# Simplifying algebraic expressions with brackets: Insights into Grade 10 learners' structure sense through a study of their errors

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Developing structure sense is an important part of learning algebra. We investigated learners' structure sense of algebraic expressions involving brackets. This led us to propose the constructs *surface structure sense* and *systemic structure sense*. Using a random sample of 58 Grade 10 learners scoring above 40% in a test, we coded incorrect responses for surface and systemic structure errors. The initial analysis revealed that the presence of more brackets supports surface structure sense. However, learners overgeneralised the presence of brackets to represent multiplication in situations involving subtraction. The arrangement of brackets also led to errors in the order of operations. Further analysis suggested that regular application of procedures on familiar algebraic structures may give the illusion of systemic structure sense. We recommend that the teaching of algebraic expressions must emphasise what the arrangement of an expression means before focusing on how to operate on the expression.

**Contribution:** The research contributes to mathematics teaching by suggesting teaching strategies to improve on learners' understanding of the role of brackets in algebraic expressions, by considering the arrangement of the components in structures.

**Keywords:** structure sense; algebraic expressions; brackets; Grade 10; algebra.

## Introduction and background

Many learners consider the 'essence of algebra' (Kieran, 1992, p. 390) to consist of memorising different procedures and applying these procedures in algebraic structures (Kieran, 1992). Relationships between components such as signs, operations and brackets are, however, affected by their arrangement (Kieran, 1989), thus requiring different procedures for different arrangements. Therefore, learners need to pay attention to the structure of expressions which involves developing algebraic structure sense in order to employ appropriate procedures.

Brackets are not isolated components possessing static functions in algebraic structures. Rather, an understanding of brackets and structure are intertwined: researchers claim that in order to understand brackets, learners must understand structure (Papadopolous & Thoma, 2023; Subramaniam & Banerjee, 2004). There is evidence that simplification of algebraic expressions with brackets and negative symbols is particularly challenging and leads to errors (Linchevski & Livneh, 1999; Vlassis, 2004), because it increases working memory load (Ayres, 2000).

The purpose of this study was to investigate the structure sense of Grade 10 learners based on their interpretation of brackets in algebraic expressions. The research questions guiding the study were: What errors do Grade 10 learners make when simplifying algebraic equations with brackets? What does an analysis of learners' errors reveal about their (systemic and surface) structure sense? Through an analysis of learners' written test responses and their errors, we identified possible misinterpretations involving brackets as a *tool* and brackets as a *signifier* (Vygotsky, 1978) for operations in different types of algebraic expressions. Given that our data set was restricted to written responses with no interview data, we do not make claims about what structure sense learners actually have; rather, we discuss what learners did and did not do which resulted in errors, and infer possible reasons for their responses.

We begin this article with an overview of Vygotsky's sociocultural perspective which served as a theoretical framework for the study. We then discuss the literature on learner difficulties in making the transition from arithmetic to algebra, and manipulation of expressions with brackets. We discuss various definitions of structure and structure sense from the literature and link these notions to the test items used in our study. Details of the coding process of errors are explained, followed by the analysis and findings.

# Literature review and theoretical framework

## Vygotsky's sociocultural perspective

Vygotsky's sociocultural perspective emphasises that human activity is influenced by its cultural settings. Vygotsky (1978) argued that *tools* and *signs* within a cultural setting affect the mental structures and learning processes of individuals, thereby contributing to their learning and development. Further, the relationship between tools and signs encourages higher mental functioning which he referred to as *internalisation* and which reflects a progression from external to internal processes in the learner's thinking.

Mathematics has a unique cultural language consisting of objects and symbols. In different mathematical contexts, an object can be regarded as a tool with a specific purpose and a symbol can be regarded as a sign with an attached meaning (Pimm, 2002). Brackets have a range of different meanings in different mathematical contexts. For example A(2;3) refers to a point A in the Cartesian plane with $x$-coordinate 2 and $y$-coordinate 3. By contrast if we write, $x \in (2;3)$, we refer to the interval of real numbers from 2 to 3 but excluding the boundary values.

Brackets can be considered as a cultural tool which is externally oriented (Vygotsky, 1978), affecting the physical arrangement of elements. Brackets can also be considered as a cultural sign which is internally driven, having meaning within a specific context. For example, in the expression $(x + 3)\ (x - 2)$, brackets are used as a tool for grouping two binomials. However, brackets also signify the multiplication of the binomials. In this sense, the brackets are considered as a sign. Therefore, in this case, the brackets can be considered as both tool and sign. By contrast, in the expression $(x + 3) - (x - 2)$the brackets are used only as a tool for grouping to indicate that $x$ and $-2$ are being subtracted.

When required to simplify algebraic expressions, learners find it difficult to distinguish between using appropriate operations. We focus on the interpretation of brackets as tools for grouping and separation, with particular focus on multiplication and subtraction. Learners may be able to multiply and subtract pairs of terms, but recognising which operation to use in an algebraic structure depends on the correct interpretation of brackets.

## The challenges of moving from arithmetic to algebra

Algebra stems from arithmetic and involves symbolising numerical relationships using particular rules to manipulate symbols in mathematical structures (Kieran, 1992). Being able to understand relations in arithmetic is a prerequisite for understanding relations in algebra (Booth, 1988; Filloy & Rojano, 1989; Subramaniam & Banerjee, 2004). For example, the structural operation of distribution in the arithmetic expression $2(5 + 3) = 2(5) + 2(3)$, must be understood in order to understand the structural operation of distribution in algebraic expressions such as in $2(x + y) = 2x + 2y$. However, in numeric examples, calculations can mask the structure which means that explicit attention must be given to numeric expressions to prepare learners to appreciate the structure of algebraic expressions.

The abstract nature of algebra renders it challenging for many learners (Carraher et al., 2000). When progressing from arithmetic to algebra, mathematical structures that include numbers develop towards structures that include letters, demanding different types of structural operations (Warren, 2003). These include brackets as structural components in both arithmetic and algebra that group together certain parts of an expression to indicate the order of operations (Papadopolous & Thoma, 2023), but which require different types of calculations or operations. For example, brackets in an arithmetic expression, such as, $2-(6 - 1)$, indicate that the difference of $6-1$must be obtained first and then subtracted from 2. In the algebraic expression, $2-(x - 1)$, the brackets need to be removed first by 'changing the sign' of the terms in the brackets before further simplification can take place.

Numerous research studies have analysed learners' difficulties involving errors and slips when working with brackets in algebraic expressions. We consider errors as regular and systematic incorrect applications of prior knowledge (Gardee & Brodie, 2015) influenced by the educational process (Olivier, 1989). Slips occur due to careless application by both beginners and experts (Ayres, 1995; Gardee & Brodie, 2015). For example, in $(2x + 3)\ (x + 4)$, a response such as $2x^2 + 6x + 4x + 12$ indicates an error connected to inappropriately multiplying the paired terms in brackets. However, a response such as $2x + 3x + 8x + 12$ indicates a slip with $2x$ instead of $2x^2$ by omitting the squared symbol ($^2$), because multiplication is used correctly with the remaining terms in brackets. Distinguishing errors from slips is important in teaching and research because errors and slips have different sources and hence require different kinds of attention in teaching and different levels of attention in data coding and analysis.

Research studies related to learner difficulties involving errors include learners using the incorrect order of operations in expressions consisting of brackets (Linchevski & Livneh, 1999; Banerjee & Subramaniam, 2005; Gunnarsson et al., 2016; Vlassis, 2004), poor manipulation skills when simplifying algebraic expressions with grouped compound terms in brackets (Hoch & Dreyfus, 2006), using mental brackets incorrectly during simplification (Marchini & Papadopoulos, 2011; Papadopoulos & Gunnarsson, 2018; Papadopolous & Thoma, 2023), inappropriately conjoining terms involving brackets (Booth, 1982; Falle, 2007), and incorrectly using exponential notation involving brackets (De Bock et al., 2007; MacGregor & Stacey, 2007; Matz, 1980). Furthermore, research over many years has shown that learners experience particular difficulty with negatives and subtraction as they move from arithmetic of positive numbers to negative numbers and then symbolic algebra (Linchevski & Livneh, 1999; Pournara et al., 2016; Vlassis, 2004). This compounds their difficulties in working with brackets. Some talk about learners failing to 'distribute the negative' into the brackets (Gregg & Yackel, 2002) which conflates the operation of subtraction with multiplication. Following Vlassis (2004), we make an explicit distinction between the operation (subtraction) and the sign (negative). We therefore refer to the *minus symbol* to avoid indicating operation or sign (not to be confused with Vygotsky's sign). We then determine the meaning of the minus symbol depending on the context in which it is embedded, and this brings us to a more detailed discussion of structure.

## The notion of structure

Algebraic structures can have different meanings in different contexts (Hoch & Dreyfus, 2004; Kieran, 2018). Providing a single precise definition of structure is virtually impossible since the notion of structure has been defined in various ways in the literature. In fact, Venkat et al. (2019) argue that different definitions of structure in algebra contain terms that may be either synonymous with or distinct from terms in other definitions.

We begin with Kieran's (1989) seminal work on algebraic structure where she distinguished *surface structure* from *systemic structure*. Surface structure is 'the given form or arrangement of terms and operations, subject – when arranged sequentially – to the constraints of the order of operations' (Kieran, 1989, p. 34). Surface structure therefore refers to the physical arrangement of elements in an expression. For example, in $3(x + 4)$, the surface structure is 3 multiplied with $x + 4$. Systemic structure refers 'to properties of operations, such as commutativity and associativity, and to the relationship between the operations, such as distributivity' (Kieran, 1989, p. 34). Systemic structure therefore refers to using the properties of the operations and the relationships between the operations appropriately to manipulate an expression into different equivalent forms. For example, $3(x + 4)$ can be manipulated as $3 \times x + 3 \times 4$ or $3x + 12$ or $12 + 3x$.

Linchevski and Vinner (1990) proposed the term *hidden structure* instead of surface and systemic structure. Firstly, they argued that by using Kieran's approach, the same surface structure could be evident in expressions despite elements being different. For example, the surface structure of $3y + 4$ is a hidden structure of $3(x + 2) + 4$. Secondly, they argued that according to Kieran's definition of systemic structure, numerous equivalent structures could be derived from a given expression. However, for a given context or scenario there is usually a specific preferred equivalent expression.

More recently Hoch and Dreyfus (2004) defined structure as the 'external appearance' (p. 50) of an algebraic expression, thus providing an alternative to Kieran's *surface structure*. Hoch and Dreyfus argue that the external appearance of an expression could be manipulated into an internal order, which depends on the relationship between the elements. For example, the external appearance of $(x + 3)$ $(x + 2)$ shows the product of two binomials, concealing the internal form of a trinomial $x^2 + 5x + 6$. Later, Novotna and Hoch (2008) drew on Linchevski and Vinner's (1990) idea of hidden structure and referred to the 'substitution principle' (p. 95) meaning that if a single term is substituted with a compound term, the structure of the expression is retained. For example, the structure of $x^2 - y^2$ represents the difference of two squares. If the variables $x$ and $y$ are substituted with compound terms such as $(x + 2)$ and $(y - 4)$, the structure of the expression $(x + 2)^2 - (y - 4)^2$ would still illustrate a difference of two squares.

As can be seen from the preceding discussion, attempts to define structure tended to produce additional terminology which did not lead to increased clarity about the notion of structure. We have found it more productive to work with the notion of *structure sense* rather than *structure* and have further refined it for our purposes.

## The notion of structure sense

Structure sense was first defined by Linchevski and Livneh (1999) as the ability to identify structures of expressions and to be able to use suitable operations in order to manipulate expressions into equivalent structures. However, this definition was limited to examples from arithmetic and basic algebra (Hoch, 2003; Novotna & Hoch, 2008) and did not involve algebraic expressions. Structure sense was re-defined in later research in the context of algebraic expressions by Hoch and colleagues (e.g., Hoch, 2003; Hoch & Dreyfus, 2004, Novotna & Hoch, 2008).

Hoch (2003) initially defined structure sense as being able to identify algebraic structures by accessing previous knowledge of algebraic operations. This was extended to recognising the aspects of a structure and using the sequence of previously learnt operations to manipulate an expression into an equivalent structure (Hoch & Dreyfus, 2004). In addition, the researchers argued that structure sense involves 'looking before doing' (Hoch & Dreyfus, 2004, p. 54) implying that learners need to recognise a structure before using appropriate operations. This suggests that structure sense depends on a learner's prior experience with algebraic expressions. A more detailed definition provided by Hoch and Dreyfus (2006) describes structure sense in three segments. Structure sense is the ability to:

- recognise a simple structure in a familiar form
- recognise a complex structure in a familiar form
- identify appropriate operations to manipulate a structure into an equivalent expression.

Novotna and Hoch (2008) further emphasised that 'substitution principle' (p. 95) has an important role in structure sense when learners were able to interpret compound terms as single terms making the structure familiar.

In our definition of structure sense, we begin with Kieran's (1989) definition of structure sense and expand to include *surface structure sense* and *systemic structure sense*. Both types of structure sense incorporate aspects from Hoch and Dreyfus's (2004) earlier definition of structure sense and consider Vygotsky's notion of tool and sign in the role of brackets. To summarise, our definition of structure sense is *the ability to recognise the surface and systemic structures of algebraic expressions containing brackets and to use appropriate manipulations to obtain equivalent expressions*. Structure sense would then be evident when one recognises that if the surface structure of two expressions is the same, then they will have the same systemic structure. For example, the expressions $2(z + 1)$ and $(z + 1)2$ have different surface structures but the same systemic structure, while the expressions $(a + 2) - 4$ and $4 - (a + 2)$ have different surface structures and systemic structures.

# Surface and systemic structure sense in the context of this study

## Operationalising systemic and surface structure sense

Surface structure sense refers to being able to interpret the arrangement of elements in an expression, by 'looking' (Hoch & Dreyfus, 2004) at the spatial organisation (Venkat et al., 2019) of elements. In other words, recognising the components of a structure (Hoch & Dreyfus, 2004). For example, surface structure sense of $2(x + 1)$ is being able to see the multiplication of 2 with $x + 1$ and $(x + 1)2$ as the multiplication of $x + 1$ with 2.

Brackets, as a tool (Vygotsky, 1978), impact the surface structure of expressions, and the ability to interpret the role of brackets in expressions influences surface structure sense. Firstly, brackets group more than one element to form a compound term such as $(x + 1)$. Secondly, the positioning of mathematical signs or operations in relation to brackets affects the physical appearance of an expression resulting in how the signs and operations are perceived. For example, surface structure sense of $4 - (a + 2)$ is recognising that the minus symbol between the constant 4 and compound term $a + 2$ means the subtraction of $a + 2$ from 4. Surface structure sense of $-4(a + 2)$, however, is recognising that the negated factor 4 must be multiplied by the binomial $a + 2$.

Systemic structure sense involves being able to use properties and relations of operations to manipulate expressions as equivalent expressions, through the act of 'doing' (Hoch & Dreyfus, 2004). For example, systemic structure sense of $2(x + 1)$ and $(x + 1)2$ is recognising the commutative property in both expressions because changing the order of the factors will not change the product. Using the distributive property in both expressions will result in an identical systemic structure. Brackets as a Vygotskian sign influence the systemic structure of expressions in two ways. Firstly, brackets signify how the structural operation must be used to obtain an equivalent structure. For example, in $4a - (a + 2)$, the bracketed binomial placed after the minus symbol signifies that both terms in the bracket must be negated or subtracted. By contrast, in $-4(a + 2)$ the bracketed binomial placed directly after the constant signifies that both terms in brackets must be multiplied by $-4$. Secondly, brackets affect the order of operations in an expression. For example, in $2 + 3(x + 1) - x$, the correct order of operations involves operating on $3(x + 1)$ first to obtain $3x + 3$, followed by adding 2 and then subtracting $x$.

## Test items used in the study

We now present the five test items and a frame for interpreting learners' responses to the items which are indicative of learners' surface and systemic structure sense. The first item was to simplify $(a + b)b$. The surface structure of $(a + b)b$ illustrates the monomial $b$ is positioned after the binomial $(a + b)$. A learner demonstrates surface structure sense of the expression if the brackets are recognised as a grouping tool for the binomial factor multiplied by a monomial. If the learner responds, for instance $(a + b)b = (a + b)$, then they do not recognise the surface structure. We anticipated that placing the monomial *after* the binomial would be challenging for learners because in Grades 9 and 10, a monomial is typically placed before the brackets, for example $2x(x - 3)$. Systemic structure sense is being aware that brackets signify the distributive property, and so $b$ must be multiplied by each term in the brackets. Learner responses such as $ab^2$, $a + b^2$ and $ab + 2b$ indicate difficulty in using the distributive property to write equivalent expressions. Being aware of the commutative property of multiplication where the order of the factors will not affect the result $ab + b^2$, that is, $b^2 + ab$, is also an indication of systemic structure sense.

The second item was to simplify $3a - (b + a)$. The surface structure illustrates subtraction of the compound term $(b + a)$ from the single term $3a$. Surface structure sense is demonstrated by recognising the dual function of the brackets, that is, as a grouping tool for the compound term and as a separation tool for the subtraction of the compound term from the single term. Responses such as $-3ab + 3a^2$ indicate that the surface structure was not recognised because brackets were used as multiplication. Systemic structure sense is recognising that the minus symbol before the brackets signifies a change in sign of both terms in brackets, that is, $3a - b - a$. A response such as $3a - b + a$ indicates difficulty with subtraction because only the term closest to the minus symbol was negated.

The third item was to simplify $2a(a - 4) - 8$. The surface structure sense involves recognising the dual function of brackets: the grouping of two factors, $2a$ and $a - 4$, and the separation of subtraction of the constant 8. Learner responses such as $(2a^2 - 8a) - 8 = 16a^2 - 64a$ indicate an awareness of brackets as a grouping tool of the two factors, but not as a separation tool for subtraction. Responses such as $3a - 12$ indicate that brackets are recognised neither as a grouping tool, nor as separation tool because adjacent terms are operated on. Systemic structure sense involves applying the primary operation of distribution on the two factors resulting in $2a^2 - 8a$. If a learner responds, $2a^2 - 8 - 8$ then they have difficulty in multiplying a variable with a constant. Systemic structure sense also involves appreciating that although subtraction of a constant is evident in the surface structure, no further operation is required, resulting in a final answer of $2a^2 - 8a - 8$.

The instruction was to multiply out the fourth item $(2x + 1)(x + 4)$. The surface structure illustrates the multiplication of two binomials. Surface structure sense is recognising the brackets as a grouping tool for two binomial factors. Learners do not recognise the multiplication of two binomials as indicated in responses such as $3x + 4$ because only like terms are operated on. Systemic structure sense is recognising that brackets signify the multiplication of both terms in the first binomial factor by both terms in the second binomial factor. Responses such as $2x + 8x + x + 4$ indicate difficulty in multiplying the first terms in both brackets.

The fifth item was to multiply out $2(x + 3)^2$. The surface structure illustrates a numerical factor and a squared binomial factor. Hence, surface structure sense involves recognising brackets as a grouping tool for multiplication of these factors: $2(x + 3)(x + 3)$. Responses such as $(2x + 6)^2$ and $2x + 11$ indicate that learners did not recognise three factors in the surface structure. Systemic structure sense is being aware of using the distributive law in the correct order, by first multiplying the binomial factors $(x + 3)$ and $(x + 3)$, then multiplying by 2. Using the distributive property from left to right will not result in an equivalent expression such as $(2x + 6)^2$. However, if a learner expresses the structure as $2(x + 3)(x + 3)$, before using the distributive property, then they recognise three factors. Responses such as $2(x^2 + 9x + 9)$ indicate difficulty in using multiplication correctly on terms in brackets.

# Research design and methodology

A qualitative approach was used with a sub-sample from a wider study exploring Grade 9 and 10 learners' performance in algebra, after a teacher development programme had been conducted. We identified errors made by the learners and coded the incorrect responses in relation to the role of brackets as a tool and as a sign. We then analysed the errors in learner responses, which suggested challenges with surface and structure sense of algebraic expressions.

## Design of the study

The research reported here adopts a qualitative approach using secondary data which provided the opportunity to investigate learners' responses to five test items involving brackets, which had not been considered in the primary research (Coe et al., 2017). The test scripts formed part of a larger study conducted by the Wits Maths Connect Secondary Project, (WMCS) to investigate the impact of teachers' participation in a professional development programme on their learners' performance. Therefore, the five items in focus here were not specifically designed for this study, nor were they informed by the notions of surface and systemic structure sense.

The data were collected in the last quarter of 2018 when learners were expected to have been taught the algebra necessary to complete the items analysed here. The test was administered under typical test conditions and calculators were not permitted.

## Participants

The larger WMCS study focused on secondary schools in poor and working-class communities in Gauteng. The instrument was designed for Grade 9 level and Rasch analyses on pilot versions of the instrument indicated that it was fit for the purpose of studying learning gains, However, for this study, we selected Grade 10 learners because our initial analysis of learner performance in the larger study showed that Grade 10 responses were generally more detailed and thus might reveal more about learners' structure sense.

The initial selection criterion for the sample was that learners must have attempted all five of the selected items. Thereafter, we randomly selected 155 learners who, it turned out, came from 11 different schools and had been taught by 15 different teachers. The randomised sample combined with the range of schools and teachers increases the generalisability of the findings of the study (Creswell, 2014).

As we worked between the empirical data and our analytical tools, and as it became clearer that the notion of structure sense would be a useful construct for the study, we identified a large number of responses which were very difficult to code for surface and systemic structure sense. In most cases these responses involved fundamental algebraic errors such as conjoining of unlike terms, e.g. $(2x + 1)(x + 4) \rightarrow (3x)(4x) \rightarrow 12x$. Most of these responses came from learners who scored below 40% on the test, and so in this article we focus only on the responses of 58 learners who obtained a test score of 40% or more, which included 10 of the 11 schools that were sampled.

## Coding of errors

The coding scheme slowly emerged through repeated examination of scripts in relation to the theoretical ideas we had adopted, such as brackets as tool or sign. This led to us inferring how learners were interpreting the meaning of brackets in the questions and their use of brackets in their responses. This provided the basis for the set of codes shown in Figure 1.

All incorrect responses were allocated either a surface structure or a systemic structure error code. Two main surface structure errors were evident (Figure 1). Firstly,

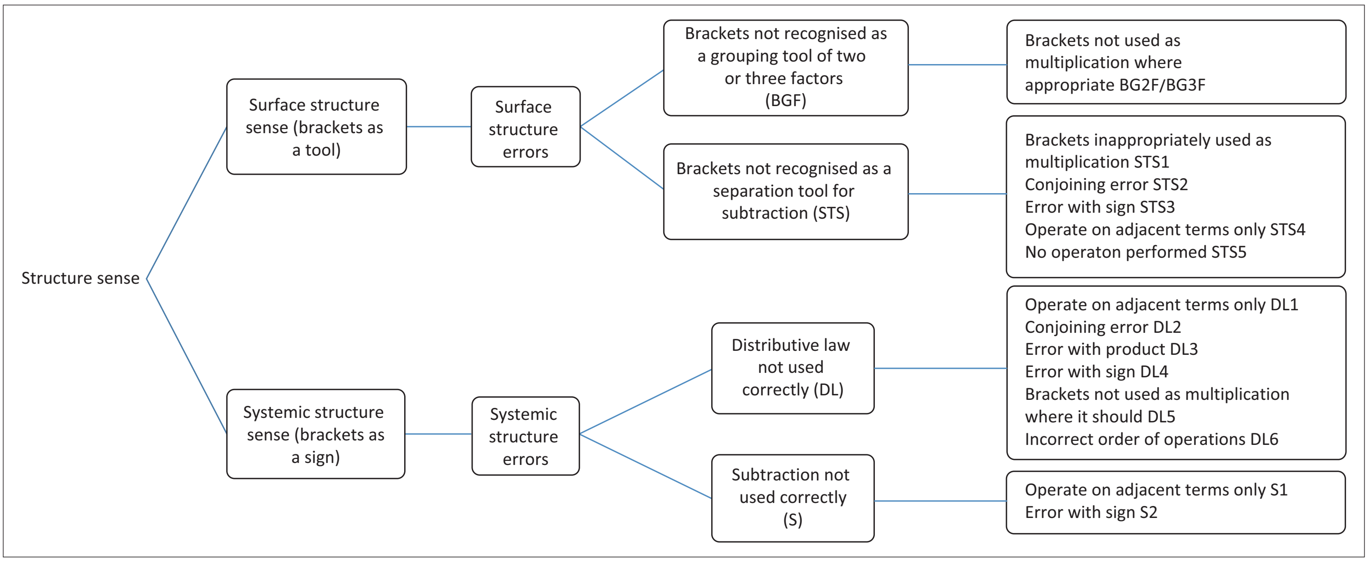


**FIGURE 1:** Surface and systemic structure sense and identified errors.

**TABLE 1:** Examples of incorrect responses illustrating surface structure errors.

| Error code | Applicable items | Incorrect response | Indicators |
|---|---|---|---|
| BG2F1 | $(a + b)b$ | $(a + b)$ | No operation of multiplication performed on $(a + b)$ by monomial $b$ |
| BG3F1 | $2(x + 3)^2$ | $(2x + 6)^2 = (2x + 6)(2x + 6)$ | Inappropriately multiplying 2 into the compound binomial $(x + 3)^2$ |
| STS1 | $2a(a - 4)-8$ | $(2a^2 - 8a) - 8 = 16a^2 - 64a$ | Inappropriately multiplying 8 after the correctly multiplied compound term $2a^2 - 8a$ |
| STS2 | $3a- (b + a)$ | $3b$ | Inappropriately conjoining $3a$ and $(b + a)$ |
| STS3 | $3a- (b + a)$ | $3a- (- b + a)$ | $b$ is inappropriately negated in the brackets |
| STS4 | $2a(a - 4)-8$ | $2a^2 - 8a$ | Omission of the constant from the final response. |
| STS5 | $2a(a - 4)-8$ | $3a - 12$ | The two pairs of adjacent terms are added:$2a$ and $a$ to obtain $3a$; and −4 and −8 to obtain −12 |

**TABLE 2:** Examples of incorrect responses illustrating systemic structure errors.

| Error code | Applicable items | Incorrect response | Indicators |
|---|---|---|---|
| DL1 | $(a + b)b$ | $a + b^2$ | Multiplication of the second term in brackets $b$ by monomial $b$ after brackets. No operation on the first term $a$ in brackets, which is positioned further from monomial $b$ |
| DL2 | $(a + b)b$ | $ab^2$ | Inappropriately conjoining $a$ and $b$ in brackets, and multiplication with monomial $b$ |
| DL3 | $(2x + 1)(x + 4)$ | $2x^2 + 4x + x + 4$ | Incorrect product of $2x$ and 4 to obtain $4x$ |
| DL4 | $2a(a - 4) - 8$ | $2a^2 + 8a - 8$ | The sign of the product of $2a$ and −4 is positive |
| DL5 | $2(x + 3)^2$ | $2(x^2 + 9)$ | Squaring the terms in brackets separately to obtain $(x^2 + 9)$ |
| DL6 | $2(x + 3)^2$ | $(2x + 6)^2$ | The monomial 2 is multiplied into the brackets from left to right to obtain $2x + 6$ |
| S1 | $3a - (b + a)$ | $3a - b + a$ | Change in sign of the first term $b$ but not the second term $a$ in brackets |
| S2 | $2a(a - 4) - 8$ | $2a^2 - 8a + 8$ | The sign of the constant 8 changes |

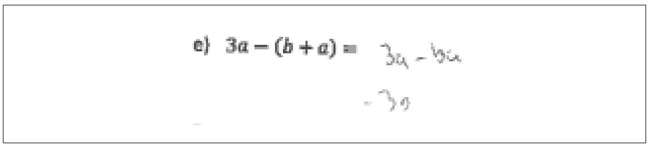


**FIGURE 2:** Learner A response to $3a - (b + a)$.

brackets were not recognised as a grouping tool of two or three factors because brackets were not treated as multiplication where this was necessary. Secondly, brackets were not recognised as a separation tool for subtraction, and learners interpreted brackets as a tool in different ways to produce incorrect answers. Sub-errors were therefore developed so that incorrect responses could be grouped for particular types of errors.

Systemic structure errors refer to the misinterpretation of brackets as a sign in algebraic expressions. Two main types of systemic structure errors were evident (Figure 1). Firstly, distributive law not used correctly and, secondly, subtraction was not used correctly. Incorrect responses involving both types of systemic structure errors illustrate operations using brackets in different ways. Similarly, sub-error codes were developed for further grouping.

We provide some examples of incorrect responses and associated error codes in Table 1 and Table 2.

Since our intent was to code errors involving the interpretation of brackets, we decided to stop coding when the errors did not involve brackets. For example, consider the response shown in Figure 2.

We have rewritten and labelled the lines of the Learner A's response for further discussion:

| | |
|---|---|
| $3a - (b + a)$ | line 1 |
| $3a - ba$ | line 2 |
| $3a$ | line 3 |

In line 2, the terms in the brackets were inappropriately conjoined indicating that the learner does not demonstrate surface structure sense of the subtraction of a compound term from a monomial. We coded the error in line 2 as a surface structure error. The second error in line 3 involves operating on unlike terms but this line does not involve brackets, so we did not code the error made in line 3. This implies that learners may have actually made more errors, but we report only on errors that relate to brackets.

## Validity and reliability

As noted above, the coding of responses involved several iterations of different kinds of codes, some of which did not operationalise our definition of surface or systemic structure sense and the use of brackets as either tool or sign. For example, the response $(2x + 1)(x + 4) = 2x + x + 1 + 4$ was initially coded as *brackets not used as multiplication where appropriate*, but this did not address brackets as a grouping tool of two factors. Once the codes described above were chosen and applied, our coding became more consistent and reliable and less dependent on the actual item being coded. All item coding was done by the first author and then checked by the other authors. Where we disagreed with a code, this matter was discussed and resolved, usually with reference to our operational definitions and what was perceived to be the key error.

By coding only the errors that involved brackets, we were able to reduce the quantity and variety of codes which also increased the reliability of our coding.

The validity of the study was strengthened through the piloting of the test items (albeit that this was for the larger WMCS study). The extent to which our findings can be generalised is increased by the initial random sampling of the 155 learners and this is retained in the reduced sample of 58 learners who were drawn from 10 different schools.

# Findings

We coded the learners' errors and classified them into either surface or systemic error types. We present the distribution of errors for each item and explain the likely sources of the high-frequency errors, followed by a discussion on the trends about learners' structure sense that were seen across items. Connections are then made with learner difficulties with brackets that have been reported in the literature.

## Surface and systemic structure errors

Following the coding of errors, we tabulated the total number of surface and systemic structure errors for each item (Table 3). We discuss the highest number of surface and systemic structure errors of the five items, which we then use to identify particular trends in errors that emerged.

The highest number of surface structure errors was evident in simplification of $3a - (b + a)$. Twenty-two incorrect responses, such as $-3a + 3a^2$, $3ab - 3a^2$, $3ab + 3a^2$ and $-3ab - 3a^2$, illustrated that $3a$ was multiplied with both the terms in brackets, where $3a$ was negated in some instances. Examples are shown in Figure 3 and Figure 4.

The second highest number of surface structure errors was evident in $2(x + 3)^2$ with 15 incorrect responses. The most common error involved the inappropriate use of multiplication on terms in brackets by squaring each term separately to obtain $2(x^2 + 9)$. Another common error was the inappropriate multiplication of the monomial 2, as the first order of operations to obtain responses such as $(2x + 6)^2$. Examples are shown in Figure 5 and Figure 6.

The remaining three items $(a + b)b$; $(2x + 1)(x + 4)$ and $2a(a - 4) - 8$ showed few incorrect responses. Most learners used multiplication appropriately suggesting they recognised the surface structures of the three items.

The highest number of systemic structure errors was evident in $2(x + 3)^2$ with 20 incorrect responses. Most errors involved using distribution where 2 had been

**TABLE 3:** Number of surface or systemic structure errors in each item.

| Error types | $(a + b)b$ | $3a - (b + a)$ | $2a(a - 4) - 8$ | $(2x + 1)(x + 4)$ | $(2x + 3)^2$ |
|---|---|---|---|---|---|
| Surface structure errors | 1 | 22 | 8 | 4 | 15 |
| Systemic structure errors | 15 | 4 | 6 | 5 | 20 |
| **Total number of incorrect responses** | **16** | **26** | **14** | **9** | **35** |

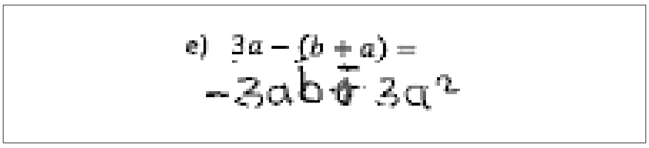

**FIGURE 3:** Learner B response to $3a - (b + a)$.

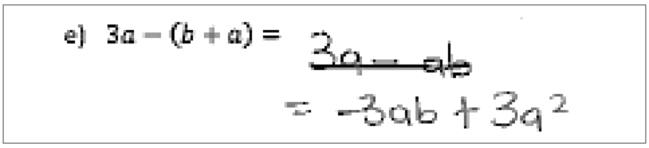

**FIGURE 4:** Learner C response to $3a - (b + a)$.

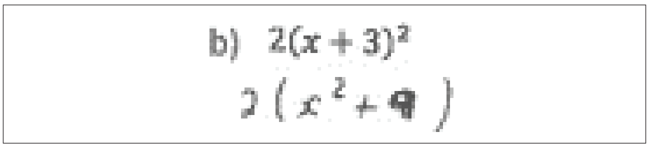

**FIGURE 5:** Learner D response to $(2x + 3)^2$.

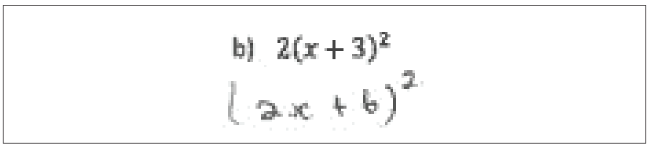

**FIGURE 6:** Learner E response to $2(x + 3)^2$.

multiplied into the brackets before squaring. An example is shown in Figure 7.

Other errors showed incorrect products after using multiplication. For example, the incorrect response $2(x^2 + 9x + 9)$ showed the product $9x$ instead of $6x$, while the incorrect response $2(x^2 + 6x + 6)$ showed an error in the constant.

The second highest number of systemic structure errors was evident in $(a + b)b$ involving incorrect distribution. In the 15 incorrect responses, errors included partial distribution of the monomial $b$ with the binomial $a + b$ such as $a + b^2$, incorrect products such as $ab + 2b$, and inappropriately conjoining all the terms such as $ab^2$. An example is shown in Figure 8.

Items $3a - (b + a)$, $2a(a - 4) - 8$ and $(2x + 1)(x + 4)$ produced fewer errors in which learners incorrectly used multiplication and subtraction to obtain equivalent structures suggesting that the systemic structure was recognised.

## Trends in errors

Particular trends in errors emerged following the analyses of learners' incorrect responses. The analysis shows that there are three common errors. These are (1) using the incorrect order of operations with brackets by prioritising operations on terms that are in close proximity to each other, (2) using the incorrect order of operations with brackets by operating from left to right and (3) not using the minus symbol correctly. We will explain these trends using learners' responses to various items.

Incorrect responses to three items, $(a + b)b$, $3a - (b + a)$ and $2a(a - 4) - 8$, indicated that only the components closer or adjacent to each other were operated on. For example, the response $(a + b)b = a + b^2$ shows that learners correctly multiplied the adjacent terms $b$ and $b$, but no operation was made on $a$. Similarly, the response $3a - (b + a) = 3a - b + a$ indicates that learners correctly changed the sign of $b$ which is positioned closer to the minus symbol, but did not change the sign of $a$ positioned further from the minus symbol.

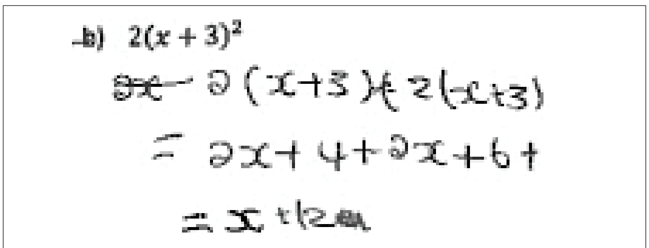


**FIGURE 7:** Learner F response to $2(x + 3)^2$.

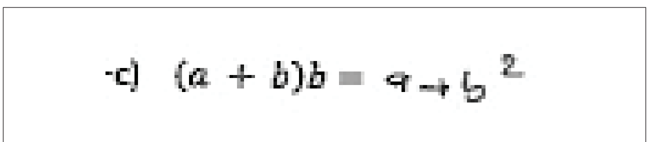


**FIGURE 8:** Learner G response to $(a + b)b$.

Similarly, in responses such as $2a(a - 4) - 8 = 3a - 12$ , both pairs of adjacent terms $2a$ and $a$ and $-4$ and $-8$ were inappropriately added.

The understanding of algebraic notation such as the squared symbol ($^2$) affected how learners operated on structures. Responses to $2(x + 3)^2$ indicated that learners did not regard the structure of $(x + 3)^2$ as $(x + 3)(x + 3)$; rather, responses such as $2(x^2 + 9)$ illustrate that each term had been squared separately, referred to as linear extrapolation errors (Matz, 1980, as cited in Olivier, 1989, p. 8) and the illusion of linearity (De Bock et al., 2007). Such operations are indications of the over-generalisation of the exponent into the bracket which we argue as operating on the terms in spatial proximity to the squared symbol.

Operating on algebraic structures is not necessarily done from left to right because the order of operations is dependent on the arrangement of components and algebraic notation. Responses to $2(x + 3)^2$ indicate that learners did not regard the algebraic notation of the square on the brackets as the first order of operations. Learners inappropriately operated from left to right to obtain responses such as $(2x + 6)^2 = (2x + 6)(2x + 6)$. The squared notation remains on the bracket $(2x + 6)$, which changes the surface structure to $(2x + 6)(2x + 6)$, which is not an equivalent structure of $2(x + 3)(x + 3)$. Similarly, an expression such as $2(x + 3)y$ indicates that 2 should be multiplied working from left to right and that $y$ should be multiplied working from right to left into the bracket. The two different approaches of multiplication could demand a higher cognitive load (Ayres, 1995). However, multiplying 2 and $y$ as the first order of operations to obtain $2y(x + 3)$ would result in a more familiar structure, which could then be less challenging for the learner to operate on further.

The minus symbol has different functions in algebraic structures depending on its positioning (Vlassis, 2004). The minus symbol is positioned before the bracket in, $3a - (b + a)$ and after the bracket in $2a - (a - 4) - 8$, rendering different meanings in the two items. Responses indicate that learners ignored the minus symbol but focused on the brackets to multiply. In $3a-(b + a)$ responses such as $-3ab + 3ab^2$ and $-3ab - 3ab^2$ indicated $3a$ and $-3a$ multiplied into the bracket. The responses showed that learners had correctly distributed $3a$ into the bracket, indicating a sense of how brackets are to be used. However, learners seemed to ignore the minus symbol between $3a$ and $(b + a)$, and saw the expression as $-3a(b + a)$ instead of $3a - (b + a)$. Similarly, in $2a(a - 4) - 8$ learners did not regard the minus symbol as separating the constant 8 from the preceding compound term. Rather learners used brackets to multiply the constant and compound term. For example, $(2a^2 - 8a) - 8 = -16a^2 - 64$ illustrates the correct multiplication of $2a$ into the bracket to obtain $2a^2 - 8a$. However, learners seemed to ignore the minus symbol between the bracket and the constant 8 and multiplied further into the brackets, as $(2a^2 - 8a) \times 8$ instead of $(2a^2 - 8a) - 8$.

## Discussion

In this section, we summarise some key observations about learners' surface and systemic structure sense inferred from their responses to the five items involving brackets. Then, we make some recommendations for teaching of brackets. We conclude with the limitations of the study and make some suggestions for future research.

### Summary of findings

Three observations were made regarding learners being able to interpret the surface structure of expressions. Firstly, more brackets seem to support surface structure sense in that learners made fewer errors in item $(2x + 1)(x + 4)$ with two pairs of brackets, compared to item $2(x + 3)^2$ with one pair of brackets. Learners more easily recognised the surface structure of the multiplication of two binomials when the brackets were side by side, instead of a squared binomial. This observation supports Hoch and Dreyfus's (2004) finding that the presence of brackets assists learners to see structure.

Secondly, there is an over-generalisation of treating brackets as multiplication to situations involving subtraction. For example, learners treated brackets as multiplication even in cases of $3a - (b + a)$ and $2a(a - 4) - 8$. In $3a - (b + a)$ most learners did not recognise the urinary nature of the minus symbol as a structural signifier to change the signs of the terms (or objects) in brackets (Sfard, 2000; Vlassis, 2004). Rather, learners multiplied $3a$ into the bracket with both terms and inappropriately used the minus symbol to negate 3. In $2a(a - 4) - 8$, learners successfully multiplied $2a$ into the brackets to obtain $(2a^2 - 8a) - 8$; however, learners did not recognise the urinary nature of the minus symbol that made the constant $-8$ an isolated number (Vlassis, 2004). Similarly, the constant was multiplied into the bracket with both terms.

Thirdly, the familiarity of expressions is supportive of surface structure sense. Item structures commonly encountered in the school curriculum were less prone to errors, such as the multiplication of a monomial and binomial in the order $(a + b)b$ and $2a(a - 4) - 8$ and the multiplication of two binomials such as $(2x + 1)(x + 4)$. Generalising learnt rules and procedures on unfamiliar structures is known to be difficult for learners (Hoch & Dreyfus, 2005; Kieran, 1992). We regard the less familiar items as the subtraction of a monomial and compound terms in $3a - (b + a)$ and multiplication of an expression with a hidden structure in $2(x + 3)^2$.

Two key observations were made regarding learners using systemic structure sense to obtain equivalent structures. Firstly, regular application of algebraic procedures on specific types of structures gives the illusion of systemic structure sense. Before doing the analysis, we assumed that there would be more errors in simplifying $(2x + 1)(x + 4)$ because there are more operations of multiplication required, and on $2a(a - 4) - 8$ because of the complicated nature of the structure. Both items, however, indicated a low number of systemic structure errors. We then considered the emphasis placed on the multiplication of two binomials in Grade 9 such as $(2x + 1)(x + 4)$ and on the multiplication of a monomial and binomial in Grade 8 such as $2a(a - 4) - 8$ which involves a specific spatial arrangement of components using a specific procedure for multiplication. Item $(a + b)b$ which also involves multiplication of a monomial and binomial, and which requires fewer operations of multiplication, was surprisingly more prone to errors. Errors indicate that learners had difficulty operating on the expression because the monomial was placed after the bracket instead of before. Banerjee and Subramaniam (2005) argue that a structural understanding of expressions is evident only if learnt procedures are able to be used in various types of expressions. This made us question if learners truly have a structural understanding of the multiplication of two binomials, and of a monomial and binomial, or if learners are only able to apply procedural knowledge on commonly encountered structures.

Secondly, systemic structure sense using the correct order of operations is affected by the arrangement of components in relation to terms in brackets. Banerjee and Subramaniam (2005) found that learners use the incorrect order of operations when brackets are included in an expression by operating from left to right within the brackets. We found that learners made errors by using incorrect operations especially when like terms were adjacent to each other such as in $2a(a - 4) - 8$ and $(a + b)b$. Errors also indicate that when the same number was positioned on either side of a bracket, learners operated from left to right. Item $2(x + 3)^2$ illustrates a 2 before the brackets and the square on the brackets. Incorrect responses suggest that the learners' decision-making process regarding which operation to do first was affected by the increased memory load of a squared bracket and multiplication of brackets because learners operated from left to right.

### Recommendations for teaching and further research

Learners need to be exposed to both familiar and unfamiliar structures with different arrangements of components so that structure sense in algebra can be better developed. Teachers need to emphasise structural aspects of brackets and mathematical symbols and operations such as grouping, multiplication and subtraction. Interviews with learners to access their thinking would help teachers to understand and address learners' misconceptions and misinterpretations of algebraic structures.

We recommend that teachers actively address structure sense of algebraic expressions by emphasising *what* the arrangement of an expression is before teaching *how* to operate on the expression, which Hoch and Dreyfus (2004, p. 54) refer to as 'looking before doing'. This could take the form of emphasising the need to identify the positioning of components in relation to other components and symbols, being attentive to operations and identifying hidden structures. Greater attention to operations would include revising the correct order of operations and distinguishing multiplication from subtraction when dealing with grouped terms in brackets.

We propose a task involving three pairs of examples that have the potential to bring some of the above aspects into focus (Figure 9). All examples use the same components but have a different surface structure. We have deliberately paired examples (A and B, C and D, and E and F) together to draw learners' attention to the similarities and differences in surface structure.

The first question requires learners to identify variance amid invariance – to focus on *what* the external appearance of an expression involves, hence the surface structure. Learners do not simplify the items at this stage but focus on the arrangement of components. For example, A and B contain the same monomial and binomial but these components must be operated on differently. The second question explicitly draws attention to the operations with the expectation that learners will see the commutative law at work in A and C, and subtraction in B and D. We suspect that many learners will not recognise that B and E will have the same answer because their surface structures are different. However, the items have the same systemic structure.

# Conclusion

In this study, we identified trends in learners' errors in the interpretation of brackets. The items used in this study did not allow us to pursue Kieran's (1989) definition of structure to its fullest because the structures in the testing instrument focused mainly on multiplication and subtraction. We defined the notions of surface and systemic structure sense to unpack the sources of learners' errors to five items involving different and sometimes dual functions of brackets. The analysis of learners' errors indicates that the surface and systemic structure errors were dependent on the position of the brackets, their functions, the operations on terms, and the familiarity with the structures encountered in school textbooks. These errors indicate the limited structure sense among learners and therefore the need to explicitly focus on developing both surface and structure sense in teaching of algebraic expressions. We recommend more research be conducted to identify further trends that limit structure sense, which can be used to develop teaching strategies to improve on learners' mathematical knowledge and skills. Our recommendation for future research is for researchers to consider the arrangement of the components in structures so that more aspects of structure sense can be deliberately investigated.

Six algebraic expressions are given below:

A. $3x(x-4)$ C. $(x-4)3x$ E. $-(x-4)+3x$

B. $3x-(x-4)$ D. $(x-4)-3x$ F. $(x-4)+3x$

(i) Look at each pair (A-B, C-D and E-F).
- What is the same?
- What is different?

(ii) In which of the six expressions are the terms $3x$ and $(x$-4) being:
A) multiplied B) subtracted C) added

(iii) In which of the expressions will you need to 'change the sign' inside the brackets?

(iv) Which expressions will give the same answer on simplification?

(v) Check your answer for question (v) by simplifying all the expressions.

**FIGURE 9:** List of contrasting items to draw learners' attention to the similarities and differences in surface structure.

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## Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

## Authors' contributions

This article is based on the master's research report of N.M.T., supervised by C.P. N.M.T. conducted the initial data analysis which was extended in consultation with C.P. and S.T. N.M.T. wrote the majority of the article. S.T. contributed to the sections on theoretical framework, literature review, findings and recommendations. S.T. and C.P. provided regular feedback on several drafts. C.P. was the principal investigator of the larger project from which the data set was drawn.

## Ethical considerations

Since human participants (teachers and their learners) as well as schools were involved in this study, the study adhered to ethical research principles. The relevant ethics clearance was obtained from the university human research ethics committee.

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## Data availability

The data that support the findings of this study are available from the corresponding author, N.M.T., on reasonable request.

## Disclaimer

The views and opinions expressed in this article are those of the authors and do not necessarily reflect the official policy or position of any affiliated agency of the authors.

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