Threshold Concepts in Quantitative Reasoning

Judith Canner
California State University, Monterey Bay, jcanner@csumb.edu
Jennifer E. Clinkenbeard
California State University Monterey Bay, jclinkenbeard@csumb.edu

Recommended Citation

Threshold Concepts in Quantitative Reasoning

Abstract

The idea of "threshold concepts" has been used to identify discipline-based concepts that are critical to that academic area. Threshold concepts are often difficult for students to assimilate in a meaningful way but, once done, can be powerful for the learner. In general, threshold concepts are 1) transformative to learner thinking; 2) bounded by the discipline; 3) integrative with other concepts; and 4) irreversible once understood (Meyer and Land 2003). This paper presents five threshold concepts in quantitative reasoning (QR) developed by transdisciplinary faculty workgroups that may be applicable for non-mathematics disciplines as well. They are as follows: 1) QR is an iterative process; 2) Abstract patterns can represent relationships between variables or objects; 3) There is a bidirectional translation between the concrete and the abstract; 4) Effective comparison depends on proportional reasoning; and 5) Different visual representations can communicate varying perspectives on the same quantitative information. The purpose of this paper is twofold: to explore and justify the proposed concepts as threshold concepts in QR-based courses and to offer practical examples which might assist students in developing and understanding these proposed threshold concepts.

Keywords

quantitative reasoning, threshold concepts, assessment

Creative Commons License

Cover Page Footnote

Dr. Judith E. Canner is a Professor of Statistics and serves as the Statistics Program Coordinator for the Statistics B.S., Statistics Minor, and Data Science Minor at California State University Monterey Bay. She earned her B.S. in Mathematics from Shippensburg University, PA, in 2004 and her Ph.D. in Biomathematics and Zoology from North Carolina State University. She teaches courses on modeling, data visualization, and design/analysis and her research areas include quantitative reasoning, education, ecology, psychology, and the health sciences. Her current professional interests include expanding access to statistics and data science education, data literacy, ethical practice, and data for good. She served as the Quantitative Reasoning Assessment Coordinator at CSUMB from 2014 to 2020.

Dr. Jennifer Clinkenbeard is an Associate Professor Mathematics in the Department of Mathematics and Statistics at CSUMB. She earned her BS and MS in Mathematics from California State University Channel Islands and her PhD in Mathematics Education from Claremont Graduate University. She coordinates and supervises corequisite first-year Quantitative Literacy and Finite Mathematics courses. She is also the current Quantitative Reasoning Assessment coordinator for the campus (2020 - present), following Dr. Judith Canner. Her research areas include quantitative reasoning, mathematics education, and mathematics history.

Introduction

The idea of "threshold concepts" has been used to identify discipline-based concepts that are critical to their academic area. Threshold concepts are generally considered 1) transformative to learner thinking; 2) bounded by the discipline; 3) integrative with other concepts; 4) irreversible once understood; and 5) troublesome for learners to understand (Meyer and Land 2003). Threshold concepts that develop skills in quantitative reasoning (QR) are often taught in the context of a general education course in mathematics or statistics. While mathematics courses certainly provide a natural setting for QR development, threshold concepts can also be applied to non-mathematics-based disciplines to meaningfully develop QR skills.

We propose that threshold concepts in quantitative reasoning are not exclusive to mathematics and statistics courses and that there is an advantage to defining threshold concepts in quantitative reasoning generally that are not mathematics-discipline specific. Therefore, we posit five threshold concepts in QR that can be applied across other disciplines.

1. QR is an iterative process;
2. Abstract patterns can represent relationships between variables or objects;
3. There is a bidirectional translation between the concrete and the abstract;
4. Effective comparison depends on proportional reasoning; and
5. Different visual representations can communicate varying perspectives on the same quantitative information.

These five threshold concepts for QR were identified by a transdisciplinary faculty workgroup based on the Meyer and Land (2003) threshold concept framework. We provide justification based on faculty discussions and the literature that each proposed concept meets the definition of a "threshold concept." We also provide examples of how these threshold concepts have been embedded into first-year general education mathematics and statistics coursework, pedagogy, and curricular structures at our institution. Because every first-time freshman student is required to complete a QR-based course for general education credit, the focus on threshold concepts in such courses supports deeper learning in QR across the disciplines. Thus, this paper has two specific goals: 1) to explore and justify the proposed concepts as threshold concepts in QR-based courses and 2) to offer practical examples which might assist students in developing and understanding these proposed threshold concepts.

Literature Review
Threshold Concepts and Quantitative Reasoning

The American Association of Colleges and Universities defines quantitative literacy as:

> A "habit of mind" or competency and comfort in working with quantitative data, results, or forms and the ability to reason or solve problems in a wide variety of authentic contexts and everyday life situations (Rhodes 2010, 25).

The terms QR, quantitative literacy, and numeracy tend to be used interchangeably, although Vacher (2014) gives a framework for the distinctions between these terms. With respect to this framework, the above definition for quantitative literacy is also appropriate for QR for our purposes. However, QR is *not* synonymous with or limited to mathematics. In the "Pedagogy in Action" toolkit, Grawe (2022) describes how QR emphasizes context and real-world data. Because of its contextual nature, it is present across disciplines, not just in mathematics. In *Achieving Quantitative Literacy*, Steen (2004) writes:

> If quantitative literacy remains the responsibility solely of mathematics department—especially if it is caged into a single course such as 'Math for Liberal Arts'—students will continue to see it as something that happens only in the mathematics classroom (18).

While QR is not bounded by a particular discipline, within each discipline are specific concepts and applications at its core that are troublesome, yet transformative for students. These are the threshold concepts.

The framework for threshold concepts was introduced by Meyer and Land (2003). Threshold concepts differ from learning outcomes or core concepts in that "a threshold concept is discipline-specific, focuses on understanding of the subject and … has the ability to transform learners' views of the content" (Zepke 2013, 98). Threshold concepts describe a process and experience with which students need to actively engage to achieve academic outcomes in a particular discipline or course. According to Meyer and Land (2003), a threshold concept has the following five characteristics:

- Troublesome: There are many forms of troublesome knowledge, including ritual knowledge (e.g., giving a "canned" response to a particular type of question), conceptually difficult knowledge, and idiosyncratic language and notation. Each of these relates to troublesome areas of acquiring quantitative knowledge. The difficulty and frustration that students experience while encountering a threshold concept speaks to its troublesome nature.
- Irreversible: Once acquired, the student is unlikely to forget or "unlearn" the concept. Practitioners refer to "crossing the threshold" with the intention that the student is unlikely to revert to their earlier perspective after doing so.
- Integrative: The concept "exposes the previously hidden interrelatedness of something" (Meyer and Land 2003, 416). Because QR is inherently multidisciplinary, it is a fertile ground for integrative concepts.
- Transformative: Once the student understands the threshold concept, their perception of the subject shifts. This characteristic can also include an affective component when the students' values, feelings, or attitude changes as a result of their change in perspective. The affective component is particularly relevant in acquiring quantitative skills because of the

> identity and power dimensions that students often encounter through experiences with school mathematics (Gutiérrez 2009).

- Bounded: This characteristic helps to define the conceptual areas of a particular discipline; in other words, how one discipline differs from another.

Meyer and Land (2003) note that this notion of boundedness is not necessarily present in every context for threshold concepts. We discuss this tension for QR more in the following section.

Current State of Research

Proportional reasoning is well-established as a fundamental threshold concept in QR. More than sixty years ago, Inhelder and Piaget (1958) established proportional reasoning as a key concept in the formal operations stage of thinking. Proportional reasoning as a threshold concept is not limited to developmental psychology; it has also resonated as a challenge for adults in popular culture. In 1988, John Allen Paulos' bestselling book, *Innumeracy,* lamented the ways in which our ability to deal with very large numbers and the probabilities associated with them have led to a host of negative societal outcomes. A lack of understanding of proportional reasoning and comparison is at the heart of this argument. Building on this work, Steen's landmark 1990 article, "Numeracy," described the "rising tide of numbers" and described examples of struggles with comparing fractions, evaluating loan options, and playing lotteries—all of which require proportional reasoning skills. Lamon (2007) identified fractions, ratios, and proportions as simultaneously the most difficult to teach, yet the most essential to success, in higher mathematics and science. More recently, Ryan and Gass (2017) identified conversions as one of six troublesome areas in QR. Frith and Lloyd (2016) specifically justified proportional reasoning as a threshold concept in quantitative literacy. They also noted that most of the prior research has focused on the development of proportional reasoning in children and adolescents, and very little in higher education. Taken together, this history of proportional reasoning at the center of conversations around numeracy (or innumeracy), the development of quantitative skills for children and for adults, suggests that it is clearly a threshold concept in quantitative literacy.

While proportional reasoning is a threshold concept in quantitative literacy or QR, it is not the only one. In addition to conversions, Ryan and Gass (2017) identified five additional troublesome areas: switching context, understanding math, student reflection, graphing and statistical software, and data literacy. Ryan and Gass categorized these troublesome areas into two broad learning thresholds: (a) the ability to apply QR across a range of contexts and use it as a tool or a form of language for scientific problem solving, and (b) fluency in data literacy which enables a student to effectively work through the scientific process. However, Ryan and Gass state that specific threshold concepts have yet to be determined under these learning thresholds.

Additional recent work on the notion of threshold concepts in higher education come from the partner disciplines of statistics (Beitelmal et al. 2022), finance education (Hoadley et al. 2016), and computer science (Farjudian 2023). The discipline of statistics has considerable overlap with QR, but they are not interchangeable (similar to the relationship between QR and mathematics). Notably, the Beitelmal et al. study (2022) argues that the concept of *boundedness* does not apply to the statistical context. We adapt this perspective for QR as well, since, like statistics, QR spans across disciplines. We therefore justify each of our proposed threshold concepts in terms of the other four characteristics of a threshold concept.

The current state of the literature suggests that some specific threshold concepts in QR in addition to proportional reasoning have yet to be identified. Based on the literature and our own research in this area, this paper proposes the possible additional threshold concepts for QR in higher education based on Meyer and Land's 2003 framework. The proposed threshold concepts were developed by a transdisciplinary faculty working group, with each person contributing their expertise in their field and examples of where and how students struggled with QR in their courses. We use the term "transdisciplinary" to describe a holistic approach, where the output is created as a result of disciplines communicating and integrating their contexts. We then present examples of how these concepts are introduced and developed in a first-year QR mathematics and statistics course context to demonstrate building a foundation for students over their undergraduate careers, with the understanding that many students will likely "cross the threshold" in later courses in their own chosen major.

Method
Context

California State University Monterey Bay (CSUMB) is a four-year public university and one of the twenty-three campuses in the California State University system. CSUMB is a primarily undergraduate institution, with about 7,000 students and 26 undergraduate degree programs. The university maintains an outcomes-based curriculum with four undergraduate learning outcomes that are foundational to all undergraduate degree programs and regularly assessed within each program. The first of these learning outcomes is labeled Intellectual Skills and includes critical thinking, written and oral communication, information literacy, and QR. In 2017, as part of ongoing projects assessing intellectual skills campus-wide, CSUMB determined that it would benefit the campus to develop Threshold Concepts for each of the intellectual skills. The purpose was to help faculty to understand more deeply how students learn Intellectual Skills which would, in turn, inform their teaching practices. Hendrawati et al. (2021) identified the most

commonly used methods to identify threshold concepts as (in order) interviews with students, surveys of students and teachers, focus-groups of students, content analysis, tests, workshops, concept maps, and quizzes. In line with Loertscher et al. (2014) and others (see Hendrawati et al. 2021), the university chose the facilitated workshop approach to identify threshold concepts in QR and the other Intellectual Skills.

Process

A one-day workshop brought faculty from across disciplines together to develop a shared understanding of threshold concepts and to initiate the development of threshold concepts for each of the intellectual skills. The QR working group was composed of six members from the disciplines of mathematics, statistics, biology, business, music, and health sciences. The core premise of any threshold concept is that it arises from student struggles and is not necessarily what an expert might identify as a "difficult concept." Therefore, the student perspective was the primary focus when proposing threshold concepts. As student struggles were identified, a flow chart authored by the workshop facilitators (Figure 1) provided a mechanism for the evaluation of struggles as something other than a threshold concept, such as a technical or mechanical issue or a struggle that was not transformative.
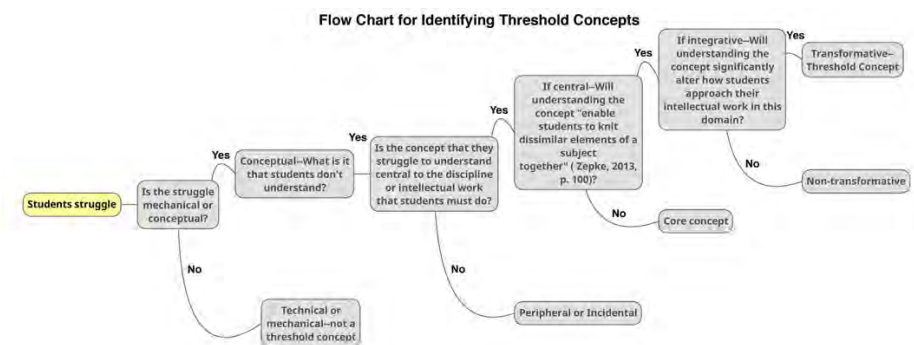
Figure 1. Flow chart for identifying threshold concepts created by workshop facilitators.

After the one-day workshop, the transdisciplinary QR cooperative faculty group worked to further refine the proposed threshold concepts by evaluating their own understanding, identifying examples within their own disciplines, and evaluating if the proposed ideas truly met the definition of a threshold concept. Through this process, the group was able to come to a consensus and to also reject some initially proposed ideas on the basis that they didn't progress on the chart (Figure 1). For example, "Selecting appropriate tools to answer a question about data, interpretation, and justification" was originally proposed, but was deemed technical or mechanical and thus not a threshold concept. Other initially proposed

concepts were absorbed into others throughout the discussion. The process yielded the five following proposed threshold concepts for QR.

Proposed Threshold Concepts

Based on the results from the working group, we propose the following "metadisciplinary" threshold concepts (Carter 2007) in QR.

1) Quantitative reasoning is an iterative process;
2) Abstract patterns can represent relationships between variables or objects;
3) There is a bidirectional translation between the concrete and the abstract;
4) Effective comparison depends on proportional reasoning; and
5) Different visual representations can communicate varying perspectives on the same quantitative information.

As QR transcends the boundaries of mathematics, although each concept may look different in different disciplines, the foundation remains the same. As stated earlier, Meyer and Land (2003) proposed that threshold concepts are most often transformative, integrative, irreversible, troublesome, and/or bounded. We take a similar view to Beitelmal et al. (2022), who focused on statistical reasoning, and Hoadly et al. (2016), who focused on finance, that the metadisciplinary nature of QR makes boundedness a non-applicable characteristic of threshold concepts for QR. Therefore, this paper focuses on the transformative, integrative, irreversible, and troublesome nature of the proposed threshold concepts for QR. For each threshold concept we will describe the proposed concept and definition, provide justification based on both the faculty workgroup discussions and existing literature, and give examples of ways that a practitioner might engage students with the threshold concepts within the classroom. In Table 1 we justify the ways in which each proposed threshold concept addresses the criteria of *transformative*, *irreversible*, *integrative*, and *troublesome*.

Five Threshold Concepts in Quantitative Reasoning: Definition, Justification, and Ways to Engage Students
1. Quantitative Reasoning Is an Iterative Process

Definition. QR is a logical, reflective, intuitive, and iterative practice, not a linear process, which includes the consideration of context, authority, and appropriateness of the quantitative information under consideration. It is an intellectual discourse designed to eventually converge, or at least get the practitioner closer to consensus while reducing uncertainty in their reasoning. QR requires not just the correct selection of appropriate tools, but also a recognition that different tools can lead to

Table 1
Outline of Proposed QR Threshold Concepts and Ways They Meet the Definition of a Threshold Concept

Threshold Concept	Troublesome	Irreversible	Integrative	Transformative
QR is an iterative process	Once an "answer" is found, additional review and revision is rarely considered necessary by the student.	As a "habit of mind," it is hard to break once developed.	The process of refining results and recognizing errors reveals unexpected similarities in different QR applications and areas.	Recognizing that QR is an iterative process moves from simple assertions to more sophisticated reasoning.
Abstract patterns can represent relationships between variables or objects	In general, real relationships are not perfect and variability unexplained by the relationship is difficult to interpret.	Once comfort with uncertainty and the ability to identify assumptions are developed, a model "feels" incomplete until those issues are addressed.	Different types of models or relationships can be extended beyond one area of application to many others, bringing together a variety of apparently unconnected contexts.	The ability to identify and describe patterns (or to recognize when they do not exist) allows the student to develop more sophisticated QR.
There is a bi-directional relationship between the concrete and the abstract	Bidirectional translation requires the student to move seamlessly between contexts and to present given information in a variety of ways. It cannot be done through memorization or arithmetic tricks.	Once the reasoning necessary to translate between the concrete and abstract is developed, the student is capable of creating representations through derivation rather than memorization.	Translation allows one to describe qualitative elements in a way that operates on them in order to draw conclusions (turns qualitative information into quantitative information, or vice-versa) or make predictions.	Formulas represent types of problems connected to real world situations. They can use the formula to draw conclusions, test ideas, and make predictions. Similarly, observation can be used to construct models.
Effective comparison depends on proportional reasoning	Additive reasoning is simpler than multiplicative. Struggles with fractions and percentages at an early age may impact one's ability to make comparisons between groups that differ at baseline.	The habit of questioning the ability to compare groups is established and proportional reasoning is approached through multiplicative thinking first, not additive thinking.	Proportional reasoning allows students to evaluate claims across contexts through simple interrogation of the information (e.g., did they control for differing group sizes?)	Proportional reasoning impacts the students' ability to understand probability, draw comparisons between groups, and generalize results to populations based on subsets of observations.
Different visual representations can communicate varying perspectives on the same quantitative information	Interpretation of visual representations as well as choices made in the creation of visual representations are challenging as they require a meaningful understanding of the context and the limitations of a given scenario.	Creating and refining visual representations becomes part of making sense of given information, even when it is not otherwise prompted for.	The ability to create and interpret allows students to apply visual representations across contexts and integrate creative modifications to the representation to enhance understanding and communication.	Visual representation is a form of communication that allows students to develop sophisticated reasoning through multivariate thinking.

different outcomes. Recognition that QR is an iterative process also impacts how one views "facts" across disciplines. In addition, the iterative process often involves recognizing and addressing errors or gaps in one's understanding.

Justification. Since QR is an iterative process, students necessarily encounter difficulties both in recognizing their own errors and misconceptions and correcting them, as well as the process of refining a result given new information. In the faculty workgroup discussions, faculty members shared how students are prone to stop once an "answer" is acquired from a calculation, and that they seldom assess if it is reasonable in the context of the situation. Ryan and Gass (2017) also identify student reflection as a troublesome concept. In this context, reflection refers to self-assessment of one's findings for reasonableness or consistency. Interrogating one's own thinking and other metacognitive practices is transformative and likely to be irreversible in that it results in the development of a habit of mind. It is integrative because as a student builds the habit of mind to refine one's results and to recognize errors, they can use similar techniques to do so even when learning an unfamiliar concept. Identifying QR is an iterative process as a threshold concept also invites students into a culture of making mistakes and fixing them which can help to cultivate a growth mindset and persistence in developing quantitative skills (Dweck 2008). Students may also identify others' mistakes or misleading quantitative claims, especially in media, as part of this process.

Ways to Engage Students. Telling students that QR is an iterative process is not sufficient to help students develop the habit of mind necessary to pass over the threshold. The habit must be cultivated. In *Example 1.1,* we provide a prompt that helps students to consider errors as part of the problem-solving process and encourages discussion about the iterative nature of fixing one's errors. In *Example 1.*2, students collect their own data on a geometric probability experiment with a known theoretical outcome. They explore how their experimental results converge to the theoretical outcomes with more and more repeated trials.

Example 1.1 Sample Task: Math Hospital. Make two columns on a sheet of paper. Write an incorrect solution to a problem that we did today on the left. Then, on the right-hand side, explain what went wrong, how to fix the mistake, and why a person might have made this error.

Example 1.2 Sample Task: Geometric Probability. Students conduct a geometric probability experiment of randomly throwing a dart onto a board with a circle inscribed in square. They find that a small set of collected data may have a high margin of error when compared with the expected value. They discuss how iterating the result via more trials should decrease this error. They then use a computer applet to generate thousands of trials and observe how it indeed converges to the expected outcome, per the Law of Large Numbers.

2. Abstract Patterns Can Represent Relationships between Variables or Objects

Definition. Quantitative systems often define the relationships among variables or objects in terms of abstract patterns, such as correlation, covariation, or even causal models. Models necessarily include assumptions and/or uncertainty in the identified relationship that must also be described and qualified.

Justification. The struggle students have with relationships between variables or objects is two-fold. First, the relationships are never perfectly defined, which creates challenges of uncertainty, variability, and/or assumptions necessary to ascribe a pattern. Second, and in contrast to the first struggle, cognitive biases tend to push individuals to look for and discover patterns even when they may not exist, as well as to attribute causality as a result of correlation. Beitelmal et al. (2022) found that correlation, association, and causation as well as uncertainty were both educator and learner-identified threshold concepts for statistical reasoning. In addition, prior research has identified regression (modeling relationships between variables) and the statistical description of relationships as threshold concepts in statistical reasoning (Beitelmal et al. 2022). Hoadley et al. (2016) identified the time value of money as a threshold concept in finance. Students encounter this concept throughout their education. In a study of fourth-grade students, Kuhn identified students' struggle with making prediction judgments while reasoning with multiple variables, even when students had previously successfully identified causal and noncausal variables (2007). The umbrella of "defining relationships between variables" captures the discipline-specific applications of the QR threshold concept. In faculty workgroup discussions at California State University Monterey Bay, similar identification of concepts such as the time value of money by a business/finance faculty member or the relationship between supply and demand are often troublesome for students to ascertain.

Recognizing these abstract patterns is irreversible because once comfort with uncertainty and the ability to identify assumptions are developed by a student, a model may "feel" incomplete until those issues are addressed. Developing more sophisticated reasoning as a way to address these issues is transformative. It is integrative because students are then able to apply the same habits across disciplines.

Ways to Engage Students. In the example below, the data are not perfectly linear, which presents the challenge of discovery described above. Students are asked to then predict gasoline consumption twelve years later, which invites discussion about if there is reason to believe the pattern should continue over time.

Example 2.1 Sample Task: Linear Model. Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below in Figure 2. How would you describe this relationship? Construct a model to predict the amount of gasoline consumed in 2016.
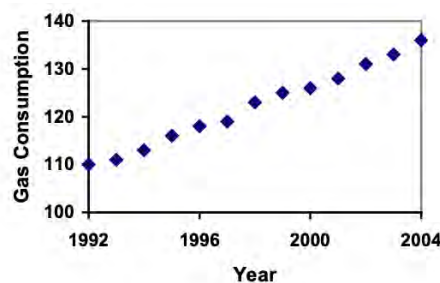
Figure 2. Graph for linear model sample task.

After the students have constructed their model, the instructor leads a discussion about what the model predicts in terms of gasoline consumption by year compared to what the recorded data show. This supports the discussion of a clear linear pattern emerging, even when the data are not perfectly linear.

3. There Is a Bidirectional Translation between the Concrete and the Abstract

Definition. QR involves transitioning back and forth between the concrete (e.g., context and observations) and the abstract (e.g., representations) in the creation, interpretation, and evaluation of models of reality. It also involves constructing multiple representations (e.g., formulas, graphs) and choosing an appropriate representation for the situation.

Justification. Bidirectional translation between the concrete and abstract allows the learner to move seamlessly between different contexts and to present given information in a variety of ways. It is integrative in the sense that it requires the student to relate multiple elements together to create a larger picture. For example, a student might construct a mathematical model for a given situation. With more information, including how well their model performs, they then adjust their model over time. This also relates to the first threshold concept presented for QR as an iterative process. The first step of the initial model from the concrete to the abstract, and vice-versa, is troublesome for many students; the further iteration to refine one's model can be even more so. Ryan and Gass (2017) identified "understanding math" as a troublesome concept specifically when students were asked to take a mathematical result and relate it back to the "bigger picture" or context. In mathematics, for example, a student might be able to perform a calculation that includes a logarithm, but when asked to interpret that calculation in a specific context (e.g., modeling population growth), the concept of logarithms may not be fully understood. In conversations with finance faculty at CSUMB, and further supported by Hoadley et al. (2016), students struggle to switch between the contexts

of financial formulas and the time-value of money. A student adept at calculation may not be able to relate their calculations back to the context of the original application, or vice versa (Verschaffel et al. 2020). Once a student crosses the threshold of bidirectional translation, it is irreversible because they can derive, and not just apply, formulas or other abstract representations. It is transformative because they are no longer limited to memorized or previously encountered relationships; they can apply their understanding across multiple contexts.
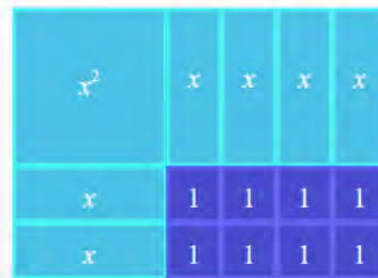
Ways to Engage Students. Instead of teaching rote memorization of calculation "tricks" in algebra or "tips" for deciphering the language of a textbook word problem, teachers can directly engage students in bidirectional translation. In Example 3.1, students demonstrate translation of an abstract idea (a symbolic quadratic equation) to a concrete one (algebra tiles representing the equation). They use the concrete representation to create a second symbolic representation of the factored form equation to solve. By first using concrete materials and then writing an abstract representation, this example uses the "concreteness fading" technique that combines the advantages of both concrete and abstract instructional materials (Fyfe et al., 2014). In Example 3.2, students discover the (abstract) mathematical constant e as it naturally arises by carrying out an application of compounding interest as the number of compoundings in a given time period increases.

Example 3.1 Sample Task: Visual Factor. Factor $f(x) = x^2 + 6x + 8$ using the diagram provided in Figure 3. Describe how the product of 14 and 12 can be ascertained from this model.

The long side can be represented by $(x + 4)$ and the short side by $(x + 2)$. By multiplying the length and width of the rectangle, we represent the total area with the factored form expression:

Figure 3. Graph for visual factoring sample task

$$f(x) = (x + 4)(x + 2).$$

Let $x = 10$ in the model. Then the product of 14 and 12 is: $100 + 6(10) + 8 = 168$.

Example 3.2 Sample Task: Discovering e. Suppose you invested \$1 for one year, at a 100% interest rate (lucky you!). Calculate the total amount of money in the account when compounding interest annually, monthly, daily, and hourly. Based

on your findings, what does it mean to compound interest continuously? Why is the number e used as the exponential base in this formula?

4. Effective Comparison Depends on Proportional Reasoning

Definition. The comparison of two or more quantities often requires the use of multiplicative—not additive—thinking, via proportions, percentages, ratios, base rates, or probabilities. The interpretation of the comparison of quantities also depends on the context of the problem. Lamon (2007) defines proportional reasoning in terms of supplying reasons in support of claims made about the structural relationships among quantities, including the ability to identify a relationship between two quantities and the ability to extend the same relationship to other pairs of quantities.

Justification. It is not surprising that a transdisciplinary team identified proportional reasoning as a threshold concept within their disciplines, as proportional reasoning is a commonly identified threshold concept for numeracy (Frith and Lloyd 2016). Lamon (2007) makes the assertion that:

> Fractions, ratios and proportions are the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science.

Similarly, unit conversion (Ryan and Gass 2017) and probability (Beitelmal et al. 2022) have also been commonly identified as troublesome or threshold concepts in scientific and statistical reasoning, and both fall under the umbrella of proportional reasoning and comparisons. In general, proportional reasoning is foundational to data literacy and numeracy, as the ability to make comparisons between populations that differ at baseline often drives misinformation (and disinformation), such as comparisons of quantity of COVID-19 cases or the total quantity of 5G cell towers instead of looking at rates per 100,000 people in a given area. Therefore, it is not surprising that much of the literature on threshold concepts in QR explores proportional reasoning as a threshold concept (see Current State of Research). Once the threshold is crossed, students develop the habit of asking the question, "Is there a difference at baseline?" The ability to reason with proportions allows the student to grow in the sophistication of their QR while able to apply this reasoning across disciplines.

Ways to Engage Students. Because students so often misinterpret or misunderstand proportional reasoning, providing an incorrect interpretation as a discussion prompt can help them to engage in this threshold concept. Example 4.1, for instance, invites a discussion of the comparison of two populations with unequal

sample sizes, as well as absolute and relative differences. Example 4.2 gives students the opportunity to grapple with incomplete information and to connect their thinking to data literacy and environmental issues.

Example 4.1 Sample Task: Comparing Quantities. Suppose, in a sample of 10,000 people, 100 of the survey participants are smokers. In the smoking group, 43 have developed lung cancer. In the nonsmoking group, 84 have developed lung cancer. Explain why this does not imply that lung cancer is more common among non-smokers when compared to smokers.

Example 4.2 Sample Task: Comparing Costs. A standard shower head uses about 2.5 gallons of water per minute. How much money does it cost for a person to take a 10-minute shower? If clean water was not available, and bottled water was used instead, how much would the shower cost? Look up costs of municipal water and bottled water online and cite your sources.

5. Different Visual Representations Can Communicate Varying Perspectives on the Same Quantitative Information

Definition. Visual representations of quantitative information are typically created or constructed and used in the processes of inquiry and argumentation. Quantitative information dictates which visual representations are appropriate, and each appropriate visual representation provides a different perspective on the same information. The creator's point of view is often reflected in the design choices for the visual representation and must be considered in the interpretation of the visual representation. Visual representations may also be used in the process of contextualizing and making sense of quantitative data, such as sketching out a model of a given scenario. Visual representations may also be dynamic, using an applet or software for real-time manipulation.

Justification. Beitelmal et al. (2022) identified threshold concepts related to visualization around the concepts of histograms, scatterplots, and bar charts. The interpretation of visual representations is often troublesome for students when asked to match a visualization to a description of a relationship, scenario, or mathematical equation, a phenomenon which was observed by many in the faculty working group. While some aspects of visualization as a threshold concept may relate to issues of bidirectional translation and/or the discovery and description of relationships between variables, the creation and interpretation of visual representations may be, in itself, troublesome. In the context of QR in the sciences, Ryan and Gass (2017) identified graphing and statistical software as a troublesome concept and emphasized that the troublesome nature of visualization was in the

development of the visualization—choosing an appropriate form and/or discerning the more appropriate form when more than one option might be possible (e.g., using boxplots for comparison of groups versus multiple histograms). The creation of visual representations is also often iterative, as several drafts may be necessary to create a reasonable representation. This practice is irreversible; once a student is in the habit of creating and refining visual representations as they make sense of quantitative information, they are unlikely to stop doing so. It is integrative because it allows them to apply visual representations across various contexts and integrate appropriate modifications to the representation. It is transformative because visualization is a form of communication that allows students to develop sophisticated reasoning through multivariate thinking.

Ways to Engage Students. Most quantitative situations involve the creation and interpretation of visual representations of quantitative information. This means that there are many opportunities to engage students. Example 5.1 involves dynamic experimentation to choose an appropriate visual representation using software. Example 5.2 has a kinesthetic component and prompts students to visually represent characteristics of a given graph using different colors.

Example 5.1 Sample Task: Representing Data. Consider a set of data including mileage and carbon footprint for several hundred different types of vehicles. Experiment with different visual representations of the given data (in a linked Excel file). Try a histogram, bar chart, and scatterplot to begin. What does each one tell you? Which would you use to try to establish a relationship between these two variables? What other visuals could you use?

Example 5.2 Sample Task: Interpreting Graphs. Given several large graphs of functions on posters, students are asked to put different color sticky notes to represent key features of each graph, including intervals of increase and decrease, inflection points, intervals of concavity, non-differentiable points, discontinuities, etc. A sample photo from a calculus class is given below in Figure 4. Note that this demonstrates various perspectives about the same quantitative information. For example, the point (-1, 2) has been identified as both a maximum and a non-differentiable point.
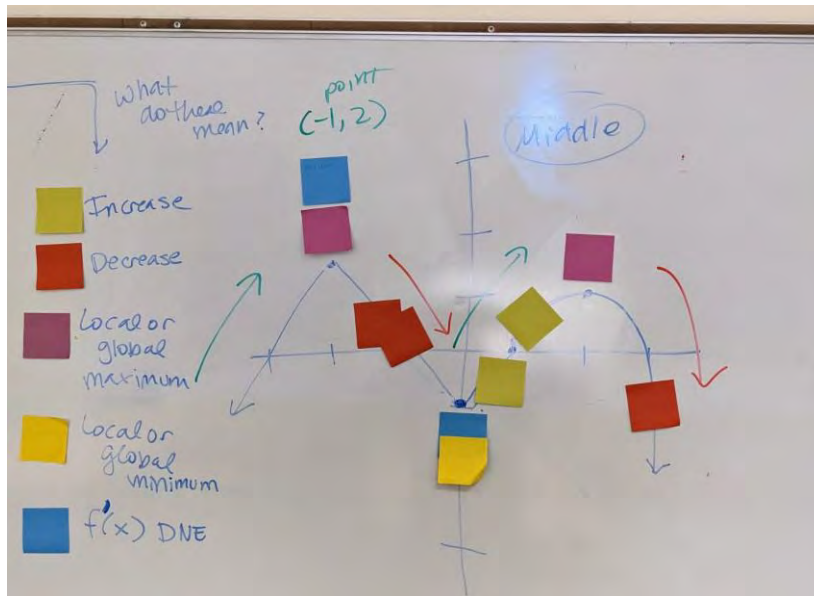
Figure 4. Example of Interpreting Graphs task

Discussion and Conclusion

This paper has proposed five threshold concepts for QR for ongoing consideration:

1) Quantitative reasoning is an iterative process;
2) Abstract patterns can represent relationships between variables or objects;
3) There is a bidirectional translation between the concrete and the abstract;
4) Effective comparison depends on proportional reasoning; and
5) Different visual representations can communicate varying perspectives on the same quantitative information.

Based on the work of Meyer and Land (2003), these concepts have been examined based on their troublesome, irreversible, integrative, and transformative characteristics (Table 1). Since QR threshold concepts are meant to be broadly applicable across disciplines, we did not include boundedness as a factor. The diverse collaboration of faculty in the workgroup and subsequent cooperative who developed the proposed threshold concepts allowed us to identify threshold concepts that may be applicable across domains that require QR as an intellectual skill within the discipline.

Engaging with threshold concepts over time, and at different levels, helps students to develop a deeper understanding and sophistication level in reasoning, become more comfortable with uncertainty, and transfer their knowledge to other domains. Moreover, directing students' attention to these threshold concepts and away from more mechanical tasks creates space for them to develop their own mathematical identity and to cultivate a growth mindset. This could be particularly

impactful for students who may have struggled with school mathematics in the past. It is important to attend to threshold concepts within mathematics and statistics courses as we believe that a focus on threshold concepts in QR reinforces the transfer of knowledge from general education mathematics and statistics, generally completed within the first year of college, to subsequent discipline-specific quantitative courses. Samples of student tasks were given as examples of how threshold concepts might be applied in a QR-based course and are not prescriptive in any way. Rather, the information presented in this paper is meant to serve as a catalyst for further discussion about the use and development of threshold concepts in QR-based courses.

Furthermore, threshold concepts in QR connect to the larger context of reasoning. Identifying core concepts in reasoning and how to teach them has been a subject of debate and research throughout human history (Holyoak and Morrison 2012). Threshold concepts are specific foundational ideas within a discipline that, once understood, transform a learner's understanding of that field. Reasoning, on the other hand, is a broader cognitive process that involves the ability to think critically, analyze information, and draw conclusions in various contexts (Khemlani 2018). While threshold concepts can be a part of the content learners reason about, reasoning encompasses a broader set of cognitive skills and processes that are used in learning and problem-solving across different domains. Therefore, the proposed threshold concepts in QR should be placed within the larger landscape of core concepts in reasoning. However, they are distinct because of the specific nature of threshold concepts (Meyer and Land 2003) and the context of QR.

The work presented here has several limitations. First, while there is justification for each of these five threshold concepts in QR, other threshold concepts may exist as well. Second, although we were able to develop these concepts together with a transdisciplinary team of faculty, the discussion took place at a single institution which might limit the scope of experiences of the faculty that engaged in the process. Engaging with practitioners across disciplines from multiple institutions across various academic levels and communities would help to strengthen and refine these concepts. Third, soliciting students' perspectives on how they have experienced these threshold concepts would provide further support or refinement to the specific aspects of the proposed threshold concepts that are troublesome or transformative, perhaps even identifying additional threshold concepts in QR. Finally, assessment of student work on tasks that involve these threshold concepts would help us to better understand how students experience and make sense of these concepts so that we, as practitioners, can attend to the diverse needs of our students as they engage with the troublesome nature of the threshold concepts.

A next step for practitioners may be to consider how to use these threshold concepts to facilitate student learning; specifically, to help identify and support

pedagogical approaches to engage threshold concepts. We can support students by helping them experience cognitive dissonance (Inhelder and Piaget 1958) and uncertainty as positive indicators of thinking and learning, and by providing learners with concrete strategies for finding and working with confusion and uncertainty. In addition, faculty need support to choose and adapt pedagogies that give students the tools to identify strategies for finding and working with confusion. For example, CSUMB has an ongoing faculty cooperative group in Reading Apprenticeship. Reading Apprenticeship is based on the ways that the social, personal, cognitive, and knowledge-building dimensions interact to support both academic and social-emotional learning (WestEd 2023). Just as students need to engage with threshold concepts over time, so too do faculty need ongoing exploration and support from experts and peers to guide them. The development of curricular and pedagogical approaches that attend to the proposed threshold concepts for QR will require time and iteration, just like QR itself.

Acknowledgments

References

Beitelmal, Wesam H., Ray Littlejohn, Paul C. Okonkwo, Israr U. Hassan, El Manaa Barhoumi, Fatemeh Khozaei, Abbas M. Hassan, and Khalid Ahmed Alkaaf. 2022. "Threshold Concepts Theory in Higher Education—Introductory Statistics Courses as an Example." *Education Sciences,* 12(11). https://doi.org/10.3390/educsci12110748

Carter, Michael. 2007. "Ways of Knowing, Doing, and Writing in the Disciplines." *College Composition and Communication,* 58(3): 385–418. https://www.jstor.org/stable/20456952

Dweck, Carol. 2008. "Mindsets and Mathematical Achievement." Prepared for the Carnegie Corporation of New York—Institute for Advanced Study Commission on Mathematics and Science Education. Accessed June 6, 2023 from http://www.growthmindsetmaths.com

Farjudian, Amin. 2023. "Bridging Mathematics and Computer Science through Threshold Concepts." *IEEE Transactions on Education,* 66(2): 139–145. https://doi.org/10.1109/TE.2022.3200162

Frith, Vera, and Pam Lloyd. 2016. "Investigating Proportional Reasoning in a University Quantitative Literacy Course." *Numeracy,* 9(1): 1–31. https://doi.org/10.5038/1936-4660.9.1.3

Fyfe, Emily R., Nicole McNeil, Ji Y. Son, and Robert L. Goldstone. 2014. "Concreteness Fading in Mathematics and Science Instruction: A Systematic Review." *Educational Psychology Review,* 26: 9–25. https://doi.org/10.1007/s10648-014-9249-3

Grawe, Nathan. 2022. "Pedagogy in Action: The SERC Portal for Educators." Last modified December 16, 2022. https://serc.carleton.edu/75180

Gutiérrez, Rochelle. 2009. "Framing Equity: Helping Students 'Play the Game' and 'Change the Game.'" *Teaching for Excellence and Equity in Mathematics,* 1(1): 4–8.

Hendrawati, Rini, Sri Mulyani, and Wiji Wiji. 2021. "A Review for Threshold Concept Identification Methods in Science." *Journal of Physics: Conference Series,* 1806 (1). http://doi.org/10.1088/1742-6596/1806/1/012192

Hoadley, S., Leigh N. Wood, Leonie Tickle, Tim Kyng. 2016. "Applying Threshold Concepts to Finance Education." *Education + Training* 58(5): 476–91. https://doi.org/10.1108/ET-02-2016-0035

Holyoak, Keith James, and Robert G. Morrison. 2012. *The Oxford Handbook of Thinking and Reasoning.* New York: Oxford University Press. https://doi.org/10.1093/oxfordhb/9780199734689.001.0001

Inhelder, Bärbel, and Jean Piaget. 1958. *The Growth of Logical Thinking: From Childhood to Adolescence.* Translated by A. Parsons & S. Milgram. Basic Books. https://doi.org/10.1037/10034-000

Kuhn, Deanna. 2007. "Reasoning about Multiple Variables: Control of Variables Is Not the Only Challenge." *Science Education*, 91(5): 710–726. https://doi.org/10.1002/sce.20214

Khemlani, Sangeet. 2018. "Chapter 11: Reasoning." *Stevens Handbook of Experimental Psychology and Cognitive Neuroscience*. John C. Wiley & Sons, 385–430. https://doi.org/10.1002/9781119170174

Lamon, S. J. 2007. "Rational Numbers and Proportional Reasoning: Toward a Theoretical Framework." In *Second Handbook of Research on Mathematics Teaching and Learning,* edited by F. Lester, 629–68. Charlotte, NC: Information Age Publishing.

Loertscher, Jennifer, David Green, Jennifer E. Lewis, Sarah Lin, and Vicky Minderhout. 2014. "Identification of Threshold Concepts for Biochemistry." *CBE Life Science Education*, 13(3): 516–28. http://doi.org/10.1187/cbe.14-04-0066

Meyer, J. H. F., and R. Land. 2003. "Threshold Concepts and Troublesome Knowledge: Linkages to Ways of Thinking and Practicing within the Disciplines." *ISL10 Improving Student Learning: Theory and Practice Ten Years On*, edited by C. Rust, 412–24. Oxford Brookes University.

Paulos, John Allen. 1988. *Innumeracy: Mathematical Illiteracy and Its Consequences.* New York: Hill and Wang.

Rhodes, T. L. (Editor). 2010. "Assessing Outcomes and Improving Achievement: Tips and Tools for Using Rubrics." Washington, DC: Association of American Colleges and Universities.

Ryan, Anne-Marie, and Susan E. Gass. 2017. "Quantitative Reasoning: Exploring Troublesome Thresholds." *Discussions on University Science Teaching: Proceedings of the Western Conference on Science Education,* 1(1): 1–16. https://ir.lib.uwo.ca/wcsedust/vol1/iss1/16

Steen, Lynn Arthur. 1990. "Numeracy." *Daedalus,* 119(2): 211–231. https://www.jstor.org/stable/20025307

Steen, Lynn Arthur. 2004. *Achieving Quantitative Literacy: An Urgent Challenge for Higher Education.* Washington, DC: Mathematical Association of America.

Vacher, H. L. 2014. "Looking at the Multiple Meanings of Numeracy, Quantitative Literacy, and Quantitative Reasoning." *Numeracy,* 7(2): 1–14. https://doi.org/10.5038/1936-4660.7.2.1

Verschaffel, Lieven, Stanislaw Schukajlow, Jon Star, and Wim Van Dooren. 2020. "Word Problems in Mathematics Education: A Survey. *ZDM,* 52: 1–16. https://doi.org/10.1007/s11858-020-01130-4

WestEd. 2023. "Reading Apprenticeship." Accessed June 6, 2023 from https://readingapprenticeship.org/

Zepke, Nick. 2013. "Threshold Concepts and Student Engagement: Revisiting Pedagogical Content Knowledge." *Active Learning in Higher Education,* 14(2): 97–107. https://doi.org/10.1177/1469787413481127