ELECTRONIC JOURNAL FOR RESEARCH
IN SCIENCE & MATHEMATICS EDUCATION
VOL. 27, NO. 3, 1-19

Teachers' Evaluations of Geometry Problems That Use Visual Arts Contexts

Gloriana González
University of Illinois Urbana-Champaign

Christine N. Rinkenberger
Urbana School District #116

ABSTRACT

This study investigates geometry teachers' evaluations of problems that use visual arts contexts. We ask, *how do teachers' evaluations relate to the four arguments justifying the geometry course?* and *How do teachers draw on the APPRAISAL system to evaluate sample geometry problems from textbooks?* Nine high school teachers were convened in three focus groups. We analyzed the teachers' discussions using systemic functional linguistics and identified 676 evaluations. Ninety percent of the evaluations pertained to the system of appreciation, including teachers' reactions to the problems, their stances about the problems' compositions, and their opinions about the problems' values. The teachers valued problems where students could appreciate the relevance of math in real-world scenarios, engage in math explorations, and become intrigued.

Keywords: curriculum, geometry, art, real-world contexts, teachers' evaluations

Introduction

As a result of mathematics education reform efforts, curricular developers have aimed at designing math problems that will engage students in meaningful learning. Researchers in turn have analyzed the effects of problem-based curricula on students. For example, Boaler (1998) found that an open-ended approach enabled students to develop conceptual understanding and flexibility in solving a novel problem. Ridlon (2009) found that sixth-grade students who experienced a problem-based curriculum improved their achievement and their attitudes towards math. Cai et al. (2013) found that high school students who were exposed to a problem-based curriculum during middle school became good at problem posing. Nevertheless, the quest for research that yields a better understanding of how math curricula can support students' development of problem-solving skills continues to be relevant considering the educational challenges that the pandemic has made salient (Bakker et al., 2021). In the case of geometry instruction, there are curricular studies focusing on students' reasoning opportunities when solving textbook problems, especially proof tasks (Hunte, 2018; Otten et al., 2014). Some research studies also focus on teachers' implementation of the tasks (Sears & Chávez, 2014; Thompson & Senk, 2014). However, there are limited studies regarding teachers' perspectives about the geometry curriculum. Gooya's (2007) study about geometry teachers' perceptions regarding curricular reform in Iran is an exception. Developing an understanding of geometry teachers' perspectives about geometry problems in textbooks is critical for future work that attempts to change the content and the ways in which geometry is taught and learned in school.

In this article, we focus on investigating geometry teachers' perspectives about textbook geometry problems that are situated in visual arts contexts in the U.S. Math reform efforts call for curricular changes that would foster high school students' appreciation for the beauty of mathematics (National Council for Teachers of Mathematics [NCTM], 2018). Proponents of "STEAM" education have pushed to integrate science, technology, engineering, and math with art by extending the "STEM" acronym to include art (Stewart et al., 2021). STEAM-education initiatives emphasize problem-based and maker-space approaches (e.g., Lavicza et al., 2018; Martínez et al., 2022; Quigley & Herro, 2016; Quigley et al., 2020), which are well aligned with interdisciplinary approaches to math education (Bakker et al., 2021). Geometry instruction can offer special opportunities for students to develop such an appreciation for math in the world through connections with art, and U.S. geometry textbooks already attempt to show examples of problems situated in art-based contexts (González, 2020). These contexts, including drawing, architecture, and crafts, have the potential of increasing students' motivation and engagement. Moreover, the selection of problem contexts for students to reinvent math ideas, namely for *mathematizing*, is a key notion within the Realistic Mathematics Framework (Freudenthal, 1991).

Our goal in this study is to learn more about geometry teachers' perceptions and appreciations of geometry problems that integrate math and visual arts. While visual arts contexts have the potential for mathematizing, teachers' use of curricular materials can amplify or diminish this potential (Remillard, 2005). Therefore, understanding what teachers value about geometry problems situated in visual arts contexts can inform curriculum developers who intend to draw on teachers' perspectives in designing new curricula, and teacher educators who wish to build on teachers' knowledge to create learning opportunities to promote STEAM approaches. Theoretically, we rely on the "practical rationality of mathematics teaching" to unpack teachers' perspectives (Herbst & Chazan, 2003, p. 407). Methodologically, we apply linguistics and specifically appraisal theory, to analyze teachers' evaluations of geometry problems (Martin & White, 2005). We situate this study within the traditional justifications for the geometry course (González & Herbst, 2006) with an understanding that current discussions about new goals for the U.S. geometry curriculum do not happen in a vacuum and set expectations for why students should learn geometry.

Theoretical Considerations

The Practical Rationality of Mathematics Teaching

The notion of the practical rationality of mathematics teaching entails that teachers share an understanding of teaching practices related to specific content areas. For instance, geometry teachers in the U.S. have similar curricular resources and teach the same content. As a result, geometry teachers have shared perceptions and appreciations about teaching geometry. Drawing from Bourdieu's (1980) notion of "habitus," Herbst and Chazan (2003, p. 2) propose that teachers' "feel for the game" becomes explicit in conversations among teachers. Further elaboration of this construct has led to the identification of four professional obligations toward math teaching that guide teachers' decisions (Herbst & Balacheff, 2009; Herbst & Chazan, 2011). These obligations are toward the discipline of mathematics (*disciplinary*), *individual* student needs, developing and nurturing *interpersonal* relationships in the classroom, and school-specific rules and regulations (*institutional*). For example, geometry teachers emphasize the development of visualization skills related to geometric thinking (disciplinary obligation), while at the same time, they follow guidelines in their school math curriculum (institutional obligation). To elicit the practical rationality of mathematics teaching, researchers show videos or cartoon-based examples of instances of teaching that they have hypothesized to be typical (Herbst & Chazan, 2011). In teachers' discussions of these examples, they identify implicit norms in teaching that guide their decisions as well as what they value (or do not value) in teaching. The notion of the practical rationality of mathematics teaching facilitates identifying teachers' perspectives about instruction that make sense to mathematics teachers, although they may be difficult for others to understand, even mathematicians. According to Brousseau (1997), the didactical contract connects teachers, students, and content so that when implementing academic tasks in the curriculum, teachers can support student learning of specific content. The examination of the practical rationality of mathematics teaching centers on specific courses of study, such as an algebra or geometry course, so that the instructional demands when teaching specific content become explicit.

Geometry Instruction in the U.S.

Our focus on geometry teachers' perspectives about curricular materials is guided by historical considerations about the geometry curriculum that place special demands on teaching geometry. In the U.S., the geometry course has been a curricular requirement since the 1840s (Quast, 1968) and has overcome attempts to merge it with other courses (Stanic & Kilpatrick, 1992). A study of the justifications for the geometry course in the 20th century reveals that there are four arguments supporting its distinct contributions to the geometry curriculum (González & Herbst, 2006). The *mathematical argument* implies that geometry students will develop their reasoning skills as mathematicians. From this perspective, asking students to make mathematical conjectures and prove those conjectures is the main goal of the course. Proponents of the *intuitive argument* hold that geometry uniquely allows students to establish connections with real-world applications, and geometry curricula instills in students an appreciation for geometric patterns in their surroundings. The *formal argument* justifies the learning of geometry as valuable for developing logical reasoning that can be applied to everyday situations. The intent of teaching proofs using the two-column proof format is for students to apply logical reasoning. Finally, the *utilitarian argument* implies that the geometry course can provide knowledge and skills for students to apply to future employment. These arguments at times present conflicting values as they relate to curricular goals. For example, a geometry problem situated in the context of architecture could develop skills that architects apply in reading a floorplan, thus aligning the problem's context with the utilitarian argument. In contrast, by using geometry to study an architectural piece, students can come to appreciate geometrical patterns in the world, thus fulfilling the learning goals implied by the intuitive argument.

In our study, we focus on teachers' perceptions and appreciations of art-based geometry problems that are aligned with each of these four justifications for the geometry course. Geometry teachers' knowledge of the subject matter, the curriculum, and their students uniquely positions them to assess the potential value of using an art-based approach to geometry instruction. Since the teaching of geometry has been justified by competing discourses, teachers may hold various views about the purpose of learning geometry. Recent research pertaining to the U.S. Geometry curriculum has revealed the need to strengthen the connections between geometry and real-world applications (Desai et al., 2021). Considering new demands for changing the high school math curriculum, Geometry teachers' evaluations of textbook problems that are situated in an art-based context may elucidate their perspectives about these traditional justifications for the geometry course. By applying linguistic techniques for identifying appraisals, we elaborate how teachers' evaluations of art-based contexts used in geometry textbooks reveal teachers' perceptions and appreciations about teaching geometry when a curriculum includes competing justifications.

Teachers' Evaluative Stances and the APPRAISAL System

Systemic functional linguistics is a theory of language that proposes that speakers' meanings can be identified by the choices they make in their talk (Halliday & Matthiessen, 2004). To this end, systemic functional linguists aim to identify taxonomies that map speakers' choices according to the purpose of their talk. These choices are typically displayed as system networks that illustrate a system of linguistic choices. See Table 1 for this formation.

Table 1

System of APPRAISAL by Martin and White (2005)

Appreciations	Affects	Judgements
Reaction	Un/happiness	Normality
Composition	In/security	Capacity
Valuation	Dis/satisfaction	Tenacity
		Veracity
		Propriety

The theory is based on the principle that there are three overarching functions of language, called *metafunctions*, which are simultaneously accomplished in a text (oral or written). The *ideational metafunction* pertains to the goal of communicating experiences and focuses on the content of a text. The *interpersonal metafunction* is about establishing relations, including the function of proposing evaluations. The *textual metafunction* pertains to resources for organizing a text. Martin and White (2005) further develop an understanding of the APPRAISAL system, which describes speakers' choices when making evaluations in the English language. According to the theory, evaluations can be categorized according to the target of the evaluation, which constitutes the domain of ATTITUDE. Specifically, there are three subsystems: APPRECIATIONS for evaluating things, AFFECTS to show feelings, and JUDGEMENTS to evaluate people. Within each subsystem of ATTITUDES, there are other subsystems (see Table 1). The evaluations can be positive or negative, and speakers have more options in some domains to refine their evaluations than in others.

The theory of appraisal has been applied to various contexts, including evaluations in newspaper editorials (Achugar, 2004), communications in political discourse (Ross & Caldwell, 2020), tourist websites (Kaltenbacher, 2006), reflective prose in higher education assignments (Szenes, & Tilakaratna, 2021), and scientific reports (Stosic, 2021). Other studies have relied on systemic functional linguistics to analyze teachers' evaluations of teaching episodes by using other elements in the theory, such as modality (Herbst & Kosko, 2014; Kosko & Herbst, 2012). In this study, we apply Martin and White's (2005) approach to geometry teachers' discussions of problems from geometry textbooks. As artifacts of teaching and learning, textbook problems illustrate how the content of a discipline has been adapted to achieve the goals of schooling; a case of the *didactic transposition* (Chevallard, 1985). Our examination of geometry teachers' evaluations of geometry problems aims to identify the elements of the practical rationality of mathematics teaching that are at play when considering the possibility of integrating geometry and the visual arts. In alignment with the notion of the practical rationality of mathematics teaching (Herbst & Chazan, 2011), what teachers appraise demonstrates their categories of perception, since others without geometry teachers' specialized knowledge may be unable to discern the elements in a math problem that geometry teachers see. Teachers' evaluations constitute their categories of appreciation about what they value (or disregard) in geometry problems situated in visual arts contexts. Our investigation is guided by the goal of understanding what justifications geometry teachers value in problems situated in the visual arts so that future work that attempts to integrate math and the arts will consider teachers' perspectives.

Two research questions frame our examination of teachers' evaluative stances toward geometry problems situated in visual arts contexts. First, *how do teachers' evaluations relate to the four arguments justifying the geometry course (i.e., the formal, intuitive, mathematical, and utilitarian)?* Second, *how do teachers draw on the APPRAISAL system to evaluate geometry problems situated in visual arts contexts?* With the first question, we are interested in identifying the connection between what teachers perceive and what they value about geometry problems situated in visual arts in relation to the arguments justifying the geometry course. With the second question, we focus on the nature of the evaluations and whether and how teachers make use of various resources from the APPRAISAL system. Collectively, we are interested in learning more about aspects of the practical rationality of mathematics teaching, using the case of geometry teachers' evaluative stances toward a set of geometry textbook problems.

Methods

Participants and Data Collection

We conducted three focus group sessions with a total of nine high school geometry teachers. All teachers taught at public high schools in a state in the midwestern region of the U.S. that had adopted the Common Core State Standards for Mathematics and were recruited through announcements to their mathematics departments. The group had a wide range of teaching experience in the classroom, from novice (approximately three years) to veteran (over ten years). Each teacher had taught geometry during their career as a mathematics teacher. The focus group methodology allowed us to understand the practical rationality of mathematics teaching since teachers could share ideas that other teachers might deem acceptable (Herbst & Chazan, 2003). The two authors facilitated the sessions following the model described by Nachlieli (2011). Facilitator 1 (the second author) was in charge of managing the session, conducting the slide show, and asking the guiding questions. Facilitator 2 (the first author) asked probing questions to elicit and contrast various perspectives. Each session was video-recorded and transcribed for analysis. The participants were from seven different schools. See Table 2 for specific information about participants. School size varied from large to small schools.

Table 2

Focus Group Participants

Session	Participants	School	Approximate number of students
1	Curtis Maxwell	Violet HS	1,400
1	Gia Michaels	Honeydew HS	900
1	Charles Rankin	Umber HS	300
1	Emma Smith	Honeydew HS	900
1	Libby Walker	Violet HS	1,400
2	Chloe Baxter	Magenta HS	500
2	Renee Fedderly	Periwinkle HS	500
3	Skyler Beck	Catalina HS	1,400
3	Charity Oberlin	Byzantium HS	2,000

Note. Following institutional review guidelines, we use pseudonyms for participants and institutions.

Each session had four main parts. We started by framing the discussion around recent NCTM documents calling for revisions to the high school geometry curriculum and a clarification of terms pertaining to the strands of mathematical proficiency framework (Kilpatrick et al., 2001). Next, we engaged the participants in a 5- to 10-minute *wish list* activity, where we asked them to identify the characteristics of geometry lessons that promote mathematical proficiency through art and design. The participants had the opportunity to revise the list at the end of the session. In the third part of the session, we showed the teachers nine sample art-based problems from five different geometry textbooks aligned to the Common Core State Standard. See Table 3 for this information. The textbooks, published by mainstream publishers in the U.S., were selected based on the first author's study about the visual arts in geometry textbooks and represented various justifications for teaching geometry.

Table 3

Textbooks Used for Selecting Sample Problems

Acronym	Authors	Year	Title	Publisher
CME	Center for Mathematics Education Project	2013	*CME Geometry Common Core*	Pearson
CPM	Dietiker, L. & Kassarjian	2014	*Core connections Geometry*, 2^{nd} *edition*	College Preparatory Mathematics
Glencoe	Carter et al.	2018	*Glencoe Geometry*	McGraw-Hill
Holt	Larson et al.	2012	*Geometry*	Holt McDougal
Pearson	Charles et al.	2015	*Geometry Common Core*	Pearson

The problems were typical short exercises where students are asked to apply their knowledge of geometric properties to a situation. In this case, we selected situations that involved art, such as a pottery design or an architectural structure. In the sessions, we presented the problems as they were written in the textbooks. We did not have examples of students' solutions to the problems or further information about the problems from the curricular materials. Our intention was for teachers to evaluate the problems with the same information that a student solving the textbook problems would have. The geometry teachers would apply their knowledge and experiences to their evaluations. Table 4 shows a description of the sample problems introduced.

Table 4

Descriptions of the Sample Problems

No.	Art Context	Math Topic	Standard	Argument	Textbook	Page No.
1	Sculpture	Right Triangles	HSG.SRT.B.5	Intuitive	Holt	449
2	Pottery	Circles	HSG.C.A.2	Utilitarian	Holt	707
3	Painting	Circles	HSG.C.B.5	Utilitarian	Glencoe	787
4	Calligraphy	Reflections	HSG.CO.A.5	Intuitive	Pearson	559
5	Drawing	Dilations	HSG.SRT.A.2	Intuitive	CME	323
6	Drawing	Congruence	HSG.SRT.B5 HSG.CO.C9	Mathematical Formal Intuitive	Glencoe	304
7	Architecture	Symmetry	HSG.CO.A.3	Mathematical Intuitive	Holt	616
8	Architecture	Pythagorean Theorem, Law of Cosines	HSG.SRT.B.5	Utilitarian	CPM	446
9	Architecture	Volume	HSG.GMD.A.3	Intuitive	Glencoe	813

Most of the problems were aligned with only one argument for justifying the geometry course, with the exceptions of problems six and seven. These problems had various parts that were aligned with different arguments; specifically, they required students to appreciate math in the world and comprised questions compelling students to complete a proof (formal argument) or propose a mathematical conjecture (mathematical argument). We selected these problems for the focus group sessions since problems with various underlying justifications are typical in geometry textbooks. In addition, there were limited problems in the textbooks aligned with the formal argument, and we wanted all arguments to be represented in the session. The participants answered the following guiding questions: (1) *Do you think that this problem is engaging for your students? Why or why not?* (2) *Does the context provide an entry point for students to learn math? How?* (3*) Is the problem promoting students' mathematical proficiency (i.e., conceptual understanding, procedural fluency, strategic competence, productive disposition, and adaptive reasoning)?* (4) *In what ways would this problem be valuable for teaching your students geometry?* (5) *Would you use this problem in your classroom? Why? If not, how would you adapt this problem?* After presenting each problem individually, we showed the problems in sets of three for the teachers to establish contrasts between them. In the last part of the session, the teachers analyzed the problem-based lessons that we created with different art-based contexts in mind. This analysis is beyond the scope of this article, and we report the results elsewhere (González et al., 2022).

Data Analysis

The sessions were fully transcribed. The transcripts show changes in turns of speech by the speakers, enumerated according to the sequence of participation in the talk. Following Martin and White (2005), the authors independently coded each turn by identifying (1) the appraising item (in bold), (2) the appraiser for that appraising item (italics), and (3) what was appraised (underlined). For example, 44 seconds after introducing problem four, Chloe said, "*I* think that <u>this one</u> is **kind of interesting** to think of someone who's always writing in a mirror image" (Session 2, Turn 117). Here, the appraising item is "kind of interesting," which is used to appraise "this one," a pronoun in reference to problem four. Chloe offered the evaluation, which is signaled when she said, "I think." The comment "to think of someone who's always writing in a mirror" provides a circumstance for considering this problem. Since what is appraised is a thing, the appraisal is an appreciation. Under appreciation, there are three subsystems: REACTION, COMPOSITION, and VALUATION. We coded this appraisal as "reaction," signaling that the problem grabbed her attention. In this case, "interesting" is a positive marker, although lessened by the modifier "kind of." The example illustrates Chloe's use of pronouns to refer to appraised items. In some cases, the appraiser or the appraised item was omitted or implied. For example, when discussing problem seven, Charity said, "So, **potentially engaging**" (Session 3, Turn 202). Here, it is unclear whether the appraiser was the teacher or if Charity implied that the problem could be engaging for the students, so we did not identify the appraiser. However, the appraised item could be recovered from the text since there was a discussion of problem seven.

In our analysis, we also considered *tokens* of evaluation. According to Martin and White (2005), appraisals could be inscribed or invoked. When an appraisal is inscribed, there is a clear link between the appraising item and what is appraised. In contrast, when an appraisal is invoked, there is an indirect connection. We found invoked appraisals when teachers used projected clauses to voice hypothetical classroom-based scenarios. For example, when discussing the three different parts of problem two, Charles said, "*I* really **hate** <u>A, B, C</u>. [Laughter.] Because **it** <u>takes all thought out of the process</u>. 'Here's the procedure.' 'What process to do you want me to memorize with the same problem on the test?' 'Here it is.' 'Memorize that.' 'I'm not going to think.'" (Turns 107-109). Here, "hate" describes a feeling, but it is a token of appreciation. That is, while the appraising item "hate" is a negative inscription of affect, the appraisal invokes a negative appreciation of the problem, a thing. The description of a hypothetical teacher-student exchange about asking for a procedure and giving a procedure in the quote is a token for an appreciation, showing a negative take on the composition of the problem because it lacks complexity. Following Martin and White (2005), we identified cases where projected speech was used to invoke evaluations as tokens because the speakers were invoking attitudes that, as analysts, we had to infer.

Another case where we identified tokens pertains to hybrids in the evaluations. According to Martin and White (2005), there is a hybrid when the inscribed and invoked attitudinal meanings differ. For example, in session one, during the teachers' discussion of problem one, Libby said that the problem was not engaging for students, and Curtis replied with the metaphor, "jump through this hoop." Libby then revoiced Curtis's comment and elaborated by stating, "Yeah. 'Jump through this hoop and show us now and independently master it.'" We identified two appraisals—"jump through this hoop" and "show us now" and "independently master it"—and we coded them as hybrids, combining JUDGEMENTS and APPRECIATIONS. The appraisals suggest a dual target for the evaluation. On the one hand, students need to show that they are capable of solving problems. Therefore, the evaluation is an inscription that makes a judgement about students' capabilities. On the other hand, the evaluation evokes a negative appreciation for the problem, specifically regarding its composition, since problem one lacks complexity. The comment suggests that according to the teachers, if students were to work on problem one, they would apply a procedure without necessarily showing their learning. The hybrid evaluations enabled the teachers in this example to offer an appraisal of problem one through the examination of the capabilities that students would need to solve the problem and demonstrate their learning.

We independently coded all of the transcripts and held subsequent meetings to resolve disagreements and refine the coding. We began by coding session one independently and checked the reliability of identifying appraising and appraised items. In addition, we checked if we agreed on the subsystem of evaluation (AFFECT, JUDGEMENT, or APPRECIATION), the subsystems within it, and if it was positive, negative, or a token. We agreed on 65% of the appraising items and 62% of the appraised items from the coding of the first session. When considering how many types of evaluations we had agreed upon, we found that we agreed on 87%. We realized that we had difficulties identifying appraising and appraised items. At times, it was difficult to recover textual references when speakers used pronouns to identify where tokens and hybrids took place. We continued to independently code the remaining sessions. Overall, when considering the appraising item, appraised item, and type of evaluation, the reliability was 72%, 69%, and 86%, respectively. We discussed our coding decisions, resolved disagreements, and reached a consensus.

Results

We start by reporting the findings regarding the evaluations that the teachers used in relation to the arguments justifying the geometry course to answer the first research question. Next, we answer the second research question by describing the resources from the subsystem of appreciation that the teachers used and their use of hybrid appraisals. Finally, we present the findings pertaining to the positive valuations that the teacher used to evaluate the problems, as these provide emerging evidence for what the teachers valued in the geometry problems situated in art-based contexts.

Teachers' Evaluations of Art-based Problems in Relation to the Arguments Justifying the Geometry Course

Overall, there seems to be evidence that the teachers preferred problems aligned with the utilitarian argument. Table 5 shows the results of all the appraisals offered by problem, aggregating the three sessions.[1]

[1] The results of appraisals per session are 217, 257, and 202 for sessions one, two, and three, respectively.

Table 5

Evaluations per Problem According to the System of APPRAISAL

Problem	Argument	Affect		Judgement		Appreciation		Total		
		Pos	Neg	Pos	Neg	Pos	Neg	Pos	Neg	Total
1	Intuitive	2	3	0	0	16	68	18 (20%)	**71 (80%)**	89
2	Utilitarian	3	2	2	1	75	26	**80 (73%)**	29 (27%)	109
3	Utilitarian	0	0	0	0	12	37	12 (24%)	**37 (76%)**	49
4	Intuitive	9	3	0	1	64	46	**73 (59%)**	50 (41%)	123
5	Intuitive	1	2	3	0	37	42	41 (48%)	**44 (52%)**	85
6	Mathematical, Formal, & Intuitive	0	2	0	1	19	44	19 (29%)	**47 (71%)**	66
7	Mathematical & Intuitive	0	0	0	0	13	14	13 (48%)	**14 (52%)**	27
8	Utilitarian	0	1	3	0	39	15	**42 (72%)**	16 (28%)	58
9	Intuitive	0	0	0	0	11	17	11 (39%)	**17 (61%)**	28
Other		0	1	0	1	9	4	9 (**60%**)	6 (40%)	15
Total		**15**	**14**	**8**	**4**	**295**	**313**	**318**	**331**	**649**

Note. "Other" refers to the general appraisals or combinations of appraisals toward more than one problem. "Pos" stands for positive, and "Neg" stands for negative. These results do not include hybrids.

The four problems with the highest number of evaluations were problem four (123 appraisals), problem two (109 appraisals), problem one (89 appraisals), and problem five (85 appraisals), which had the art-based contexts of calligraphy, pottery, sculpture, and drawing, respectively. Three of these four problems represented the intuitive argument (problems one, four, and five). Five problems (problems one, two, three, six, and eight) triggered evaluations with more than 70% positive or negative appraisals. Problems two and eight were evaluated mostly as positive (73% and 72% of the appraisals, respectively). These two problems represented the utilitarian argument. Specifically, problem two used the context of pottery for an archeologist to find the diameter of a plate by using a broken circular piece. Problem eight used the context of architecture, sharing the case of a person, Lashayia, who wishes to redesign a kitchen according to construction guidelines. In solving the problem, students would have to apply the Pythagorean theorem and trigonometry (i.e., the law of cosines) to determine whether the design meets the guidelines. In contrast, problems one, three, and six were mostly viewed as negative (80%, 76%, and 71% of the appraisals, respectively). These problems represented various arguments. Problem one concerned finding the height of a monument using trigonometry and represented the intuitive argument, as it was an example of how students could use geometry in their surroundings. Problem three was aligned with the utilitarian argument and shared the case of an artisan who had to rely on properties of circles to estimate the area of a mural. Problem six, situated in the context of drawing, would require students to examine a geometric figure that created an optical illusion. The problem represented the intuitive argument by asking students to appreciate the configuration of the visual arts piece. The problem uniquely represented the formal argument by asking students to complete a triangle congruence proof. Additionally, the problem requested students to explain their reasoning for establishing the relationship between two lines in the diagram, thus aligning the question with the mathematical argument. A further analysis of the evaluations proposed provides more nuance to the teachers' preferences.

Teachers' Uses of Resources from the Subsystem of APPRECIATION for Evaluating the Problems

The teachers used various resources from the system of appreciation to evaluate the problems, demonstrating complex analyses. Approximately one-third of the total appreciations offered by the teachers were from each subsystem of APPRECIATION, with slightly more appreciations coded as "composition." See Table 6 for this information.

Table 6

Evaluations of Problems According to the Subsystem of APPRECIATION

	Reaction			**Composition**			**Valuation**			Total
	Pos	**Neg**	**Total**	**Pos**	**Neg**	**Total**	**Pos**	**Neg**	**Total**	
Problem										
1	2	16	18	6	25	31	8	27	35	84
2	27	4	31	27	14	41	21	8	29	101
3	2	6	8	10	20	30	0	11	11	49
4	27	8	35	21	13	34	16	25	41	110
5	16	13	29	8	19	27	13	10	23	79
6	11	10	21	6	26	32	2	8	10	63
7	7	1	8	5	5	10	1	8	9	27
8	18	9	27	5	2	7	16	4	20	54
9	6	4	10	3	9	12	2	4	6	28
Other	2	1	3	1	0	1	6	3	9	13
Total	118 (**19%**)	72 (12%)	190 (31%)	92 (15%)	133 (**22%**)	225 (37%)	85 (14%)	108 (**18%**)	193 (32%)	608

Note. "Other" refers to the general appraisals or combinations of appraisals toward more than one problem. "Neg" stands for negative, and "Pos" stands for positive. These results do not include hybrids.

The findings show that the "reaction" appraisals were mostly positive. In contrast, the "composition" and "valuation" appraisals were mostly negative. These findings suggest that the teachers assumed a more critical stance through detailed analyses of the problems in terms of their characteristics and worth.

As an example of how the teachers' uses of resources from the system of appreciation allowed for a more sophisticated view of a problem, we discuss the teachers' evaluations of problem four. The problem, situated in the context of calligraphy, showed a Leonardo da Vinci illustration, where his handwriting appears in a mirror image. Students were asked to write the mirror image of the sentence, "Leonardo da Vinci was left-handed," and to discuss the possible reasons for his ease of writing mirror images of conventional text (Center for Mathematics Education Project, 2013, p. 559). There were a total of 110 appreciation appraisals for problem four. The reaction evaluations were mostly positive, with 27 positive reaction appraisals (25%) versus eight negative reaction appraisals (7%). The composition appraisals were also mostly positive, with 21 positive appraisals of composition (19%) and 13 negative appraisals of composition (12%). However, the valuation appraisals for problem four were mostly negative, with 25 negative appraisals of valuation (23%) and 16 positive appraisals of valuation (15%).

The positive reaction appraisals of problem four stated that the problem was "cool" "interesting, and "fun." The teachers also used tokens to state that the problem "is going to be intriguing" and that students "mostly heard of da Vinci." Some negative reactions were "why am I" writing backward? and that the problem was "never going to give you buy-in." With these negative evaluations, the teachers anticipated their students' reactions to the problem. The composition appraisals provided a more detailed evaluation of the characteristics of the problem. The positive composition appraisals included tokens to state that the problem "sneaks the math in" and "gets students thinking, writing." In terms of the problem's complexity, Libby stated that it was "something that anybody can try." Nevertheless, the composition appraisals were mostly negative. For example, the teachers stated that writing backwards would be "hard to do" and "a struggle." In addition, they stated that the problem "would take a lot of paper." The teachers noted that the statement of the problem did not include the required mathematical concepts to solve it. The teachers said, "not that it says reflection anywhere in that problem." Specifically, the problem did not discuss the concepts pertaining to reflections, such as the distance between the object and the mirror line, and instead stated "with it being that far from the line or that far." These examples show that by using composition appraisals, the teachers assessed specific characteristics of the problems, such as the language used or the mathematical concepts involved, thus revealing complex evaluations. Some positive valuations were that the problem could be "memorable," a "gateway," and "could lead you into talking about reflections." Nevertheless, some examples of negative valuation were "didn't really teach that much about symmetry," "dumbest thing ever," "not very mathematical," "is about being able to read it backwards," "more about being left handed," "the math's not really there," "wondering the math in it?," "not something that they're going to be graded on," and that writing backwards "might become a detriment later on." Skyler discussed how it was a problem "where you need a bunch of kids to do it and see what they do." The examples of evaluations toward problem four show how the teachers used various resources from the system of APPRECIATION to provide a multifaceted evaluation of the problems. Moreover, by using composition and valuation appraisals, the teachers changed their initially positive evaluation of a problem into a negative view with specific critiques.

Teachers' Uses of Hybrids for Evaluating the Problems

One characteristic of teachers' evaluations was the use of hybrids that combine APPRECIATIONS with AFFECTS or JUDGEMENTS. See Table 7 for this information.

Table 7

Hybrid Appraisals per Session

Session	Affect/Appreciation	Judgement/Appreciation	Total
1	8	4	12
2	11	3	14
3	1	0	1
Total	19	7	27

We found a total of 26 hybrid appraisals, mostly combining AFFECT and APPRECIATION. While the number of these types of appraisals is small, they speak to the nature of teachers' knowledge in terms of how they integrate their knowledge of their students into discussions about the mathematics curriculum. Session two was the session with the most hybrid appraisals (14), and session three was the session with the least hybrid appraisals (1). Hybrid appraisals of affect were more frequent than hybrid appraisals of judgement.

Teachers' hybrid appraisals of AFFECT and APPRECIATION mostly anticipated their students' feelings in relation to the problems. For example, when discussing problem six, Renee said that "*the kids* **hate** <u>triangles</u>," which we coded as dis/satisfaction and as a token of negative appreciation. In contrast, with regard to problem four, Charity said, "I think that *they* will be **curious**, that <u>it</u> actually works." We coded "curious" as a positive appraisal of satisfaction and as a token of positive appreciation for reaction. All of the hybrid appraisals of JUDGEMENT and APPRECIATION described capabilities, while also providing an appreciation for the problem. For example, when discussing writing backwards, Chloe said, "I **can't even do** that." We coded this appraisal as a token of negative judgement of capacity because of the apparent limitations in capabilities for writing backwards. At the same time, we coded this evaluation as a token of positive appreciation for composition, since there is suggestion that the process of writing backwards is complex. Earlier, we provided another example of a hybrid appraisal of judgement and appreciation with the phrase "jump through this hoop." Overall, the hybrid appraisals allowed the teachers to position themselves or their students in terms of their feelings or their characters when evaluating the problems.

Teachers' Positive Valuations of Problems Situated in Arts Contexts

By using reaction appreciations, the teachers stated their initial takes on the problems. By using composition appreciations, the teachers evaluated the sense of balance or the complexity of the problems. Ultimately, the valuation appreciations signaled whether, according to the teachers, the problems were worthwhile or not. That is, would the teachers keep or eliminate a problem and why? The teachers stated 85 positive valuation appraisals. We were interested in learning more about what was appraised with valuation. We listed the items that the teachers appraised, recovering the meanings from the transcription when they used pronouns. See Table 8 for these items and their appraisals.

Table 8

Positive Valuation Appraisals about Geometry Problems Situated in the Visual Arts

Item No.	Problem or alternative	Appraisal	Appraised
3	1A	might have been one that we went over	it [the problem similar to problem one]
204	1A	Like you don't have a choice	[doing word problems the whole day]
205	1A	you can't skip	it [doing word problems the whole day]
206	1	a little nice application of that	this [problem one]
207	1	put, on the worksheet about word problems	that [problem one]
<u>264</u>	1A	they're getting to do that themselves	[measuring the height of the statue in problem one]
268	1	matters	your context
<u>269</u>	1	matters	the measurements, when you're the one out there measuring the thing
<u>42</u>	2	really cool of discovery learning	this [problem two]
<u>43</u>	2	very different than, it's not just, "here's this figure, do the math skill that you do"	it [problem two]
63	2	you can't trace	crazy things that are really large
66	2A	I could incorporate that into the construction class	a problem like that
70	2A	integrity	the building
74	2	when you are talking about cross-curricular	[problem two]
75	2	introduced into like a world history class	having some of this [archaeology]
80	2	would like, more in a cross-curricular thing; where they could see the point better	it [archaeology]
210	2	like a real thing that could potentially happen	it [problem two]
212	2	an actual application that's like a real thing that a human would do normally	this [problem two]
214	2	using, to do the math	the pottery

Table 8

Positive Valuation Appraisals about Geometry Problems Situated in the Visual Arts

Item No.	Problem or alternative	Appraisal	Appraised
231	2	a good one to say, "When am I ever going to use this in life?"	it [problem 2]
232	2	"if you're an archaeologist, you might use it"	it [problem 2]
233	2	always a good utilist for some of those	it [problem 2]
235	2	maybe	that's what archaeologists do
240	2	cool	discussion [of different strategies for part "c"]
256	2	the math is situated in the actual use of the context	[problem 2]
438	2	see where that could be useful	that [problem 2]
439	2A	a little more of tying into something that they might want to do in the future	[a video of an archaeology dig]
440	2	might not have thought about, "Oh, I could use this for that"	[problem 2]
444	2	I could use	that [problem 2]
125	4	like	the concept of this problem
129	4	lets you open up to a lot of people that did stuff like this	it [da Vinci's reference]
185	4	gateway, to further mathematics	problem
186	4A	a compare and contrast	that [a problem with the word spelled backwards]
286	4	I see the mathematical part	in this [problem 4]
299	4	still would include	it [problem 4]
302	4	could lead you into talking about reflections	it [problem 4, part "a"]
306	4A	use, to talk about reflection	it [discussion of problem 4 at the beginning of the lesson]
307	4	does [provide an entry point for students to learn math], if it comes first, not #30	[problem 4]
311	4	memorable enough to remember	[problem 4]
312	4	could be a trigger.	[that da Vinci thing]
313	4	"Oh yeah!"	[problem 4]
465	4	could talk about reflection	[problem 4]
466	4	I think it does [promote mathematical proficiency]	[problem 4]
474	4	reflection is there	[problem 4]
545	4	I see, fitting	that [problem 4]
329	5A	"Ohhh, now I'm doing"	something with my art class and my Geometry class
335	5A	see	the connection [Geometry & art]
336	5A	might see, "Ohhh, they're related"	[Geometry & art]
486	5	get them [students] to think about what similar means	[problem 5]
487	5	could generate some good discussion	[what similar means]
488	5	there's a lot of exploration that could be done	two pictures [problem 5]
498	5	There's gotta be some math involved if you're looking at where the center of dilation is	[problem 5]
501	5	is definitely art design	it [problem 5]
502	5	is definitely used in logos, which is art	resizing an image
504	5	is drawing	it [problem 5]
535	5	there's more math going on in that one	number five

Table 8

Positive Valuation Appraisals about Geometry Problems Situated in the Visual Arts

Item No.	Problem or alternative	Appraisal	Appraised
538	5	has almost all of it	five
546	5	does for sure [have an opportunity to be a bigger problem]	five
517	6	is where you do the math	B [problem 6, part "b"]
518	6	is the naked math	there [problem 6, part "b"]
385	7A	an entry point	[a building known by students]
161	8	relevance	it [problem 8]
162	8	relevant	it [problem 8]
165	8	"Oh, I can see value in why you want to solve"	it [a problem with buy-in]
166	8	more relevance	[problem 8]
387	8	a real-world application	[problem 8]
390	8	interested	construction and building houses and stuff
391	8	"Oh yeah, I know that about this. And now I'm going to solve this triangle to see if it really does fit"	[problem 8]
392	8	I know	the math behind it [problem 8]
394	8	that you could use later to engage the kids	a question [problem 8]
398	8	more prevalent	blueprints
400	8	be good	this [problem 8]
573	8	at least shows that math is used somewhere	it [problem 8]
574	8	a good thing	that [showing that math is used somewhere]
581	8	a little less forced though than some of the other like, quote—unquote, real-world applications	it [problem 8]
584	8	less forced	This [problem 8 vs. another problem about making a ramp following building code]
585	8	I don't think is forced	this one [problem 8]
415	9	more cultural	problem [9]
596	9	want	kids to find the volume of a cone
260	1 & 3	you're actually doing math, I guess	[problems 1 & 3]
261	1 & 3	you're like using numbers and stuff	[problems 1 & 3]
361	4 & 6	have more of an entry point than five	[problems 4 & 6]
362	4 & 6	useful	the context [problems 4 & 6]
583	general	No [does not sound forced]	[another problem about making a ramp following building code]
587	general	"hey there's a ramp"	[another problem about making a ramp following building code]

Note. The appreciation appraisals are numbered by session as they appear in the transcripts: appraisals 1–187 pertain to session one, appraisals 188–418 pertain to session two, and appraisals 419–608 pertain to session three. We use "A" to denote when the appraised item is an alternative to the problem versus the provided item. Appraisals pertaining to usefulness are bolded, to mathematics are italicized, and to discovery are underlined.

The list of appraised items using positive valuation appraisals includes 13 items that are alternative to the problems provided (15% of the positive valuation appraised items). This is relevant because at times the teachers altered the problems that we provided them to discuss the characteristics that they would value in problems. Thirty of the positive valuation appraisals (35% of the positive valuation appraised items) pertained to statements regarding the opportunity to use math in real-world situations. These appraisals were aligned with the utilitarian argument. Eighteen of the positive valuation appraisals (21% of the positive valuation appraised items) pertained to the mathematical content of a problem. For example, the teachers named specific valuable content (e.g., the center of dilation in appraisal 498) or how a problem provides an entry point to the mathematical ideas in a lesson (e.g., reflection in appraisal 402). The attention to the mathematical content of the problem was aligned with the mathematical argument. Six appraisals (7% of the positive valuation appraised items) revealed that the teachers valued problems where students can discover a new idea and apply multiple solution strategies. The attention to discovery-oriented opportunities was aligned with the intuitive argument. Altogether, the teachers' evaluations were related to pedagogical decisions regarding how students come to learn a new idea through problem solving.

The teachers' positive valuation appraisals also included comments regarding the desirable characteristics of geometry problems that are situated in visual arts contexts. Namely, the teachers valued problems that coherently connected the mathematical content and the problem's context (e.g., "a little less forced though than some of the other like, quote—unquote, real-world applications" appraisal 581). The teachers also valued opportunities to use a context to motivate students (e.g., "could be a trigger," appraisal 312). The teachers saw the promise in adapting problems where students would use their knowledge of their surroundings (e.g., choosing a building known to the students as an example of architecture, appraisal 385) and interests (e.g., "construction and building houses and stuff" are contexts for students' interests, appraisal 390). The teachers also stated that cross-curricular problems (e.g., "would like it more in a cross-curricular thing; where they could see the point better," appraisal 80) and problems that include cultural connections (e.g., "more cultural," appraisal 415) are valuable. Overall, the positive valuation appraisals showed that in evaluating the problems, teachers contended with a problem's contexts, the mathematical ideas in the problem, and the pedagogical aspects of how a problem provides opportunities for students to learn geometry. The teachers' valuations revealed elements of the practical rationality of mathematics teaching by illustrating what teachers perceive and appreciate.

Discussion

According to the teachers in this study, the integration of geometry and art is possible. Nevertheless, their appraisals of the sample problems yielded a complex picture of what they value in geometry problems. These appraisals revealed the practical rationality of mathematics teaching in that the teachers considered how the problems target specific math concepts, thus showing their responsibility to portray the discipline of mathematics. Additionally, their appraisals revealed their anticipations of their students' feelings towards the problems—students' likes, dislikes, motivation, curiosity—as well as their capabilities. In the teachers' consideration of their students' perspectives, they were revealing their obligation towards their individual students, which is a component of the practical rationality of mathematics teaching. Additionally, there is an interpersonal component in the teachers' attention to students' feelings because they considered students' motivation and engagement during problem-solving. The teachers' suggestions to adapt problems to their students' local context is another example of how they strived to attend to their individual students' interests. While there was less evidence of how the teachers attended to the institutional obligation when reviewing and making curricular choices, all of the problems were aligned with the Standards as established in their school curriculum. The various "obligations" that mathematics teachers have to fulfill include teaching mathematics and taking care of their students (Bieda et al., 2015; Herbst & Balacheff, 2009; Herbst & Chazan, 2011). Our study is consistent, showing how in geometry teachers' evaluations of problems situated in art-based contexts, they attended to the mathematical content to fulfill their disciplinary obligations and to their students' interests and needs to fulfill their obligations toward individual students. We were able to elicit the practical rationality of teaching by holding discussions of curricular materials and learning more from teachers about what they find appropriate to use in their classroom and why.

In terms of the arguments justifying the geometry, the teachers evaluated the sample problems aligned with the utilitarian argument as valuable. This finding is relevant considering current work that seeks to connect math and design in STEAM education (Bush et al., 2018). It seems that the problems that are aligned with the utilitarian argument would help teachers to answer a question that students often ask, "Why do I need to learn this?" The sample problems with art-based contexts that were embedded in jobs such as being an archeologist, a painter, or an architect positioned the students as someone who must use geometry in their professional practice. Curricular designers who are seeking authentic opportunities for students to appreciate geometry may want to investigate contexts aligned with the utilitarian argument. At the same time, the teachers had positive evaluations towards problems aligned with the intuitive argument for various reasons, including the opportunity to engage in discovery-oriented investigations as well as the chance to appreciate math in the world. Additionally, the teachers' attention to the math content of the problems (or their lack of math content), whether an art-based context provided students an entry point to examine math ideas, and the opportunity for students to display various solution strategies to a math problem were aligned with the mathematical argument. In contrast, there was limited evidence of teachers' preference for problems aligned with the formal argument. In their discussions of the problems, the teachers did not seem to argue in favor of problems that promote opportunities for *doing proofs*, which typically rely on a two-column format for writing statements and reasons that justify the solution of a problem (Herbst, 2002), as one of the goals of the geometry course. Therefore, it seems that teachers are less fond of proof tasks that appear in the geometry curriculum (Otten et al., 2014). Further investigation is needed to see whether and how the formal argument justifying the geometry course is one that teachers continue to support.

Overall, the teachers demonstrated a sophisticated analysis of the sample geometry problems. The linguistic methods revealed a complex picture of their evaluations. With REACTION, the teachers anticipated what their students would say about the problems. With COMPOSITION, the teachers analyzed the problems' complexity and coherence. Then with VALUATION, the teachers showed worthwhile characteristics of the problems. The teachers' evaluations were sophisticated and unpacked a deep analysis of the problems. For example, the teachers valued coherence between the math content and the art-based context. In contrast, the teachers critiqued cultural contexts that trivialized the authenticity of the problem. Therefore, teachers' involvement in designing geometry problems that use art-based contexts would bring attention to important issues that are close to their students.

Conclusion

Many voices are calling for changes to the geometry curriculum (NCTM, 2018) and for investigating interdisciplinary approaches to math instruction (Bakker et al., 2021). There are suggestions for a new geometry curriculum that can leverage students' experiences and use geometry to represent their noticings and wonderings (Desai et al., 2021). The proponents of STEAM have looked into connections between math and art, and most recently, using design thinking as a unifying theme (Bush et al., 2018). With a design thinking theme, curricular developers may be able to target the utilitarian argument by extending students' use of geometry concepts to authentic problems. At the same time, students may be drawing on their intuition to see geometry embedded in real-life spaces and situations.

Geometry teachers possess knowledge of math and their students that become crucial in curricular adaptations. According to the Standards by the Association of Mathematics Teacher Educators (AMTE, 2017) in the U.S., well prepared teachers can anticipate students' thinking, and are also knowledgeable of contexts that shape students' learning. Professional development initiatives can build on that knowledge and promote connections between geometry and art. For example, Verner et al. (2019) report on an initiative in Israel for teachers to use ethnomathematics in crafting tasks for their students to learn more about geometry with cultural artefacts. Work on ethnomodeling also shows the potential for incorporating culturally sustaining teaching practices in geometry (Desai et al., 2022).

Methodologically, our study provides insights about teachers' linguistic choices when making evaluations about curricular materials with their peers. It may be relevant to investigate prospective teachers' evaluations, mirroring other work that focuses on their notions of *good* problems (e.g., Crespo & Sinclair, 2008). We recognize the various limitations of our study, including the small number of participants and the limited number of problems that we showed to the teachers. Nevertheless, our study does not concern a particular curricular approach but rather the more general use of the visual arts in relation to the traditional justifications for the geometry course. To make sustainable changes to the geometry curriculum, it is crucial to understand teachers' perspectives. The quest for meaningful contexts for students to learn and enjoy mathematics can benefit from teachers' knowledge of their students, the curriculum, and mathematics.

This work was supported by the by the Office of the Vice Chancellor for Research at the University of Illinois Urbana-Champaign through a Campus Research Board award to Gloriana González for the project GeoDesiC-Geometry and Design in Classrooms [RB19058].

Gloriana González (ggonzlz@illinois.edu) is a Professor in the Curriculum & Instruction Department at the University of Illinois Urbana-Champaign and University of Illinois Scholar. Dr. González is currently investigating lesson study adaptations to support teachers' implementation of a human-centered design approach for geometry problem-based instruction. She was a middle school and high school teacher in Massachusetts and Puerto Rico.

Christine N. Rinkenberger (crinkenberger@usd116.org) is an assistant principal of Curriculum-Instruction at Urbana High School. She has a master's degree in Curriculum Instruction, as well as Educational Policy from the University of Illinois Urbana-Champaign. Christine was a high school math and ESL teacher prior to becoming a school administrator.

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