed originated by Leonardo Pisano.

Mangueira et al. (2021b) developed the hybrid numbers of the Leonardo sequence, obtaining their matrix form, Binet's formula, and the generating function. Subsequently, by generalizing the coefficients of the recurrence formula of the hybrid Leonardo sequence, Mangueira et al. (2022b) presented the k-Leonardo hybrid numbers.

Continuing the generalization of the Leonardo numbers, Mangueira et al. (2021c) present the duals, by inserting the dual unit into the Leonardo sequence, and the sedenions, which form a 16-dimensional algebra over the Leonardo numbers. In this work, the authors presented the matrix form, the generating function, and the Binet formula of these numbers, building the generalization of the duals and sedenions for terms with non-positive integer indices. In the same perspective, other works such as Vieira et al. (2022) and Ozimamoglu (2023) brought other contributions to the mathematical development of Leonardo's sedenions.

In the mathematical field, there are the biquaternions, or complex quaternions, which are an extension of quaternion numbers and have complex components. In parallel, there are the set of elliptic biquaternions, which encompasses the sets of complex and real quaternions and are presented in Cartesian form. Based on these definitions, Mangueira et al. (2021a) presented the development of the elliptic biquaternions of Leonardo, which constitute a new numerical set.

In Mangueira et al. (2022a), a study was conducted on the complexification of the Leonardo numbers, portraying their quaternions (numbers presented in four dimensions). Mangueira et al. (2022a) associated quaternions with the Leonardo hybrids, resulting in the Leonardo hybrid quaternions, presenting their recurrence, Binet's formula, generating function, and identities related to this association.

Bivariate numbers involve the use of two simultaneous variables, while polynomial studies utilize letters, powers, and coefficients. To associate these two fields with the Leonardo sequence, Mangueira et al. (2022c) presented the bivariate polynomials of Leonardo, formulating a new recurrence, its Binet's formula, its generating function, and its matrix form, contributing to the mathematical evolutionary process of this sequence.

Many branches have emerged in mathematical literature based on the association between the Leonardo numbers and their sequence and other topics in mathematics, as well as other sequences. We have noticed a significant growth in scientific production in the year 2023, where a large set of works on the subject was published.

In literature, some authors have been working on Pauli coefficients associated with a numerical sequence, for example. With this in mind, Isbilir et al. (2023) originated the Pauli-Leonardo numbers, presenting their quaternions. Additionally, Isbilir et al. (2023) investigated these numbers based on the relationships between Pauli-Fibonacci and Pauli-Lucas numbers.

In another work, Gokbas (2023) presents a new family of numerical sequence: the Leonardo-Alwyn numbers. These numbers were dedicated to Leonardo Pisano and Alwyn Horadam and are presented similarly to the Leonardo numbers. The elements of their sequence are derived from other sequences, where the first two terms are equal. Additionally, the properties presented by the author seem to have great potential for future mathematical applications.

Regarding the k-Leonardo numbers, Prasad et al. (2023) introduced new families of generalized k-Leonardo numbers, as well as Leonardo Gaussians, bringing new properties, generating functions, and identities.

Nurkan and Guven (2023) introduce a sequence of quadruple numbers derived from the Leonardo numbers, called ordered Leonardo quadruple numbers. Properties involving these numbers were developed, relating them to Leonardo, Fibonacci, and Lucas numbers. Classical identities were also presented, demonstrated through symmetric and antisymmetric properties of Fibonacci numbers.

Mersin and Bahsi (2023) extended the hybrid numbers of Leonardo to hyper-Leonardo hybrid numbers. Mersin and Bahsi (2023) refer to these numbers as hyper because they are a generalization of the hybrid-Leonardo numbers. Additionally, Kara and Yilmaz (2023) relate Leonardo Gaussian numbers to hybrid numbers, resulting in hybrid numbers with Gaussian coefficients of Leonardo. Furthermore, Vieira et al. (2023) continued the complexification of Leonardo numbers, presenting their Gaussians and octonions.

Diskaya et al. (2023) introduce the hyperbolic numbers of Leonardo and bring to the literature a new definition, called the Francois sequence, related similarly to the sequences of Leonardo and Lucas. Diskaya et al. (2023) developed the Binet formula for these numbers, their generating functions, and presented binomial sums and identities such as d'Ocagne, Catalan, and Cassini.

In terms of works focused on teaching situations, Brazilian authors have been pioneers, associating the numbers and the Leonardo sequence with teaching theories (Alves et al., 2021) in the didactics of mathematics, such as TDS (Brousseau, 2008) and DE (Artigue, 2020). The authors seek, with this approach, to involve the student, the teacher, and mathematical knowledge, valuing the knowledge mobilized by the students and their involvement in the construction of knowledge.

Indeed, Alves et al. (2021) addressed the complexification of the Leonardo sequence, using the sequence in its bidimensional form (dimensional increase of the sequence), with the insertion of the imaginary unit i. In this study, a didactic situation was developed in a Mathematics teacher education course, where students built the bidimensional relationship of the Leonardo sequence based on their initial values, previous knowledge, and intuition. The research used TDS as a teaching theory and DE as a methodology, aiming to assess the intuitive and investigative side of the students. The authors concluded in their study that the students, despite difficulties, achieved the intended result, and the lead teacher conducted an internal validation of the experiment with satisfactory results.

Similarly, Mangueira et al. (2021d) adopted the same teaching perspective, using DE and TDS in an investigative process around the hybrid numbers of Leonardo. Two didactic situations were developed and applied remotely via the Google Meet platform in a teacher education class. The experience allowed for a historical understanding and insight into the mathematical evolution of these numbers.

It is worth noting that Alves et al. (2021) and Mangueira et al. (2021d) were pioneering research in the classroom and are part of a section of Mangueira's (2022) master's thesis. In her thesis, Mangueira (2022) developed didactic situations about various numerical sequences, including the Leonardo sequence, in a teacher education class in the History of Mathematics. The research was conducted at the Federal Institute of Education, Science, and Technology of Ceará, remotely due to the COVID-19 pandemic scenario. Mangueira (2022) conducted an internal validation, indicating that the theories in didactics of mathematics used allowed for important discussions, as well as enabled students to be active in knowledge construction and sparked interest in the topic.

Finally, these were just some of the areas, where the Leonardo numbers and the Leonardo sequence have generated interest and research. As the field continues to evolve, it is likely that new mathematical branches and applications will be discovered and explored.

FINAL CONSIDERATIONS

In this state of the art, we conducted a bibliographic survey on the Leonardo sequence, its history, its numbers, mathematical evolution, complexification, and generalization. To achieve this, we cataloged and discussed works on the subject, both in the field of pure mathematics, which encompasses its evolution in the epistemic-mathematical field, and in the classroom, with a view to the possibility of its teaching in initial training courses.

From the construction of this work, it can be said that the Leonardo sequence is a new sequence in mathematical literature, with its first works published from 2019 onwards. The study of these numbers is constantly progressing, especially in their mathematical evolutionary process. However, it is worth noting that in the field of teaching the topic, research has been developed and published by Brazilian authors, with a contribution from Portuguese authors.

Finally, considering perspectives for future investigations, we recognize a wide range of issues to be explored and deepened in the scope of the Leonardo sequence. Potential areas of interest include the creation of works that expand the mathematical understanding of these numbers, offering visual representations that illustrate their essence. Additionally, there is room for a didactic approach that transforms this content into practical teaching materials, suitable for application in introductory courses for teacher training, especially in the discipline that addresses the study of numerical sequences.

Author contributions: All authors have sufficiently contributed to the study and agreed with the results and conclusions.

Acknowledgements: The authors would like to thank the National Council for Scientific and Technological Development (CNPq).

Funding: This study was partially financed by Portuguese Funds through FCT–Fundação para a Ciência e a Tecnologia, within the projects UIDB/00013/2020, UIDP/00013/2020, & UIDB/00194/2020 and National Council for Scientific and Technological Development (CNPq).

Ethical statement: The authors stated that the study did not require ethics committee approval. The study does not involve live subjects and human beings. The authors further stated that the highest ethical practices of scientific research were followed during the study.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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