A case study of proving by students with different levels of mathematical giftedness

María J. Beltrán-Meneu[1], Rafael Ramírez-Uclés[2], Juan M. Ribera-Puchades[3], Angel Gutiérrez[4], Adela Jaime[4]

[1] Department of Mathematics, Jaume I University, Castelló de la Plana, Spain

[2] Department of Mathematics Education, University of Granada, Granada, Spain

[3] Department of Mathematics and Computer Sciences, Les Illes Balears University, Palma, Spain

[4] Department of Mathematics Education, University of Valencia, Valencia, Spain

mmeneu@uji.es, rramirez@ugr.es, j.ribera@uib.es, angel.gutierrez@uv.es, adela.jaime@uv.es

***Abstract**: We present a case study of proving by three 12–13-year-old students with different levels of mathematical giftedness. After analysing students' proofs, we conclude that: there was a relation on the consistency and the students' levels of mathematically giftedness, being the least consistent the student not mathematically gifted and the most consistent the student with the highest level of mathematical giftedness; the variability was greater in the arithmetical problems; the quality of the proofs produced increased as the level of mathematical giftedness did; the two students with a lower level did better proofs in the arithmetical than in the geometrical problems, while the student with the highest level did not show significant differences between the two areas.*

Keywords: mathematical giftedness; deductive proofs; empirical proofs; arithmetical problems; geometrical problems.

INTRODUCTION

Many authors consider proving a fundamental aspect of mathematical education and emphasise its importance to promote mathematical understanding (Hanna, 2000). In fact, national curricula from various countries, researchers, and educators have made a call to pay more attention to proving (Campbell et al., 2020, NCTM, 2000).

Mathematical proving is recognized to be difficult to teach and learn, and different studies concluded that only a minority of students can build consistent deductive proofs at the end of high school (Gueudet, 2008). However, with instruction, mathematically gifted students (m-gifted students hereafter) may master inference principles, which are precursors to proof, as early as the fifth grade and, in grade nine, their proof processes and intuitive notions of proof show similarities to those of mathematicians (Sriraman, 2004).

M-gifted students are those who show 'a unique aggregate of mathematical abilities that opens up the possibility of successful performance in mathematical activity' (Krutetskii, 1976, p. 77). Their problem solving abilities are unusually higher than those of their peers. Their identification is highly controversial, and we can find in the literature intelligence tests, creativity tests, mathematical achievement and ability tests, although research indicates that multidimensional approaches are the most adequate to identify them (Pitta-Pantazi et al., 2011; Dündar et al., 2016; Yazgan-Sağ, 2022).

A part of the literature on mathematical giftedness (m-giftedness hereafter) has paid attention to the differences between ordinary and m-gifted students, but, to our knowledge, there are few studies analysing differences between the problem-solving processes carried out by students with different levels of m-giftedness.

Some studies analysing such differences are those by Fritzlar and Karpinski-Siebold (2012) in the context of linear patterns and generalization, showing differences in algebraic thinking between m-gifted students and others not m-gifted but with good grades in school mathematics, and those by Leikin et al. (2014, 2017) and Paz-Baruch et al. (2022) on cognitive attributes. However, there are no studies focusing on differences when learning to prove.

Housman and Porter (2003) observed that undergraduate female students with good college mathematics marks produced different types of proofs even during one short time span, exhibiting even the least sophisticated types. In our paper we explore the types of proofs produced by students considering different grades of mathematical talent as a variable of the study. We consider as proofs any mathematical argument raised to justify the truth of a mathematical statement, not only the formal proofs made by mathematicians, in the line of authors like Balacheff (1988), Harel and Sowder (2007), and Fiallo and Gutiérrez (2017).

There is no agreement regarding possible differences on student's performance in making proofs in different areas of mathematics: some works revealed differences between arithmetic and geometry (Healy & Hoyles, 2000, Hoyles & Küchemann, 2000), while others (Buchbinder & Zaslavsky, 2018, Recio & Godino, 2001) did not. There are also few studies on mathematical content that might more likely promote students' proofs (Lin, 2016). In this paper we continue exploring this issue.

We present a part of a larger research aimed to analyse the proving abilities of students with different levels of m-giftedness. It consists of a case study with three 12–13-year-old students who excel in school mathematics (grades higher than 90%) and are nominated by their teachers as students with a very good competence in mathematics, but have different levels of m-giftedness. Our specific research objectives are:

1. To analyse the consistency of the types of proofs produced by each student across the experiment and relate it to the student's level of m-giftedness.
2. To analyse possible differences in the types of proofs produced by each student in arithmetical and geometrical problems.
3. To compare the proofs produced by the three students and relate them to students' levels of m-giftedness.

The results of our study can contribute to the description of characteristics of m-giftedness associated with proving and help teachers and researchers when designing educational interventions that adjust to the diversity of their pupils, allowing to develop the proving competence of m-gifted students according to their mathematical ability.

LITERATURE REVIEW

Mathematical giftedness

There is not a commonly accepted definition of m-giftedness (Paz-Baruch et al., 2022; Yazgan-Sağ, 2022) and since the study by Krutetskii (1976), researchers have deepened in its characteristics. Jaime and Gutiérrez (2014) described various of them based on the characteristics proposed by Freiman (2006), Greenes (1981), Krutetskii (1976) and Miller (1990). These characteristics often differentiate between mathematical abilities, such as high mathematical memory, or atypical problem solving, and general personal traits, like perseverance or interest in challenging tasks (Singer et al., 2016). Related to the abilities to prove, researchers recognise as indicators of m-giftedness the abilities to abstract and generalise (Greenes, 1981, Krutetskii, 1976; Sriraman, 2004), to see mathematical patterns and relationships (Miller, 1990), to use heuristic thinking, and to appreciate mathematical proofs (Sriraman, 2004).

Mathematical abilities relate to the potential to do mathematics (Leikin, 2018). So, excellent grades in school mathematics are not an indicator of m-giftedness, since they may not reflect students' independent mathematical reasoning (Leikin, 2018; Paz-Baruch et al., 2022). Nor does m-giftedness imply excellent grades (Juter & Sriraman, 2011). Leikin (2018, p. 3) includes these ideas in her definition of m-giftedness, indicating that a student is m-gifted 'if s/he exhibits a high level of mathematical performance within the reference group and is able to create mathematical ideas which are new with respect to his/her educational history'. This definition takes into account creativity, considered by many authors as a component of m-giftedness (see also Dündar et al., 2016, and the references therein).

Mathematical proofs

There is a general agreement that the formal texts produced by mathematicians to communicate their results are proofs, but there is an open discussion about the texts produced by students which do not fit the requirement of mathematicians' proofs (Stylianides et al., 2017). We align with Balacheff (1988), Harel and Sowder (2007), Fiallo and Gutiérrez (2017), and other authors in considering as proofs any mathematical argumentation raised to justify the truth of a mathematical statement, not only the formal proofs made by mathematicians.

Since Polya (1945), problems that ask to prove a conjecture are named proof problems. The conjecture may be given in the statement or may have to be found by the solver as part of the solution.

Some researchers described students' work when solving proof problems and identified several types of empirical proofs –using examples as the main element of conviction– and deductive proofs –based on abstract properties and logical deductions– (Asghari et al., 2018, Balacheff, 1988, Harel & Sowder, 1998, Marrades & Gutiérrez, 2000). In many studies, secondary school students were not successful completing deductive proofs, even with instruction (Clements & Battista, 1992). Even when students seem to understand the function of proofs and to recognize that they must be general, they prefer to rely on empirical methods (Hoyles & Küchemann, 2002) and on a few examples for proving a general claim (e.g., Balacheff, 1988, Harel & Sowder, 2007, Healy & Hoyles, 2000).

Mathematical giftedness and proving

The learning and understanding of proofs by m-gifted students has attracted attention of researchers (Housman & Porter, 2003). Sriraman (2004) suggested that m-gifted students may have an intuitive notion of proof and its role in mathematics, even without instruction, and that their processes to construct a proof show similarities to those of mathematicians.

However, Kwon and Song (2007) showed that, although m-gifted students in their study were confident in their abilities and had a balanced idea about the role of proofs, they failed to distinguish between empirical evidence and deductive proofs. Moreover, Housman and Porter (2003) observed that above-average mathematics students produced different types of proofs even during one short time span and even students with prior experience exhibited the least sophisticated types, in line with the results obtained by Harel and Sowder (1998). In our work we want to deepen into this aspect by exploring possible differences in the consistency in the types of proofs produced by students with different levels of m-giftedness.

Solving arithmetical and geometrical proof problems

An issue present when investigating students learning of transversal topics, like proof, generalisation, etc., is whether the specific mathematical content used in the teaching experiments influences students' outcomes and performance. Some studies on students' conceptions of proof revealed differences in performance and treatment of (counter-)examples between arithmetic and geometric contexts (Arcavi, 2003, Healy & Hoyles, 2000, Hoyles & Küchemann, 2000, Zodik & Zaslavsky, 2008), others did not observe differences (Buchbinder & Zaslavsky, 2018). On the other hand, Recio and Godino (2001) suggested that the arithmetical and geometrical content of problems had little influence on mathematical proof capacity (Harel & Sowder, 1998).

Related to students' ability of proving, Lin (2016) observed that there are insufficient studies on the mathematical contents that might more likely promote students' argumentation. In our study we deepen into this issue in the context of students with different levels of mathematical giftedness.

THEORETICAL FRAMEWORK

Levels of mathematical giftedness

We are interested in exploring the context of proving on the three levels of mathematical giftedness considered by Leikin et al. (2014) on their studies on cognitive attributes: super-mathematically gifted students (super m-gifted students hereafter), m-gifted students, and students with expertise in mathematics not generally gifted.

They consider m-giftedness as a combination of general giftedness (IQ>130) and expertise in mathematics (Leikin et al., 2017, Paz-Baruch et al., 2022), where expertise is determined by student's scores in school mathematics and in an assessment test in mathematics. At an operational level, they define expertise in mathematics as high performance in school mathematics, so in our study we consider it as excellence in school mathematics (grades higher than 90%) and being nominated by the teacher as a student with very good competence in mathematics. M-giftedness, giftedness, and expertise in mathematics are interrelated but different constructs, and not any gifted is an expert in mathematics and not any expert in mathematics is gifted (Leikin et al., 2017; Paz-Baruch et al., 2022). Super m-gifted students are defined as m-gifted students nominated by their teachers as having exceptional talent in mathematics, and who displayed exceptional achievements such as membership in national Olympiad teams.

Student's proofs classification

Considering our specific context, we distinguish the following constructs: a *conjecture* is a mathematical statement the veracity of which is doubtful; an *argument* is a verbalization aimed to explain how a conjecture was identified, to convince that it is plausible, or to be part of a proof; a *proof* is a mathematical argumentation, not necessarily formal, produced to justify the truth or untruth of a conjecture (Balacheff, 1988; Harel & Sowder, 2007; and Fiallo & Gutiérrez, 2017); *proof problems* are problems asking to prove a conjecture which may be given in the statement or may have to be found by the solver as part of the solution (Polya, 1945).

Marrades and Gutiérrez (2000) proposed a framework to classify students' proofs focusing on their production processes. The framework is based on the integration of previous categories by Balacheff (1988) and Harel and Sowder (1998), and it is adequate to evaluate students' proving skills along a learning period, although in the context of our experiment, it is necessary to clarify some aspects. We have made a refinement of Marrades and Gutiérrez's framework to correct a gap and some imprecisions:

- Crucial experiment proofs consist of just checking the conjecture in some examples, so we do not consider Marrades and Gutiérrez subtypes analytical and intellectual, since they include the production of abstract statements of definitions, properties, etc.
- Generic example proofs include abstract statements of properties identified after having manipulated some examples, so we do not consider the subtypes of example-based and constructive proofs, as they do not require to do any abstraction.
- Students do not jump from thought experiment proofs (based on getting information from examples) to formal proofs in mathematical language. Instead, they progress by doing deductive proofs that are not fully detached from examples and lack the detail and rigour of formal proofs. To fill this gap in, we have defined the *informal deductive* proofs as deductive proofs expressed in informal, mostly verbal, ways.
- We have reworded some Marrades and Gutiérrez's definitions to include proofs of the untruth of conjectures based on counter-examples.

Taking these points into account, we define the following categories of proofs (Fig. 1):

- *Empirical proofs*: are 'characterised by the use of examples as the main element of conviction' (Marrades & Gutiérrez, 2000, p. 91). Students use examples or relationships observed in them to prove the (un)truth of a conjecture. Depending on the ways examples are selected, there are several types of empirical proofs and, depending on the way the examples are used, the types have some subtypes:
    - *Naive empiricism*: proofs where one or more examples, selected without any particular criterion, are used to show that the conjecture is true or false in them.
        - Perceptual proofs: the examples are used to check the conjecture only by visual or tactile perception.
        - Inductive proofs: the checking of the conjecture includes the use of mathematical elements or relationships identified in the examples.
    - *Crucial experiment*: conjectures are proved (refuted) by showing that they are true (do not work) in a specific, carefully selected, (counter-)example or sequence of examples. Students are aware of the need for generalisation, so they choose the examples as non-particular as possible, although these are not considered as representatives of their family of examples. Students assume that, if the conjecture is true in the chosen example, then it is universally true.
        - *Example-based proofs*: the proof only consists of showing the existence of an example or the lack of counter-examples (for positive proofs), or the existence of a counter-example (for negative proofs).

        - *Constructive proofs*: the proof focuses on the way of getting the examples or counter-examples.
    - *Generic example*: students select one or more specific examples, seen as representatives of their class, and the proof includes abstract statements of definitions, properties or relationships of the family empirically observed after operations or transformations on the examples.
        - *Analytical proofs*: the properties used are stated in a general and abstract way but remaining linked to the specific examples.
        - *Intellectual proofs*: the properties used in the proofs are, at least partially, decontextualized from the examples, since the proofs include some deductive parts in addition to statements based on the examples.
    - *Failed*: proofs lacking the coherence or the detail necessary to assign them to the previous categories but showing signs of an empirical reasoning.
- *Deductive proofs* are 'characterised by the decontextualization of the arguments used, [and] are based on generic aspects of the problem, mental operations, and logical deductions, … to validate [or refute] the conjecture in a general way' (Marrades & Gutiérrez, 2000, p. 93). Examples, when used, are a help to organise the proofs, but the particular characteristics of an example are not considered in the proof. Depending on the level of formalisation, we differentiate three types of deductive proofs:
    - *Thought experiment*: abstract deductive processes supported by previous observations on specific examples used to organise the proof. The examples are not part of the proof, but an aid to find properties and relationships to construct the proof. Depending on the way the proof is organised, we consider:
        - *Transformative proofs*: based on mental operations producing a transformation of the initial problem into another equivalent one. The role of examples is to help foresee convenient transformations.
        - *Structural proofs*: sequences of logical deductions derived from the data of the problem, axioms, definitions, etc. The role of examples is to help organize the steps in the sequence of deductions.
    - *Informal deductive*: abstract deductive processes expressed informally, combining verbal expressions with mathematical language, using statements assumed to be obviously true, etc., and based on mental operations that can be carried out with the help of specific examples or not. The informal deductive proofs may be *transformative* and *structural proofs*, defined like for the thought experiment proofs, but with the examples having a more limited role or even, not being used.
    - *Formal deductive*: abstract deductive processes expressed in a formal way (using mathematical language, justifying all the steps, etc.) and based on mental operations performed without the help of examples. The formal deductive proofs may be

transformative and *structural proofs*, defined like for the thought experiment proofs, but without the presence of examples.
  - *Failed*: proofs lacking the coherence or the detail necessary to assign them to the previous categories but showing signs of a deductive reasoning.

- *Non-codifiable proofs*: when there is no answer, or the answer does not allow characterising it as an empirical or deductive proof.

Failed proofs do not necessarily refer to incorrect proofs since some incorrect proofs can be classified in an adequate type if they show the reasoning characteristics of such type.

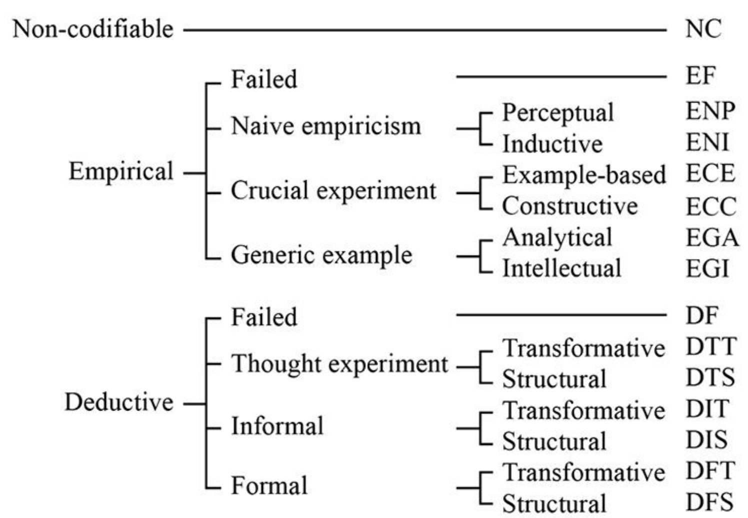


Figure 1. Scheme of the categories of proofs integrating our theoretical framework

METHOD

Description of the sample and the experiment

We are interested in exploring the context of proving in the three levels of m-giftedness considered in the study by Leikin et al. (2014) and defined in the theoretical framework: super m-gifted students, m-gifted students, and non-gifted students with expertise in mathematics. To this aim, we designed a case study of three students, each of which belonging to one of the three groups above, in order to analyse their performance when solving arithmetical and geometrical proof problems. None of the three students had received previous formation in proving.

To preserve the anonymity of the students in the experiment, we refer to them as S1 (or 'the non-gifted student'), S2 (or 'the m-gifted student'), and S3 (or 'the super m-gifted student'). All of them were selected because they excel in school mathematics and are nominated by their mathematics teachers as students with very good competence in mathematics. Student S1 was 13 years old and studied grade 7 (the first grade in Spanish secondary school); he is not generally gifted. Student S2 was 12 years old and studied grade 7, and student S3 was 13 years old and studied grade 8. Both S2 and S3 had been identified as gifted students (IQ>130), had been advanced one academic year, and attended out-of-school programs for general gifted students. These programs consisted of workshops devoted to problem-solving and to introduce non-curricular mathematical topics, such as strategy and logic games, and mathematical activities of various kinds, not focused on teaching to prove. Although both S2 and S3 participated in mathematical competitions some years after the workshop, S2 did not achieve relevant positions, while S3 ranked high in the national mathematics Olympics competitions and was nominated by their teachers as having exceptional talent. So, S3 meets the requirements indicated by Leikin et al. (2014) to be considered a super m-gifted student and S2 a m-gifted student.

The experiment was conducted by using a videoconference platform that offered group and private chats, and a whiteboard with pointing, writing, and drawing tools, available simultaneously to all participants. The students used computers to connect to the video-conference sessions, and smartphones, to share written answers through an instant messaging network, photos of their drawings, etc. One of the researchers acted as the teacher who conducted the experiment, proposed the proof problems and led the discussion, and the other researchers acted as observers. The teacher also used the private chats and the instant messaging network to communicate with each student and know the progress of their solutions. He also used the group chat to promote general discussions or to clarify students' doubts. GeoGebra was available and all students had the necessary expertise to use it when they considered it useful.

The experiment consisted of 6 sessions of about 100 minutes each. Each problem was stated verbally while displayed on the shared whiteboard. Depending on its complexity, the teacher could give a short explanation to promote understanding. After posing a problem, a time slot was left for the students to work individually; during this time, chat messages were exchanged between students and the teacher so that students showed him privately their solutions to the problem. Later, students were asked to explain their solutions to the group, by using the whiteboard if convenient to give graphical or textual support to their explanations. The teacher encouraged students to discuss the others' solutions and, when an answer was incorrect, to check whether it was correct or incorrect and justify it. Finally, the teacher institutionalised the correct answers by using adequate mathematical language. After each session, the researchers discussed the performance of the students and decided how to proceed in the next session.

The sequence of proof problems

We took the problems posed in the experiment from mathematics education literature and mathematical Olympiads training courses, and we adapted some of them to fit the aims of the research and the characteristics of our students. Most problems (Table 1) have several parts, consisting of different related questions. As for the codes, e.g., problem 1.2 was the second problem of session 1 and it has two parts, with part 1.2B also having two parts.

Table 1. Structure of the sessions of the workshop. Problems in grey cells are arithmetical, and those in white cells are geometrical.

Session 1	**Session 2**	**Session 3**	**Session 4**	**Session 5**	**Session 6**
1.2A 1.2B(a, b)	2.1(A-C)	3.1(A-C)	4.1(A, B)	5.1	6.1
1.3(A-D)	2.2(A, B)		4.2(A, B)	5.2(A-C)	6.2(A-D)

The sequence of problems (see the Appendix) alternated a balanced number of arithmetical and geometrical problems. Problem 1.1 was aimed to introduce the students to the sessions and the video-conference platform and has no interest in this paper, so we ignore it.

The several questions of the problems go from specific situations, linked to particular examples, to general contexts, to guide the students in the acquisition of processes of generalisation. They admit both empirical and deductive proofs, to identify different students' behaviours. The statements of some geometrical problems include figures and students were provided with GeoGebra files with dynamic versions of the figures, that they could drag to get more examples and generate conjectures. The GeoGebra measurement tools to calculate lengths, areas or angles allowed students to approach the solutions empirically and check properties. The arithmetical problems allowed the students to easily find particular examples to base their arguments, generate conjectures or counter-examples, and produce solutions as algebraic or verbal generalisations that they had to prove. These problems were ordered by their difficulty to generate examples and the need of deductive proofs.

Analysis of student's outcomes

The data for the analysis were obtained from the video recordings, students' written responses provided in the chats or through photos, and the field notes by the researchers. For each student, we took as units of analysis all their interventions in each problem part and assigned each students' productions to the types of proofs described in the theoretical framework. When a student showed different proofs in the same problem part, we registered the one in the highest type. We made a triangulation of our analysis. Each session was analysed independently by two researchers, focusing on the types of proofs provided by each student. Then, both analyses were reviewed by a third researcher. Finally, a consensus was reached when different types of proofs were assigned to the same student's solution.

RESULTS

In this section we analyse the types of proofs produced by the students in the different problems.

Analysis of the geometrical problems

As examples of the proofs produced in the geometrical problems, we show the answers to problem 5.1. This problem induced rich answers representative of the types of proofs produced by each student in the geometrical problems.

Fig. 2 shows S3's answers. To answer the first question (Fig. 2a), he took mentally out the common factor π and obtained the obvious equality $AB+BC+CD = AD$. In the second question (Fig. 2b), S3 used a decontextualized argument since he was able to generalise the problem to n points in the diameter without the need of drawing particular points. Therefore, this is a deductive informal structural proof (DIS).

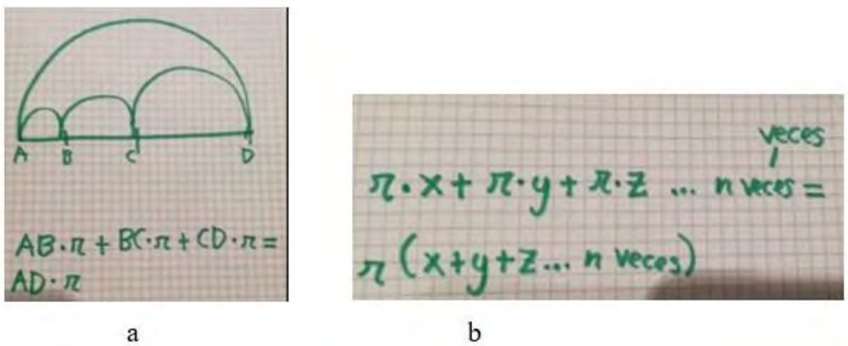


Figure 2. S3's answers to problem 5.1

S1 and S2 gave naive empirical proofs. S1 made visual comparisons by dragging points in the GeoGebra construction, providing empirical naive empiricism perceptual proofs (ENP):

> S1 [chat response]: I think that the red path is longer because if you move one of the points, like E, to A with GeoGebra, the red line occupies more.

> S1 [verbal answer]: I think that both are the same [length] because if, for example, you move all the points with GeoGebra to A or B, the red line is always just on [superposes] the blue one, or vice versa.

S2 measured the semicircles with the Length tool of GeoGebra, so he made an empirical naive empiricism inductive proof (NEI):

> S2: I think that they are the same, but because there is a tool [in GeoGebra] that is the length, I have used it to measure [...] so I have added everything [...]

Analysis of the arithmetical problems

To exemplify the proofs produced by the students in the arithmetical problems, we show the solutions to several problems. Each student produced a quite large diversity of types of proofs in the arithmetical problems, although no problem induced by itself such a diversity, so we present the proofs produced in several problems by each student.

S1 and S2 made deductive proofs in problem 4.2A:

> S1: It is odd because it would be like adding that odd number the same number of times, so it is odd. If it were to add it the number of times of an even number, it would be even: 9 times 9 = 81 but 9 times 8 = 72.

S1 made a transformative thought experiment (DTT) since he transformed n^2 into the sum of n times n and used general properties and deductions based on specific examples.

> S2: Odd, because when you multiply odd numbers, it gives an odd number. If n is odd, then n^2 is an odd number since if you multiply an odd number by another odd it gives odd.

Student S2 made a deductive informal structural proof (DIS) based on known general properties of odd numbers.

In the other arithmetical problems, S1 and S2 only made empirical proofs, although S2 produced more elaborated types. Student S3 only produced three empirical proofs, the others being deductive. So, there is a difference in students' styles of proofs, which may be well exemplified by comparing their answers to problem 4.1A:

> S1: They would be all multiples of three. 0 + 1 + 2 = 3, 1 + 2 + 3 = 6, 2 + 3 + 4 = 9. If we start with the first example, we could follow the sequence and each result would be the previous addition plus 3.
>
> Teacher: Very well. So, which numbers are the sum of three consecutive natural numbers?
>
> S1: All multiples of 3.

Student S1 began by checking a sequence of examples, that let him generalise the results to all multiples of 3. He based his argument on the recursive relationship he had observed (each addition is the previous addition plus 3) and generalised it to give a general answer. S1 did not prove the recursive relationship, so this is an empirical generic example intellectual proof (EGI).

Student S2 began solving the problem by trying several sets of consecutive numbers, all their sums, by chance, being multiple of 6. Then, he verbalised his answer:

> S2: They are multiples of 6, from what I am seeing.
>
> Teacher: Are you sure? If I give you a multiple of 6, can you tell us which are the three consecutive numbers?

S2: The result of the addition is the same if you multiply the number in the middle by 3. For example, 2 + 3 + 4 = 9.

Teacher: Note that the question I asked is the reverse. Which numbers can be written as [the sum of] three consecutive [numbers]?

S2: Aaahhhh. So, they are multiples of 3.

Student S2, reacting to the teacher's comment, noticed that the decomposition not only worked for multiples of 6, but also for multiples of 3, based on the example shown (2+3+4=9). Therefore, S2 considered some specific examples and induced the property, stating it in an abstract way but without any argument to prove it apart from the examples themselves. So, S2 produced an empirical crucial experiment example-based proof (ECE).

Student S3 found a solution and a procedure to calculate the addends:

S3 [chat response]: Given three consecutive numbers, the middle one multiplied by 3 gives the sum of these three numbers. So, given a natural number, if it is divisible by 3, it can be decomposed into three numbers like this: n/3, the previous and the next. Given that number [a multiple of 3], dividing it by three, it will give us, of those three consecutive numbers, let's say the middle one, if we put them in order, because [...] in one session we saw that (n - 1) + n + (n + 1) was 3n. So, just divide by three, the previous, and the next.

S3 proved his solution by resorting to previous knowledge and following a deductive informal structural proof (DIS).

The most elaborated level of S3's proofs (DIS in most cases) may also be observed in his answers to problem 6.2, which are representative of his way of reasoning. In part 6.2A, all students did empirical crucial experiment example-based proofs (ECE), since each of them looked for a pyramid whose apex was not multiple of 4, presenting it as a counter-example.

S3 started his solution to part 6.2B making a wrong proof, but then he changed his approach and produced a deductive informal structural proof (DIS):

S3: If the numbers in the base row are, if the first number is n, if the second, let's say, is n+2, then the third should be n+4. Why? Because if from the first to the second [number] there are 2, that difference has to be the same to the third [number]. [He wrote in the blackboard of the video-conference]

4n+8

2n+2 2n+6

n n+2 n+4

Teacher: What does 4n+8 mean?

S3: That it is multiple of 4.

Teacher: What do n, n+2, n+4 mean?

S3: That there must be the same difference between the first and second numbers and between the second and third numbers.

Teacher: What did you like to write a while ago?

S3: That, if they are n and n+x, the third [number] should be n+2x.

Teacher: So, would this be true for any difference?

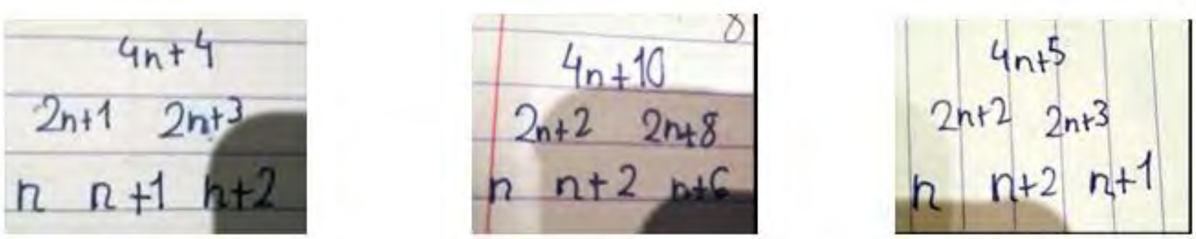
S3: Yes.

Teacher: What is in the other rows?

S3: 2n+x and 2n+3x in the second, and 4n+4x in the third.

Before listening to S3's answer to part 6.2B, S1 had provided an empirical crucial experiment example-based (ECE) by showing two specific numeric pyramids as an example and a counter-example, and S2 had produced an empirical failed proof (EF). They did not answer parts 6.2C and 6.2D.

In part 6.2C, S3 wrote a deductive informal structural (DIS) solution, including an example (Fig. 3a) and two counter-examples to show that, in the base row, the difference between the numbers must be constant (Fig. 3b) and that the numbers must be ordered (Fig. 3c). In 6.2D, S3 used algebraic language to prove that the relationship is true also in the case of integer numbers and he generalised the solution to a pyramid with four numbers in the base row (Fig. 4), thus providing another deductive informal structural (DIS) proof.



a b c

Figure 3. Student S3's answers to parts 6.2C

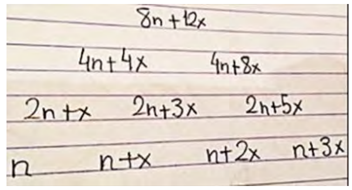


Figure 4. Student S3's answers to parts 6.2C

DISCUSSION

The research objectives stated include to analyse the consistency of the types of proofs produced by the students across the experiment and possible differences between the types produced in the arithmetical and geometrical problems. To answer these objectives, we summarise the types of proofs produced by each student in the geometrical (Fig. 5) and arithmetical problems (Fig. 6) and the frequencies of each type (Table 2).

Fig. 5 shows that student S1 did not produce any deductive proof in the geometrical proof problems, S2 did one, and S3 did eight of them. Three main types of proofs became apparent: empirical naive empiricism perceptual (ENP), based on visual arguments; empirical naive empiricism inductive (ENI), based on measures made with the GeoGebra tools; and deductive informal structural proofs (DIS), consisting of abstract deductive processes expressed as combinations of verbal and algebraic expressions. There was a considerable amount of non-codifiable answers (NC), corresponding to inconsistent responses. The students did not produce proofs in the types between empirical generic example analytical (EGA) and deductive informal structural (DIS). Student S3 was clearly the one producing the best types of proofs, and S1 was the student who did the worst types.

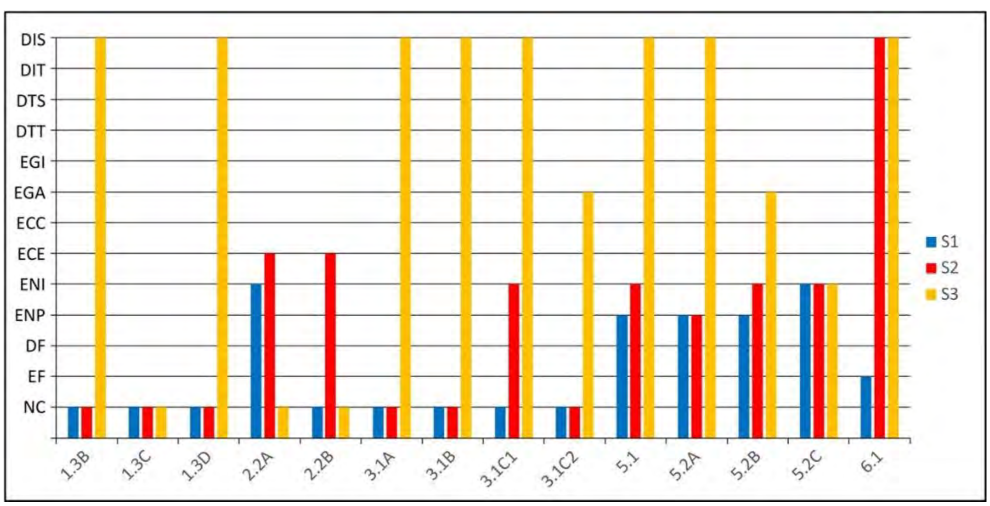


Figure 5. Types of proofs produced in the geometrical problems

All S1's proofs and all but one S2's proofs were empirical. Except on two problems where S2 used empirical crucial experiment example-based proofs (ECE), the other S1 and S2's answers were perceptual or inductive naive empiricism proofs (ENP or ENI). They did not provide codifiable answers in problem 1.3 and some parts of problem 3. In contrast, S3 produced empirical proofs only three times; in particular, he used the inductive naive empiricism (ENI) once and the analytical generic example (EGA) twice. We have differentiated part 3.1c (as 3.1c1 and 3.1c2) because S3 used first a deductive informal structural proof (DIS) to generalise the problem to simple quadrilaterals and next an empirical generic example analytical proof (EGA) for complex quadrilaterals.

Concerning the arithmetical problems, Fig. 6 shows that almost all types of proofs were produced at least by one student, in contrast with the geometrical problems, where the types of proofs were not so many. All three students produced deductive proofs in the arithmetical problems (D--codes), but while S1 and S2 did it only once, S3 showed it on ten proofs, nine structural informal proofs (DIS) and one structural thought experiment proof (DTS). Student S1 did proofs in almost all parts of the arithmetical problems, most of them being the type of empirical crucial experiment (EC-) and S2 only produced six valid proofs, which were of very diverse types, ranging from two empirical example-based crucial experiments (ECE) to one deductive informal structural (DIS) proof. Again, S3 was clearly the one producing the best types of proofs, and now S2 was the student who got the worst results in terms of the number of proofs produced, although there is not a difference between the quality of the proofs done by S1 and S2.

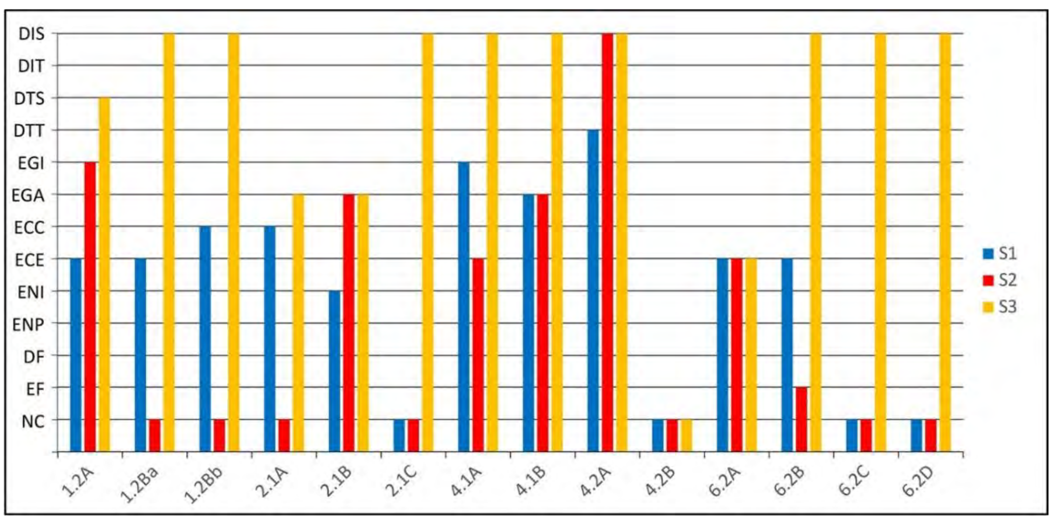


Figure 6. Types of proofs used in the arithmetical problems

Table 2 summarises the number (and percentage) of proofs of each type produced by each student in the arithmetical (A) and geometrical (G) problems. The results show inconsistencies in the types of proofs used, even in the same problem. We observe more variability in the arithmetical problems, where S1, S2 and S3 showed 6, 5, and 4 different types of codifiable proofs, respectively, while in the geometrical problems, S1 and S3 showed 3 different types, and S2 showed 4 types.

Table 2. Number (and %) of proofs of each type produced.

	S1		S2		S3		S1+S2+S3	
	A	G	A	G	A	G	A	G
NC	4 (28,6)	8 (57,1)	7 (50)	6 (42,9)	1 (7,1)	3 (21,4)	12 (28,6)	17 (40,5)
EF	0 (0)	1 (7,1)	1 (7,1)	0 (0)	0 (0)	0 (0)	1 (2,4)	1 (2,4)
DF	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
ENP	0 (0)	3 (21,4)	0 (0)	1 (7,1)	0 (0)	0 (0)	0 (0)	4 (9,5)
ENI	1 (7,1)	2 (14,3)	0 (0)	4 (28,6)	0 (0)	1 (7,1)	1 (2,4)	7 (16,7)
ECE	4 (28,6)	0 (0)	2 (14,3)	2 (14,3)	1 (7,1)	0 (0)	7 (16,7)	2 (4,8)
ECC	2 (14,3)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	2 (4,8)	0 (0)
EGA	1 (7,1)	0 (0)	2 (14,3)	0 (0)	2 (14,3)	2 (14,3)	5 (11,9)	2 (4,8)
EGI	1 (7,1)	0 (0)	1 (7,1)	0 (0)	0 (0)	0 (0)	2 (4,8)	0 (0)
DTT	1 (7,1)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	1 (2,4)	0 (0)
DTS	0 (0)	0 (0)	0 (0)	0 (0)	1 (7,1)	0 (0)	1 (2,4)	0 (0)
DIT	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
DIS	0 (0)	0 (0)	1 (7,1)	1 (7,1)	9 (64,3)	8 (57,1)	10 (23,8)	9 (21,4)
DFT	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
DFS	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
Total	14 (100)	14 (100)	14 (100)	14 (100)	14 (100)	14 (100)	42 (100)	42 (100)

The most inconsistent student was S1, who showed a great difference in performance depending of the types of problems, his proofs being more sophisticated in the arithmetical ones: in the arithmetical problems S1 mainly produced empirical proofs, of the types crucial experiments (EC-) and analytical generic examples (EGA), and also a deductive thought experiment transformative proof (DTT), but in the geometrical problems he only produced empirical failed (EF) and naive empiricism proofs (EN-).

S2 produced two deductive informal structural proofs (DIS), one in each kind of problem, and the amount of non-codifiable answers was similar in both types, but he produced more elaborated proofs in the arithmetical problems, with empirical generic example proofs (EG-) being the most used (28,5%) and empirical naive empiricism proofs (EN-) being the most frequent in the geometrical problems (35,7%).

S3 was the most consistent student since he usually did structural informal proofs (DIS) on both kinds of problems. However, his global performance is better on arithmetic problems, with only one non-codifiable answer, in contrast to the three of the geometric problems. His difficulties are related to a lack of knowledge of algebraic language. S3's variability in the geometrical problems was mainly in the last sessions, where an increment in the difficulty made him move from deductive to empirical proofs.

The global data in Table 2 show that geometrical problems were more difficult for the students than arithmetical ones, in which the three students produced better types of proofs, this difference being more noticeable in S1 and S2.

These results support the conclusion that, although the three students show inconsistencies in the types of proofs produced, there is a relation on the grade of consistency and the students' levels of mathematically giftedness, being the less consistent the non-gifted student and the most consistent the super m-gifted student. The case study also seems to indicate that the dependence of the outcomes on the content area (geometry and arithmetic in our experiments) decreases as the level of giftedness increases.

CONCLUSIONS

In this article, we have presented the results of a case study designed to analyse the behaviour of three students with different levels of m-giftedness –S1 being non-gifted and with expertise in mathematics, S2 m-gifted, and S3 super m-gifted–, when solving arithmetical and geometrical proof problems. None of them had received previous formation in proving.

A contribution of the paper is the framework of our study, a refinement of the classification of proofs by Balacheff (1988), Harel and Sowder (1998), and Marrades and Gutiérrez (2000), including the new category of *deductive informal proofs* for decontextualized deductive proofs which are not expressed with formal mathematical language. This category proved to be important because it is the highest type of proof produced by the participants in our study.

The first research objective was to analyse the consistency of the types of proofs produced along the experiment and relate it with student's level of m-giftedness. In the line of results obtained by other researchers (Harel & Sowder, 1998; Housman & Porter, 2003) we observed that the three students were not consistent, producing different types of proofs even in different parts of the same problem. The super m-gifted student (S3) was the most consistent, producing most times deductive informal structural proofs (DIS), both in the arithmetical and geometrical problems. His empirical proofs can be associated to an increment in the difficulty. The most inconsistent student was the non-gifted student (S1), showing up to six different types of proofs in the arithmetical problems (only one deductive). These results seem to indicate that as the students' level of m-giftedness increases, the more consistent their types of proofs are both on arithmetical and geometrical problems.

The second research objective was to identify possible differences between the types of proofs produced in arithmetical and geometrical problems. The data point to that the dependence of the types on the area decreases as the level of m-giftedness increases. The super m-gifted student (S3) had a similar behaviour on both types of problems, but the non-gifted student (S1) and the m-gifted student (S2) made more sophisticated proofs in the arithmetical problems. In the geometrical problems, the non-gifted student (S1) only was able to produce empirical naive empiricism proofs (EN-), whereas in the arithmetical ones he mainly did the more elaborated empirical crucial experiment proofs (EC-). Most of the m-gifted student (S2)'s solutions in the geometrical problems were also empirical naive empiricism proofs (EN-), while most of his proofs in the arithmetical problems were empirical generic example proofs (EG-). The differences observed between arithmetical and geometrical problems are consistent with the results by Healy and Hoyles (2000) and Hoyles and Küchemann (2000), but differ from those by Buchbinder and Zaslavsky (2018).

Our results also suggest that students who are able to produce deductive proofs produce the same types across mathematical areas, and that arithmetic proof problems seem to promote more sophisticated proofs on students who produce mainly empirical proofs, although more research on this issue is necessary. Some reasons for it may be: (i) the students were more familiar with arithmetic than geometry, since teachers usually devote much more time to teach this area; (ii) the dragging and measurement tools of the GeoGebra may have influenced students to make more empirical naive empiricism (EN-) and crucial experiment (EC-) proofs in the geometrical problems, as pointed by Healy (2000), Mariotti (2002) and Komatsu and Jones (2019). We noticed that some types of proofs were more related to some type of problem: the empirical naive empiricism perceptual proofs (ENP) were only produced in geometrical problems, and the empirical crucial experiment (EC-) and generic example (EG-) proofs were more frequent in the arithmetical problems. Respect to deductive proofs, the thought experiment (DT-) was produced only in arithmetical problems.

Regarding the third research objective, focused on comparing the proofs produced by the students and relating them to their levels of m-giftedness, we observed different levels of proficiency, since most proofs by the non-gifted student (S1) and the m-gifted student (S2) were empirical, while the super m-gifted student (S3) showed from the very beginning his capacity to produce deductive proofs (64% of his proofs). The less able student was the one not gifted (S1). In general, he was the slower student completing the solutions and, on many occasions, he did not give coherent proofs. He was the one who showed more variability in the types of proofs produced, both in the arithmetical and in the geometrical problems. According to theories of creativity, a variety of types of proofs is a symptom of highly creative minds. However, throughout the article, when we refer to variability, we refer to the use of different types of proofs in terms of Marrades and Gutiérrez's (2000) framework, not to different ways to approach a problem. His proofs were more sophisticated in the arithmetical problems although most of the times were of empirical type. The m-gifted student (S2) also did better proofs in the arithmetical problems, so arithmetic may be more adequate to promote the improvement of students' proving abilities. The ablest student was the super m-gifted student (S3) and had a similar behaviour on arithmetic and geometric problems, using mostly deductive informal structural proofs (DIS). He did not reach the deductive formal type (DF-) because of a lack of instruction on mathematical language and the verbal character of most responses. He was the fastest in giving solutions and he used to generalise from the very beginning, even when we did not ask for it.

Our conclusions are based on a case study with three students solving specific proof problems, so the results presented do not pretend to be generalisable. However, they contribute to the development of instructional practices to create opportunities for m-gifted students to improve their abilities to do proofs. The analysis of the types of proofs produced allows designing interventions adjusted to the diversity of these students and may help them develop their proving competence. The study also contributes to the description of characteristics of m-giftedness associated with proving, based on the types of proofs produced by students in arithmetic and geometric contexts and the consistency of the types along a sequence of proof problems.

The sequence of problems presented, aimed to introduce secondary schools students to the learning of proving and proofs, can be used by teachers of m-gifted students, but it can also be used in their ordinary classrooms, granting that the teachers guide their students, at least in the first problems, to make them aware of the kind of arguments they are supposed to produce. The arithmetic problems allow students to easily produce examples that can direct their attention to discover a general property of numbers. The geometric problems are adequate to be approached with the support of a dynamic geometry software, which will help students to identify properties of the figures evident when dragging, that students would convert into conjectures and then try to prove.

Acknowledgments

This work was supported by the R+D+I National Program of the Spanish Ministry of Science and Innovation under project PID2020-117395RB-I00 (AEI/ERDF, EU).

REFERENCES

[1] Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215–241. https://doi.org/10.1023/A:1024312321077

[2] Asghari, A. H., Dastjerdi, S. T., & Mobarakeh, F. A. (2018). Big blocks of proof. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd PME Conference.* Umeå, Sweden (Vol. 2, pp. 67–74). PME.

[3] Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). Hodder & Stoughton.

[4] Buchbinder, O., & Zaslavsky, O. (2018). Strengths and inconsistencies in students' understanding of the roles of examples in proving. *Journal of Mathematical Behavior, 53*, 129–147. https://doi.org/10.1016/j.jmathb.2018.06.010

[5] Campbell, T. G., Boyle, J. D., & King, S. (2020). Proof and argumentation in K-12 mathematics: a review of conceptions, content, and support, *International Journal of Mathematical Education in Science and Technology, 51*(5), 754–774. https://doi.org/10.1080/0020739X.2019.1626503

[6] Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). NCTM.

[7] Dündar, S., Temel, H. & Gŭnduż, N. (2016). Development of a mathematical ability test: a validity and reliability study, *International Journal of Mathematical Education in Science and Technology*, *47*(7), 1061-1075. https://doi.org/10.1080/0020739X.2016.1153734

[8] Ericsson, K. A., & Lehmann, A. C. (1996). Expert and exceptional performance: evidence of maximal adaptation to task constraints. *Annual Review of Psychology, 47*(1), 273–305. https://doi.org/10.1146/annurev.psych.47.1.273

[9] Fiallo, J., & Gutiérrez, A. (2017). Analysis of the cognitive unity or rupture between conjecture and proof when learning to prove on a grade 10 trigonometry course. *Educational Studies in Mathematics, 96*(2), 145–167. https://doi.org/10.1007/s10649-017-9755-6

[10] Fritzlar, T., & Karpinski-Siebold, N. (2012). Continuing patterns as a component of algebraic thinking – An interview study with primary school students. Paper presented at the *12th International Congress on Mathematical Education (ICME 12)*. Seoul, South Korea.

[11] Greenes, C. (1981). Identifying the gifted student in mathematics. *Arithmetic Teacher, 28*(8), 14–17.

[12] Gueudet, G. (2008). Investigating the secondary-tertiary transition, *Educational Studies in Mathematics, 67*, 237–254. https://doi.org/10.1007/s10649-007-9100-6

[13] Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics, 44*(1–2), 5–23. https://doi.org/10.1023/A:1012737223465

[14] Harel, G., & Sowder, L. (1998). Students' proof schemes: results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *CBMS issues in mathematics education* (vol. 3, pp. 234–283). American Mathematical Society.

[15] Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). NCTM.

[16] Healy, L. (2000). Identifying and explaining geometrical relationship: interactions with robust and soft Cabri constructions. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th PME Conference.* Hiroshima, Japan (Vol. 1, pp. 103–117). PME.

[17] Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396–428. https://doi.org/10.2307/749651

[18] Housman, D., & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students, *Educational Studies in Mathematics 53*, 139–158. https://doi.org/10.1023/A:1025541416693

[19] Hoyles, C., & Küchemann, D. (2000). Year 8 students' proof responses: some preliminary findings from four groups. In T. Rowland (Ed.), *Proceedings of the British Society for Research into Learning Mathematics, 20*(3), 43–48.

[20] Hoyles, C., & Küchemann, D. (2002). Students' understandings of logical implications. *Educational Studies in Mathematics, 51,* 193–223. https://doi.org/10.1023/A:1023629608614

[21] Jaime, A., & Gutiérrez, A. (2014). La resolución de problemas para la enseñanza a alumnos de Educación Primaria con altas capacidades matemáticas. In B. Gómez & L. Puig (Eds.), *Resolver problemas. Estudios en memoria de Fernando Cerdán* (pp. 147-190). PUV.

[22] Komatsu, K., & Jones, K. (2019). Task design principles for heuristic refutation in dynamic geometry environments. *International Journal of Science and Mathematics Education*, *17*, 801–824. https://doi.org/10.1007/s10763-018-9892-0

[23] Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. University of Chicago Press.

[24] Kwon, S., & Song, S. (2007). Views on mathematical proof of able students in the 3rd to 7th grades. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st PME Conference.* Seoul, Korea (Vol. 1, p. 249). PME. https://doi.org/10.1007/BF00311060

[25] Leikin, R. (2018). Giftedness and high ability in mathematics. In S. Lerman (Eds.) *Encyclopedia of Mathematics Education*. Springer, Cham. https://doi.org/10.1007/978-3-319-77487-9_65-4

[26] Leikin, R., Leikin, M., Paz-Baruch, N., Waisman, I., & Lev, M. (2017). On the four types of characteristics of super mathematically gifted students. *High Ability Studies, 28*(1), 1–19. https://doi.org/10.1080/13598139.2017.1305330

[27] Leikin, R., Paz-Baruch, N., & Leikin, M. (2014). Cognitive characteristics of students with superior performance in mathematics. *Journal of Individual Differences, 35*(3), 119–129. https://doi.org/10.1027/1614-0001/a000140

[28] Lin, P. J. (2016). The quality of students' argumentation used in a fourth-grade classroom. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th PME Conference.* Szeged, Hungary (Vol. 3, pp. 211–218). PME.

[29] Mariotti, M. A. (2002). Justifying and proving in the Cabri environment. *International Journal of Computers for Mathematical Learning, 6*(3), 257–281.https://doi.org/10.1023/A:1013357611987

[30] Marrades, R., & Gutiérrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44,* 87–125. https://doi.org/10.1023/A:1012785106627

[31] Miller, R. C. (1990). *Discovering mathematical talent* (ERIC Digest E482). ERIC.

[32] National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. NCTM.

[33] Paz-Baruch, N., Leikin, M., & Leikin, R. (2022). Not any gifted is an expert in mathematics and not any expert in mathematics is gifted. *Gifted and Talented International*. https://doi.org/10.1080/15332276.2021.2010244

[34] Pitta-Pantazi, D., Christou, C., Kontoyianni, K., & Kattou., M. (2011). A model of mathematical giftedness: integrating natural, creative, and mathematical abilities. *Canadian Journal of Science, Mathematics and Technology Education, 11*(1), 39–54. https://doi.org/10.1080/14926156.2011.548900

[35] Pólya, G. (1945). *How to Solve It*. Princeton University Press.

[36] Recio, A. M., & Godino, J. D. (2001). Institutional and personal meanings of mathematical proof. *Educational Studies in Mathematics, 48,* 83–99. https://doi.org/10.1023/A:1015553100103

[37] Singer, F. M., Sheffield, L. J., Brandl, M., Freiman, V., & Kakihana, K. (2016). *Activities for, and research on, mathematically gifted students*. In G. Kaiser (Ed.), *Proceedings of the 13th ICME* (pp. 391–395). Springer.

[38] Sriraman, B. (2004). Gifted ninth graders' notions of proof: investigating parallels in approaches of mathematically gifted Students and professional mathematicians. *Journal for the Education of the Gifted*, *27*(4), 267–292. https://doi.org/10.4219/jeg-2004-317

[39] Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237–266). NCTM.

[40] Yazgan-Sağ, G. (2022). Views on Mathematical Giftedness and Characteristics of Mathematically Gifted Students: The Case of Prospective Primary Mathematics Teachers. *Mathematics Teaching Research Journal, 14*(5), 128–140.

[41] Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*(2), 165–182. https://doi.org/10.1007/s10649-008-9140-6

APPENDIX

SESSION 1

1.2A. Calculate some products of three consecutive natural numbers. Is the product always a multiple of 6?

If the answer is yes, do you think this property holds for all products of three consecutive natural numbers? Why? Justify your answers mathematically.

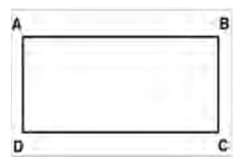
If the answer is no, explain why not. Find a condition that three consecutive natural numbers must hold for their product to be a multiple of 6. Justify your answers.

1.2B. Calculate some sums of three consecutive natural numbers.

a. Is the sum always a multiple of 6?

- If the answer is yes, do you think this property holds for all products of three consecutive natural numbers? Why? Justify your answers mathematically.
- If the answer is no, explain why not.

b. Find a condition that three consecutive natural numbers must hold for their sum to be a multiple of 6. Justify your answers.



1.3A. The polygon ABCD is a rectangle. Draw it in your notebook and draw its two diagonals.

1.3B. What is the relation between the lengths of the diagonals of rectangle ABCD? Justify your answer.

1.3C. Draw another rectangle. Is there the same relation between its diagonals? Justify your answer.

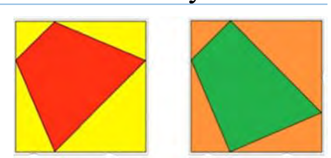
1.3D. Is there the same relation between the diagonals of any rectangle? Justify mathematically that your answer is correct.

SESSION 2

2.1A. How many numbers in the form 2n are there between 100 and 150? Justify your answers mathematically.

2.1B. How many numbers of the form 3n + 1 are there between 100 and 150? Justify your answers mathematically.

2.1C. If a is a fixed number between 2 and 49, How many numbers of the form a ·n + 1 are there between 100 and 150? Justify your answers mathematically.



2.2. Consider the following squares.

2.2A. Is the yellow area the same as the red one? Why?

2.2B. Is the orange area the same as the green one? Why?

SESSION 3

3.1. Let us consider a rhombus ABCD whose diagonals measure AC = 6 cm and BD = 11 cm.

3.1A. Find the area of the rhombus.

3.1B. Let us consider a quadrilateral EFGH, which is not a rhombus, whose diagonals are perpendicular and measure EG = 6 cm and FH = 11 cm. Find the area of this quadrilateral.

3.1C. What can you say about the previous results? Generalise and prove that property.

SESSION 4

4.1A. Find numbers that can be decomposed into the sum of 3 consecutive natural numbers.

- Deduce a general procedure to find numbers of this type. Justify the procedure.
- Have you found all the numbers that can be decomposed as a sum of 3 consecutive natural numbers? Prove that your answer is correct.

4.1B. Consider problem 4.1A with 2 consecutive natural numbers instead of 3.

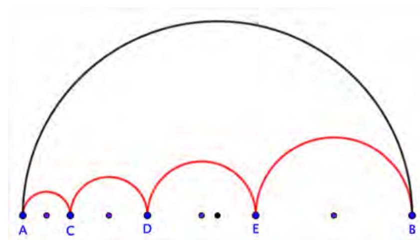
4.2A. If a number n is odd, is its square n^2 even or odd? Prove your answer.

4.2B. If a number n is even, is its square n^2 even or odd? Prove your answer.

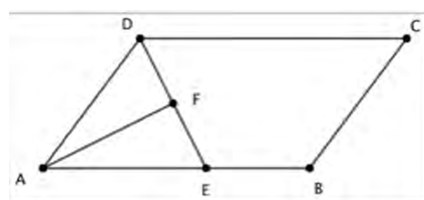
SESSION 5

5.1. The blue path goes directly from A to B and the red path does it through partial paths (from A to C, from C to D, from D to E and finally, from E to B).

- If all paths are semicircles, which path is longer, the blue one or the red one?
- What would happen if there were more points between point A and B? Explain it.



5.2. Build a parallelogram ABCD in GeoGebra. Draw the bisector of <D and denote E the point where it cuts the side AB. Draw the bisector of <A and denote F the point where it cuts the segment DE.



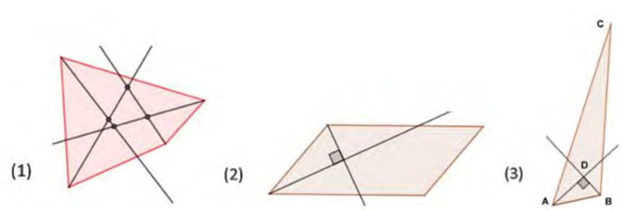
5.2A. What kind of triangle is ADF? Why?

5.2B. What kind of triangle is AEF? What is the relation between triangles ADF and AEF?

5.2C. Modify the parallelogram by dragging its vertices. Do you observe any particularity?

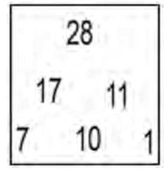
SESSION 6

6.1. In general, the bisectors of two consecutive angles of the quadrilaterals are not perpendicular (1). But, in last session we saw that the bisectors of two consecutive angles of particular quadrilaterals, the parallelograms, are always perpendicular (2).



What can you say about the bisectors of two consecutive angles of a triangle? Can they also be perpendicular? (3).

6.2. Observe the pyramid of numbers. Each number in the middle and top rows is the addition of the two numbers under it. Note that the number in the apex (28) is a multiple of 4.



6.2A. Make another pyramid with the same structure but with other numbers in the bottom row. Is the apex always a multiple of 4?

6.2B. Can you find a rule for the numbers in the bottom to get in the apex a multiple of 4?

6.2C. What if the numbers in the bottom row are natural numbers having a constant difference between them, for example 127, 134, 141?

6.2D. If the numbers in the bottom row are whole numbers, is the apex a multiple of 4?